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GRANDES DÉVIATIONS POUR LE FLUX MAXIMAL EN PERCOLATION DE PREMIER PASSAGE

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Grandes déviations pour le flux maximal en percolation de premier passage

Résumé : Le sujet de cette thèse est l'étude du flux maximal en percolation de premier passage dans le graphe $\mathbb{Z}^d$ pour $d \geq 2$. Dans les trois premières parties de la thèse nous nous intéressons au flux maximal ϕ entre le sommet et la base d'un cylindre et au flux maximal τ entre le bord du demi-cylindre supérieur et le bord du demi-cylindre inférieur. Une loi des grands nombres est connue pour τ quand les dimensions du cylindre tendent vers l'infini, et elle se généralise facilement à ϕ dans le cas des cylindres très plats. Concernant ϕ dans des cylindres droits, une loi des grands nombres beaucoup plus difficile à établir a été prouvée par Kesten en 1987, et améliorée par Zhang en 2007. Dans la première partie de cette thèse nous montrons que les grandes déviations par au-dessus de τ et ϕ dans les cas cités précédemment sont d'ordre volumique. Nous obtenons de plus le principe de grande déviation correspondant pour ϕ dans des cylindres droits. Dans la deuxième partie de la thèse nous prouvons que les grandes déviations par en-dessous de τ et ϕ dans les mêmes cas sont d'ordre surfacique, et nous montrons les principes de grande déviation correspondant. Dans la troisième partie nous considérons le cas de la dimension deux, dans lequel nous généralisons la loi des grands nombres, le principe de grande déviation par en-dessous et l'étude de l'ordre des déviations supérieures à la variable ϕ dans des cylindres inclinés. La quatrième partie de la thèse est consacrée à l'étude du flux maximal à travers un domaine connexe de $\mathbb{R}^d$ dont les dimensions tendent vers l'infini à la même vitesse dans toutes les directions. Nous prouvons une loi des grands nombres pour ce flux, et nous montrons que ses déviations supérieures sont d'ordre volumique tandis que ses déviations inférieures sont d'ordre surfacique. Ce résultat s'applique en particulier aux cylindres penchés dont les dimensions grandissent de manière isotrope, et généralise donc la loi des grands nombres pour ϕ prouvée par Kesten dans le cas des cylindres droits.

Mots-clés : Percolation de premier passage, flux maximal, coupure minimale, grandes déviations.

Large deviations for the maximal flow in first passage percolation

Abstract: The object of this thesis is the study of the maximal flow in first passage percolation on the graph $\mathbb{Z}^d$ for $d \geq 2$. In the first three parts of the thesis, we are interested in the maximal flow ϕ between the top and the bottom of a cylinder and in the maximal flow τ between the boundary of the upper half cylinder and the boundary of the lower half cylinder. A law of large numbers is known for τ when the dimensions of the cylinders go to infinity, and it can be easily extended to ϕ in very flat cylinders. As concerns ϕ in straight cylinders, a law of large numbers much more difficult to establish has been proved by Kesten in 1987, and improved by Zhang in 2007. In the first part of this thesis, we prove that the upper large deviations for τ and ϕ in the cases cited above are of volume order. Moreover we obtain the corresponding large deviation principle for ϕ in straight cylinders. In the second part of the thesis, we show that the lower large deviations of τ and ϕ in the same cases are of surface order, and we prove the corresponding large deviation principles. In the third part, we consider the case of the dimension two, in which we generalize the law of large numbers, the lower large deviation principle and the study of the order of the upper large deviations to the variable ϕ in tilted cylinders. The fourth part of the thesis is devoted to the study of the maximal flow through a connected domain of $\mathbb{R}^d$ whose dimensions go to infinity at the same speed in every direction. We prove a law of large numbers for this flow, and we show that its upper large deviations are of volume order whereas its lower large deviations are of surface order. In particular, this result applies to tilted cylinders whose dimensions grow isotropically, and hence extends the law of large numbers for ϕ proved by Kesten in the case of straight cylinders.

Keywords: First passage percolation, maximal flow, minimal cut, large deviations.

AMS Classification: 60K35, 60F10.

Contents

CHAPITRE 1

Introduction

1. Présentation du modèle

1.1. Introduction à la mécanique statistique. La physique statistique est une branche de la physique qui a pour but de comprendre le comportement d'un système par l'étude des caractéristiques et des interactions de ses constituants. Ceux-ci ont une taille très petite par rapport au système étudié, qui en comporte donc un très grand nombre. L'échelle du système tout entier est dite macroscopique, et celle beaucoup plus petite de ses constituants est dite microscopique.

Voici quelques exemples de systèmes que l'on peut étudier en physique statistique :

- *Un métal ferromagnétique :* c'est un métal qui est aimanté pour toute température inférieure à une température fixée, appelée température de Curie, et qui perd brusquement son aimantation dès que la température dépasse cette valeur critique. L'aimantation globale du métal existe si les moments magnétiques de spins des atomes microscopiques qui constituent le système s'orientent de façon privilégiée suivant une direction. Cette vision microscopique du système a donné lieu à la création du modèle d'Ising en physique statistique.
- *Une protéine dans une cellule :* pour comprendre la localisation d'une protéine à l'intérieur d'une cellule, on peut s'intéresser aux interactions que chaque acide aminé constituant la protéine a avec le milieu qui présente des inhomogénéités. L'échelle microscopique est ici celle de l'acide aminé. Ce système est étudié via différents modèles de polymères en physique statistique.
- *Une forêt infectée par une maladie :* pour expliquer pourquoi certaines maladies qui affectent des arbres déciment des forêts entières, on peut regarder comment la maladie se propage localement d'un arbre aux arbres voisins. L'étude de ce système à l'échelle microscopique, c'est-à-dire ici à l'échelle de chaque arbre constituant cette forêt, correspond à l'étude du modèle de percolation en physique statistique.

Bien sûr, beaucoup d'autres modèles existent. Comme nous pouvons le constater, le rapport entre les échelles macroscopique et microscopique d'un modèle, c'est-à-dire le nombre d'éléments microscopiques constituant le système, est toujours très grand, mais la taille d'un constituant microscopique n'a pas d'importance en soi.

Puisque le système étudié est toujours composé d'un très grand nombre de composants microscopiques, la compréhension d'un phénomène macroscopique implique la description du comportement d'un nombre important de ses constituants. Il n'est donc pas envisageable d'étudier toutes les caractéristiques de chaque élément microscopique individuellement. C'est à ce stade que les probabilités interviennent : le but de la mécanique statistique est l'étude mathématique du système afin d'en déterminer le comportement le plus probable, une fois l'ensemble des configurations du système muni d'une loi de probabilité cohérente avec les phénomènes physiques, biologiques ou chimiques en jeu. Lorsque le comportement le plus probable du système est compris, la mécanique statistique peut s'intéresser aux comportements atypiques du système, par exemple en essayant de déterminer la probabilité de leur réalisation. Ceci est le principe même de l'étude des grandes déviations d'un système.

1.2. Percolation. Revenons à présent au troisième exemple présenté ci-dessus, celui de la propagation d'une maladie dans une forêt. Pour construire un modèle mathématique intéressant, il faut simplifier suffisamment le système physique observé pour être capable d'en faire une étude fructueuse, tout en conservant suffisamment de sa complexité pour en garder les propriétés macroscopiques. Dans le cas de la transmission de la maladie entre les arbres, nous allons donc faire quelques hypothèses simplificatrices. Tout d'abord nous supposons que chaque arbre infecté ne peut transmettre la maladie qu'à un ensemble fini d'arbres donné, les arbres voisins, c'est-à-dire que la transmission de l'infection n'a lieu que localement. Nous supposons aussi que la transmission de la maladie d'un arbre infecté à l'un de ses voisins a lieu avec une certaine probabilité p qui ne dépend que de la maladie, donc ni de l'arbre infecté, ni de son voisin, ni de la position du groupe d'arbres dans la forêt, etc... Nous pouvons maintenant donner une définition mathématique du modèle. Considérons un graphe G de sommets V qui correspondent chacun à un arbre, et d'arêtes E. Deux arbres sont reliés par une arête si et seulement si ils sont suffisamment proches dans la forêt pour se contaminer. Donnons-nous un paramètre p à valeurs dans $[0, 1]$. Il va quantifier la propension de la maladie à s'étendre : plus le paramètre est proche de 1, plus un arbre voisin d'un arbre infecté va avoir de risques d'être infecté lui aussi. À chaque arête e de E nous associons une variable aléatoire $t(e)$ de loi de Bernoulli de paramètre p, de telle sorte que la famille $(t(e), e \in E)$ est indépendante et identiquement distribuée (i.i.d.). Si la variable $t(e)$ vaut 1 (on dit alors que l'arête e est ouverte), la maladie va se propager le long de l'arête e, c'est-à-dire que si l'un des sites à l'extrémité de e est infecté, l'autre le sera aussi. Inversement, si $t(e) = 0$ (on dit que l'arête est fermée), la maladie ne peut pas se propager via e. Ce que nous venons de décrire est appelé le modèle de percolation par arête. Sa formulation mathématique a été introduite par Broadbent et Hammersley en 1957 [**15**].

Tout l'enjeu de l'étude du modèle de percolation est de comprendre au mieux les propriétés du graphe aléatoire $\widetilde{G}$ composé de l'ensemble des sommets de G, et uniquement des arêtes ouvertes pour la famille de variables aléatoires $(t(e), e \in E)$. Si un arbre est infecté, quelle est la probabilité qu'il infecte plus de N arbres pour N grand, c'est-à-dire quelle est la probabilité que la composante connexe dans $\widetilde{G}$ du sommet représentant cet arbre soit de taille supérieure à N ? Quelle est la probabilité que la maladie transmise par cet arbre infecte une région donnée de la forêt, c'est-à-dire que la composante connexe dans $\widetilde{G}$ du sommet représentant notre arbre infecté à l'origine intersecte un sous-ensemble donné de V ? Dans l'étude d'un modèle de physique statistique, il est logique de se placer à la limite où les composants microscopiques sont en nombre infini, car cela traduit la différence d'échelle entre le système et ses composants. Si on se place dans ce cadre, notre graphe G est infini, et on peut se demander si une maladie a un risque de se propager sur des distances infiniment grandes à l'échelle microscopique (c'est-à-dire à travers la forêt entière à l'échelle macroscopique). Cela revient à se poser la question de l'existence d'une composante connexe infinie dans notre graphe aléatoire $\widetilde{G}$. Un des résultats fondamentaux qui ont été démontrés sur le modèle de percolation est le suivant :

THÉORÈME 1. *On considère le modèle de percolation par arêtes de paramètre p sur le graphe $G = (\mathbb{Z}^d, \mathbb{E}^d)$ en dimension $d \geq 2$, où G a pour sommets les points de $\mathbb{Z}^d$ et pour arêtes l'ensemble des arêtes entre plus proches voisins. Le système présente une transition de phase, c'est-à-dire qu'il existe un paramètre critique $p_c(d)$ dans $]0, 1[$ tel que :*
- pour tout $p < p_c(d)$ (régime sous-critique), presque sûrement, il n'existe pas de composante connexe infinie d'arêtes ouvertes dans le graphe,
- pour tout $p > p_c(d)$ (régime sur-critique), presque sûrement, il existe une unique composante connexe infinie d'arêtes ouvertes dans le graphe.

Cette propriété de transition de phase a été montrée par Broadbent et Hammersley [**15**] et par Hammersley [**36**], [**37**]. Pour une démonstration de ce théorème et bien d'autres, nous renvoyons le lecteur à l'excellent livre [**35**], qui est un ouvrage de référence sur le modèle de percolation. De nombreux autres résultats ont été prouvés à propos de ce modèle, en particulier dans $\mathbb{Z}^d$. Il serait impossible de tous les présenter ici, et par ailleurs l'étude de ce modèle n'est pas l'objet de cette thèse. Nous avons néanmoins choisi de présenter ce résultat car nous retrouverons le paramètre $p_c(d)$ introduit ici dans l'énoncé d'un grand nombre de nos résultats.

1.3. Percolation de premier passage. Hammersley et Welsh [**38**] ont introduit en 1967 une variante du modèle de percolation par arête, appelé modèle de percolation de premier passage. Reprenons la problématique de la transmission d'une maladie au sein d'une forêt. Au lieu de se demander si la transmission entre un arbre infecté et un de ses voisins a lieu ou non, on peut se demander en combien de temps elle a lieu, pour ensuite comprendre la vitesse de propagation de la maladie à l'échelle de la forêt toute entière. Nous conservons les notations du modèle de percolation et nous considérons toujours un graphe $G = (V, E)$ dont les sommets représentent les arbres. À présent, si une arête existe, cela signifie que la propagation de la maladie entre les arbres qui sont aux deux extrémités de cette arête aura lieu si un des arbres est infecté, mais l'inconnue est le temps nécessaire pour que cette transmission de la maladie ait lieu. On associe donc à chaque arête e une variable aléatoire $t(e)$ qui cette fois est à valeurs dans $\mathbb{R}^+$: c'est ce temps de propagation. On suppose à nouveau que la loi de cette variable ne dépend pas des arbres localement, i.e., que la famille $(t(e), e \in E)$ est i.i.d. Nous obtenons le modèle de percolation de premier passage.

L'enjeu de l'étude de ce modèle est de comprendre la métrique aléatoire sous-jacente : temps minimum nécessaire pour que la maladie se propage d'un sommet à un autre, forme de l'ensemble des sommets atteints par la maladie en un temps t si un seul arbre fixé était infecté au temps 0, etc... Pour une présentation plus complète des questions liées à l'étude de cette métrique aléatoire et un aperçu des résultats connus, nous conseillons la lecture du cours de l'École d'été de probabilités de Saint Flour écrit par Kesten [**40**], et pour des avancées plus récentes l'article de Benjamini, Kalai et Schramm [**8**] qui met en relief la complexité de ce modèle pourtant si simple à définir.

Dans cette thèse, nous allons étudier le modèle de percolation de premier passage considéré sous un angle différent. Nous avons introduit le modèle de percolation comme une modélisation de la propagation d'une maladie dans une forêt, mais il peut aussi modéliser une roche dont on veut savoir si elle est poreuse. Les arêtes du graphe sont alors vues comme des petits tuyaux qui laissent passer de l'eau s'ils sont ouverts, et n'en laissent pas passer s'ils sont fermés. Une roche sera donc considérée comme poreuse si l'eau peut la traverser à l'échelle macroscopique, c'est-à-dire s'il existe une composante connexe infinie d'arêtes ouvertes dans le graphe G infini. C'est en fait pour modéliser ainsi un milieu poreux que le modèle de percolation a été introduit pas Broadbent et Hammersley [**15**]. Le mot percoler en français signifie d'ailleurs pour un liquide de passer à travers des matériaux poreux. Les enjeux du modèle de percolation restent les mêmes, et le théorème 1 garde toute son importance dans cette interprétation. Nous avons choisi de présenter le modèle de percolation comme la modélisation de la transmission d'une maladie dans une forêt, car c'est en partant de la modélisation de ce problème que Hammersley et Welsh ont défini dans [**38**] le modèle de percolation de premier passage.

Dans le contexte des milieux poreux, le modèle de percolation de premier passage peut être interprété d'une autre manière : si le graphe est un ensemble de tuyaux, la variable réelle positive ou nulle $t(e)$ associée au tuyau e peut être vue comme la capacité de e, c'est-à-dire la quantité maximale d'eau qui peut traverser e par seconde dans un régime de circulation de l'eau à l'équilibre (c'est-à-dire que l'on suppose l'ensemble du système déjà immergé et l'eau en circulation). Le modèle mathématique est exactement le même, il a donc gardé le nom de percolation de premier passage, même si ce nom est lié à l'interprétation du modèle en termes de temps d'atteinte. Une troisième interprétation possible du modèle serait de considérer les arêtes du réseau comme des fils électriques dont les résistances ou conductances sont données par les variables aléatoires $(t(e), e \in E)$; cette interprétation est assez proche de l'interprétation en termes de réseau de tuyaux en ce qui concerne les problèmes soulevés, mais le traitement du modèle est un peu différent compte tenu du comportement des résistances montées en parallèle. Nous ne nous intéresserons pas davantage aux réseaux électriques, pour nous concentrer sur l'interprétation du modèle en terme d'ensemble de tuyaux de capacité aléatoire. La question qui se pose naturellement dans ce modèle est la suivante : si la roche est poreuse, quelle quantité d'eau au maximum va pouvoir traverser une couche de roche de taille macroscopique par seconde ? Il faut donc définir ce qu'est un flux maximal dans notre graphe.

2. Flux maximal : définition et état de l'art

2.1. Définition du flux maximal en percolation de premier passage.

2.1.1. *Deux formulations équivalentes.* Rappelons brièvement en quoi consiste le modèle de percolation de premier passage : nous considérons un graphe $G = (V, E)$ de sommets V et d'arêtes E (on peut penser au graphe de sommets les points de $\mathbb{Z}^d$ pour $d \geq 2$, et d'arêtes reliant les sites les plus proches voisins pour la distance euclidienne, ou à un sous-ensemble de ce graphe). À chaque arête e dans E on associe une variable aléatoire $t(e)$ à valeurs dans $\mathbb{R}^+$ de telle sorte que la famille $(t(e), e \in E)$ est indépendante et identiquement distribuée. On interprète $t(e)$ comme la capacité de l'arête e, elle-même vue comme un petit tuyau : $t(e)$ est la quantité maximale d'eau qui peut traverser e par seconde. On note F la fonction de répartition de la loi des capacités des arêtes. Soient F_1 et F_2 deux sous-ensembles non vides de V. Nous allons définir le flux maximal de F_1 à F_2 dans G. On dit qu'une arête e, notée $\langle x, y \rangle$ et d'extrémités les sommets x et y, est incluse dans un sous-ensemble A de $\mathbb{R}^d$ si et seulement si le segment joignant x à y, à l'exception éventuellement de ses extrémités, est inclus dans A. On note $\vec{E}$ l'ensemble des arêtes orientées de G, c'est-à-dire qu'un élément $\vec{e}$ de $\vec{E}$ est une paire ordonnée de sommets de G qui sont voisins pour la structure de graphe de G. Si l'arête $\vec{e}$ a pour extrémités les sommets x et y et est orientée de x vers y, on la note $\langle\langle x, y \rangle\rangle$. On considère l'ensemble $\mathcal{S}$ des couples de fonctions (g, o) avec $g : E \to \mathbb{R}^+$ et $o : E \to \vec{E}$ telles que $o(\langle x, y \rangle) \in \{\langle\langle x, y \rangle\rangle, \langle\langle y, x \rangle\rangle\}$ et satisfaisant les conditions suivantes :

(i) pour toute arête e de E, on a
$$0 \leq g(e) \leq t(e) \,,$$

(ii) pour tout sommet v de $V \setminus (F_1 \cup F_2)$ on a
$$\sum_{e \in E} g(e) \mathbb{1}_{\{o(e) = \langle\langle v, \cdot \rangle\rangle\}} = \sum_{e \in E} g(e) \mathbb{1}_{\{o(e) = \langle\langle \cdot, v \rangle\rangle\}} \,,$$

où la notation $o(e) = \langle\langle v, \cdot \rangle\rangle$ (respectivement $o(e) = \langle\langle \cdot, v \rangle\rangle$) signifie qu'il existe u dans V tel que $e = \langle u, v \rangle$ et $o(e) = \langle\langle v, u \rangle\rangle$ (respectivement $o(e) = \langle\langle u, v \rangle\rangle$). Un couple $(g, o) \in \mathcal{S}$ est un courant possible pour l'eau entre F_1 et F_2 dans G : $g(e)$ est la quantité d'eau qui traverse l'arête e par seconde, et $o(e)$ indique la direction dans laquelle l'eau circule. La condition (i) signifie que la quantité d'eau qui traverse une arête ne peut pas excéder sa capacité et la condition (ii) exprime l'absence de fuite dans le réseau de tuyaux. Ainsi l'eau ne peut entrer ou sortir de G que par les sommets de F_1 et F_2. À chaque courant possible dans le graphe on associe le flux correspondant
$$\mathrm{flux}(g, o) = \sum_{u \in G \setminus F_2 \,,\, v \in F_2 \,:\, \langle u, v \rangle \in E} g(\langle u, v \rangle) \mathbb{1}_{\{o(\langle u, v \rangle) = \langle\langle u, v \rangle\rangle\}} - g(\langle u, v \rangle) \mathbb{1}_{\{o(\langle u, v \rangle) = \langle\langle v, u \rangle\rangle\}} \,.$$

C'est la quantité d'eau qui circule entre F_1 et F_2 dans G si l'eau circule suivant le courant (g, o). Le flux maximal entre F_1 et F_2 dans G, noté $\phi(F_1 \to F_2 \text{ dans } G)$, est le supremum de cette quantité sur tous les choix possibles de courant :
$$\phi(F_1 \to F_2 \text{ dans } G) = \sup\{\mathrm{flux}(g, o) \,|\, (g, o) \in \mathcal{S}\} \,.$$

Par commodité, on notera de même le flux maximal dans des sous-ensembles continus de $\mathbb{R}^d$ en se ramenant à leur intersection avec l'ensemble des sommets du graphe.

Cette définition du flux n'est pas toujours facile à manipuler, c'est pourquoi il est intéressant d'en donner une formulation équivalente. Pour ce faire nous avons besoin de quelques définitions simples. Un chemin d'un sommet x à un sommet y dans G est une suite

$$(x = v_0, e_1, v_1, ..., e_n, v_n = y)$$

de sommets $v_0, ..., v_n$ alternant avec des arêtes $e_1, ..., e_n$ tels que pour tout $i \in \{1, ..., n\}$, $e_i = \langle v_{i-1}, v_i \rangle$. On dit qu'un ensemble d'arêtes $\mathcal{E}$ inclus dans E sépare F_1 de F_2 dans G s'il n'existe aucun chemin de F_1 à F_2 dans $G \setminus \mathcal{E}$. Lorsque le contexte le permet, on omettra de préciser les ensembles F_1, F_2 et G pour parler simplement d'ensemble de coupure. Par ailleurs, si $\mathcal{E}$ est un ensemble d'arêtes de G, on définit sa capacité, notée $V(\mathcal{E})$, par

$$V(\mathcal{E}) \,=\, \sum_{e \in \mathcal{E}} t(e)\,.$$

Le théorème du flux maximal - coupure minimale ("Max flow - Min cut Theorem" en anglais), voir par exemple [**12**], établit que

$$\phi(F_1 \rightarrow F_2 \text{ dans } G) \,=\, \inf\{V(\mathcal{E}) \,|\, \mathcal{E} \text{ sépare } F_1 \text{ de } F_2 \text{ dans } G\}\,.$$

On peut évidemment se restreindre à considérer uniquement les ensembles de coupure qui sont minimaux pour cette propriété, c'est-à-dire tels qu'aucun de leurs sous-ensembles stricts ne sont des ensembles de coupure. L'idée qui est derrière cette propriété de théorie des graphes est simple et intuitive : le flux maximal est déterminé par la capacité des arêtes qui sont saturées, c'est-à-dire qui sont traversées par une quantité d'eau égale à leur capacité. Ces arêtes saturées constituent un ensemble de coupure, sinon cela contredirait la maximalité du flux qui pourrait s'accroître via un chemin constitué uniquement d'arêtes non saturées. Finalement, certaines arêtes saturées ne vont éventuellement pas avoir un effet limitant sur le flux si celui-ci est déjà limité en amont ou en aval par d'autres arêtes saturées, c'est pourquoi il faut considérer un ensemble de coupure de capacité la plus petite possible.

2.1.2. *Flux maximal, nombre de chemins et surfaces.* Regardons un instant à quoi correspond le flux maximal dans le cas où la loi des capacités des arêtes est une loi de Bernoulli de paramètre p (c'est le cas de la percolation classique). Considérons un graphe fini $G = (V, E)$ et le flux maximal dans G entre deux sous-ensemble disjoints de V notés F_1 et F_2. On dit qu'un chemin est ouvert s'il n'est composé que d'arêtes ouvertes, i.e., de capacité égale à 1. Deux chemins sont dits disjoints s'ils n'ont aucune arête commune, même s'ils passent par des sommets communs. Un résultat simple de théorie des graphes établit que le nombre maximum de chemins ouverts disjoints entre F_1 et F_2 dans G est exactement égal au nombre minimum d'arêtes ouvertes à supprimer pour couper tout chemin ouvert de F_1 à F_2. La capacité de toute arête ouverte étant égale à 1, et toute arête fermée ne laissant passer aucun flux, ce nombre minimal d'arêtes ouvertes à supprimer pour couper tout chemin ouvert de F_1 à F_2 dans G est exactement égal à la capacité minimale d'un ensemble séparant F_1 de F_2 dans G. Par le théorème du flux maximal - coupure minimale, nous savons que la capacité minimale d'un tel ensemble est égale au flux maximal entre F_1 et F_2 dans G. Nous en déduisons que le flux maximal entre F_1 et F_2 dans G est égal au nombre maximal de chemins ouverts disjoints reliant F_1 à F_2 dans G. La figure 1 donne une visualisation du courant correspondant au flux maximal $\phi(B)$ entre le sommet et la base d'un cylindre B dans le graphe de sommets les points $\mathbb{Z}^d$ (seules les arêtes ouvertes sont représentées sur la figure).

Dorénavant, nous considérerons uniquement le graphe $(\mathbb{Z}^d, \mathbb{E}^d)$ de sommets les points de $\mathbb{Z}^d$ dont les arêtes $\mathbb{E}^d$ relient les plus proches voisins pour la distance euclidienne, éventuellement renormalisé (dans le chapitre 7), et des sous-ensembles de celui-ci. Grâce à la définition équivalente du flux maximal en terme d'ensemble de coupure minimal, nous pouvons trouver une équivalence en dimension deux entre les deux interprétations du modèle de percolation de premier passage que nous avons évoquées précédemment, celle en termes de temps d'atteinte et celle en termes de

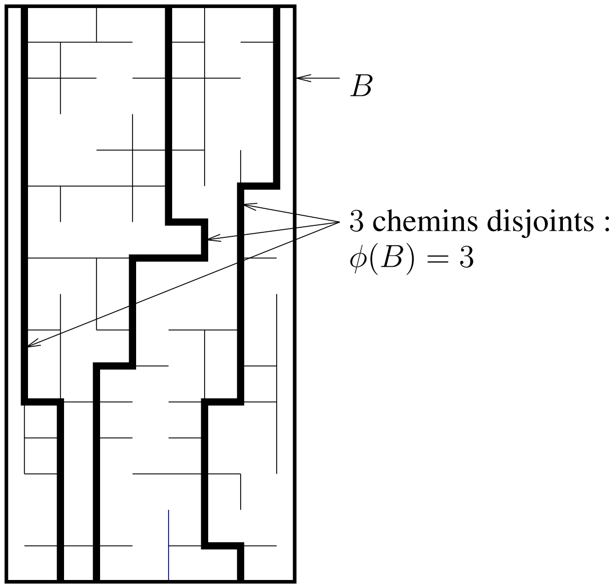

FIG. 1. Capacité des arêtes suivant une loi de Bernoulli.

capacité ou de flux. La dimension deux nous offre la possibilité de parler du graphe dual du graphe (planaire donc) que nous étudions. Le dual d'un graphe planaire a un sommet dans chacune des faces du graphe initial, et une arête entre deux sommets si et seulement si les faces correspondantes du graphe initial sont adjacentes. Le graphe dual du graphe $(\mathbb{Z}^2, \mathbb{E}^2)$ est très simple, il n'est autre que lui-même translaté par le vecteur $(1/2, 1/2)$. Chaque arête e du graphe initial est coupée par une et une seule arête e^* du graphe dual, et e^* est perpendiculaire à e et la coupe en son milieu. On définit alors $t(e^*) = t(e)$, associant ainsi à chaque arête du graphe dual une variable aléatoire positive. Pour toute arête e du graphe initial, nous interprétons $t(e)$ comme la capacité de e, en revanche pour toute arête e^* du graphe dual nous interprétons $t(e^*)$ comme le temps nécessaire pour traverser e^*. Considérons le cylindre $B = [0, n] \times]0, n[$ dans le graphe initial, et le cylindre dual $B^* =]-1/2, n+1/2[\times [1/2, n-1/2]$. D'après le théorème du flux maximal - coupure minimale, le flux maximal du sommet $[0, n] \times \{n\}$ à la base $[0, n] \times \{0\}$ du cylindre B est égal au minimum des capacités des ensembles de coupure dans B qui sont de plus minimaux pour l'inclusion. Soit $\mathcal{E}$ un tel ensemble de coupure, et $\mathcal{E}^*$ son ensemble dual (i.e., l'ensemble des arêtes duales des arêtes de $\mathcal{E}$). Alors on remarque que $\mathcal{E}^*$ est un chemin du côté gauche $\{-1/2\} \times [1/2, n-1/2]$ au côté droit $\{n+1/2\} \times [1/2, n-1/2]$ du cylindre B^* dans le graphe dual. La réciproque est également vraie, i.e., tout chemin de gauche à droite dans B^* a pour dual un ensemble de coupure pour B (voir figure 2). Le flux maximal du sommet à la base de B dans le graphe initial est donc égal au temps d'atteinte minimal entre le côté gauche et le côté droit de B^* dans le graphe dual. Cela nous donne une autre image de ce qu'est le flux maximal en percolation de premier passage dans le cas de la dimension deux. L'étude du modèle de percolation de premier passage dans cette dimension est donc essentiellement la même quelle que soit l'interprétation que l'on fait des variables $(t(e), e \in E)$. C'est la raison pour laquelle Kesten a présenté dans [**41**] le modèle de percolation de premier passage interprété en termes de flux comme une version en dimension supérieure ou égale à trois du modèle plus ancien et plus classique de percolation de premier passage interprété en termes de temps d'atteinte. Le modèle ayant été étudié depuis plus longtemps sous le point de vue des temps d'atteinte, voir par exemple [**40**] pour une présentation de quelques résultats connus, l'étude du flux maximal en percolation de premier passage a déjà été effectuée en partie en dimension deux. Le résultat de Garet que nous allons présenter et le chapitre 4 de cette thèse apporteront des compléments à cette étude en dimension deux. Dans le reste de la thèse, nous énoncerons les

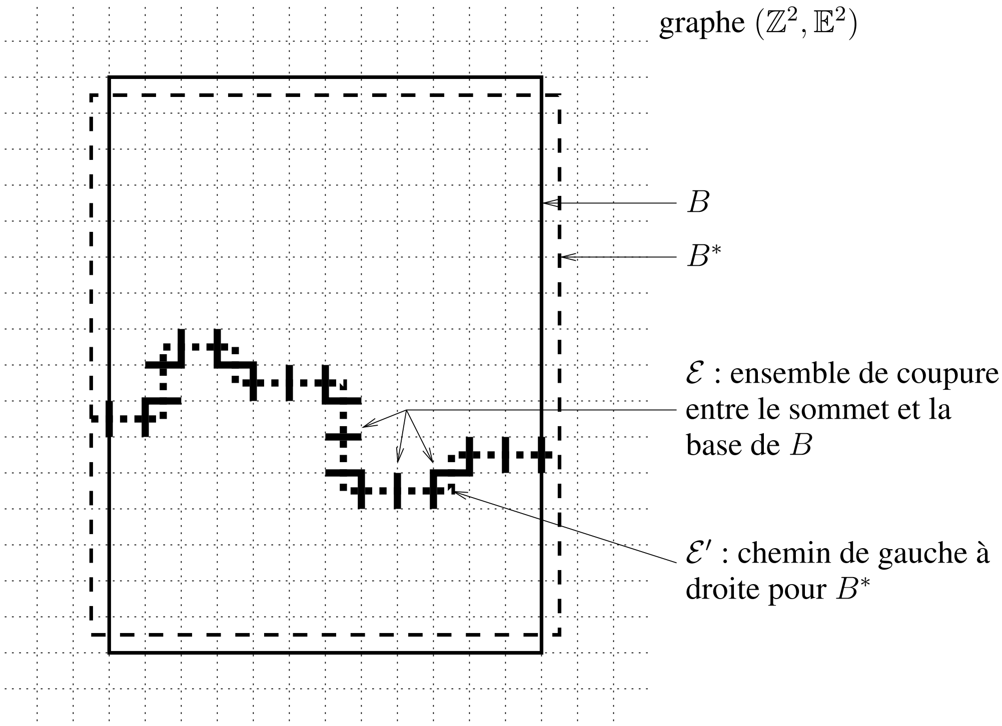

FIG. 2. Dimension deux et dualité.

résultats en dimension $d \geq 2$, sachant que les démonstrations pourraient éventuellement être simplifiées en dimension deux, et que les techniques de preuves utilisées pour la dimension trois sont en fait valables pour toute dimension $d \geq 2$ car en dimension trois le modèle révèle déjà toute sa complexité.

En effet, essayons maintenant d'imaginer et de visualiser le dual d'une arête en dimension $d \geq 3$, c'est-à-dire en un sens l'objet unitaire naturel qui intersecte une arête quelconque du graphe $(\mathbb{Z}^d, \mathbb{E}^d)$. Nous sommes amenés à définir une plaquette $\pi(e)$, c'est-à-dire un petit carré qui est un translaté par une translation à coordonnées entières d'un des éléments $[-1/2, 1/2]^i \times \{1/2\} \times [-1/2, 1/2]^{d-i-1}$ pour $i = 0, ..., d-1$, qui est orthogonal à l'arête e en question et qui la coupe en son milieu (voir figure 3). En dimension deux, comme expliqué précédemment,

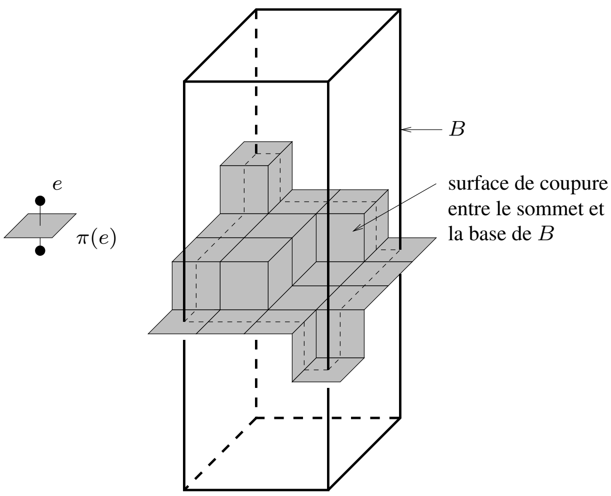

FIG. 3. Plaquette et surface de coupure.

cette plaquette est en fait plus simplement le translaté d'une arête, mais dès la dimension trois l'objet dual est différent d'une arête. En fait, on peut voir une plaquette comme un petit élément unitaire de surface. Vu sous cet angle, le dual d'un ensemble d'arêtes dans un sous-ensemble B de $\mathbb{R}^d$ qui sépare deux sous-ensembles disjoints F_1 et F_2 de B est un ensemble de plaquettes qui constitue une surface discrète qui disconnecte F_1 de F_2 dans B (nous n'essaierons pas ici de donner une définition propre du terme de surface, mais nous encourageons le lecteur à en avoir une représentation visuelle, grâce par exemple à la figure 3). Nous ferons régulièrement allusion à un ensemble de coupure en termes de "surface de coupure" en ayant en tête cette correspondance arête - plaquette implicite.

2.1.3. *Flux maximal dans des cylindres.* Nous repassons au cadre de la dimension $d \geq 2$. Nous n'avons pour l'instant parlé de flux maximal qu'en toute généralité. Il est important de définir deux cas particuliers. Soit A un hyperrectangle non dégénéré, c'est-à-dire une boîte de dimension $d-1$ dans $\mathbb{R}^d$. Sauf exception (chapitre 2, cas des cylindres droits) pour des raisons techniques, nous supposerons tous les hyperrectangles fermés dans $\mathbb{R}^d$. On note $\vec{v}$ un des deux vecteurs unitaires orthogonaux à $\mathrm{hyp}(A)$, l'hyperplan dans lequel A est inclus. Pour h un réel positif, on définit le cylindre de base A et de hauteur h, noté $\mathrm{cyl}(A,h)$, par

$$\mathrm{cyl}(A,h) \,=\, \{x + t\vec{v} \,|\, x \in A, t \in [-h,h]\}\,.$$

Soit $T(A,h)$ (respectivement $B(A,h)$) le sommet (respectivement la base) de $\mathrm{cyl}(A,h)$, i.e.,

$$T(A,h) \,=\, \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h)\,,\ \langle x,y\rangle \in \mathbb{E}^d \text{ et } \langle x,y\rangle \text{ intersecte } A + h\vec{v}\}\,,$$

$$B(A,h) \,=\, \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h)\,,\ \langle x,y\rangle \in \mathbb{E}^d \text{ et } \langle x,y\rangle \text{ intersecte } A - h\vec{v}\}\,.$$

On définit le flux maximal du sommet à la base du cylindre $\mathrm{cyl}(A,h)$, noté $\phi(A,h)$, par

$$\phi(A,h) \,=\, \phi(T(A,h) \to B(A,h) \text{ dans } \mathrm{cyl}(A,h))\,. \tag{1.1}$$

Dans le cas de cylindres droits, c'est-à-dire que A est de la forme $\prod_{i=1}^{d-1}[a_i,b_i] \times \{c\}$, si les a_i, b_i, c et h sont entiers, $T(A,h)$ (respectivement $B(A,h)$) correspond exactement aux points du graphe qui sont sur le sommet $\prod_{i=1}^{d-1}[a_i,b_i] \times \{h\}$ (respectivement la base $\prod_{i=1}^{d-1}[a_i,b_i] \times \{-h\}$) du cylindre $\mathrm{cyl}(A,h)$ vu comme sous-ensemble de $\mathbb{R}^d$; la définition ci-dessus, moins intuitive, est néanmoins nécessaire dans le cas des cylindres inclinés, car nous devons considérer une version discrète du sommet et de la base du cylindre. Via le théorème du flux maximal - coupure minimale, nous savons que $\phi(A,h)$ est aussi la capacité minimale d'un ensemble d'arêtes qui sépare $T(A,h)$ de $B(A,h)$ dans $\mathrm{cyl}(A,h)$. Le dual d'un tel ensemble de coupure est une surface de plaquettes qui sépare $T(A,h)$ de $B(A,h)$ dans $\mathrm{cyl}(A,h)$. La trace de cette surface sur les "parois verticales" $\mathrm{cyl}(\partial A,h)$ du cylindre est complètement libre. Si nous considérons deux cylindres côte à côte, par exemple $[0,n]^d$ et $[n,2n] \times [0,n]^{d-1}$, et une surface de coupure dans chacun de ces cylindres pour un flux entre le sommet et la base du cylindre (dans la direction donnée par le vecteur de coordonnées $(0,...,0,1)$), nous ne pouvons pas a priori les recoller ensemble, c'est-à-dire que l'union de ces deux surfaces ne constitue pas nécessairement une surface de coupure entre le sommet et la base du cylindre $[0,2n] \times [0,n]^{d-1}$. Nous n'avons donc pas de propriété de sous-additivité pour notre famille $\phi(\mathrm{cyl}(A,h))$, ce qui constitue une réelle difficulté pour l'étude de cette variable. Il est alors naturel de vouloir définir un autre flux maximal à l'intérieur de $\mathrm{cyl}(A,h)$ qui aurait de bonnes propriétés de sous-additivité. Pour ce faire, il faut que ce flux maximal corresponde à une surface de coupure minimale dont la trace sur les parois verticales du cylindre $\mathrm{cyl}(\partial A,h)$ soit fixée et plate, par exemple le long du bord ∂A de A. C'est ce que nous faisons à présent. L'ensemble $\mathrm{cyl}(A,h) \smallsetminus \mathrm{hyp}(A)$ a deux composantes connexes, que nous notons $\mathcal{C}_1(A,h)$ et $\mathcal{C}_2(A,h)$. Pour $i=1,2$, soit A_i^h l'ensemble des points de $\mathcal{C}_i(A,h) \cap \mathbb{Z}^d$ qui ont un plus proche voisin dans $\mathbb{Z}^d \smallsetminus \mathrm{cyl}(A,h)$:

$$A_i^h \,=\, \{x \in \mathcal{C}_i(A,h) \cap \mathbb{Z}^d \,|\, \exists y \in \mathbb{Z}^d \smallsetminus \mathrm{cyl}(A,h)\,,\ \langle x,y\rangle \in \mathbb{E}^d\}\,.$$

On définit alors

$$\tau(A,h) \,=\, \phi(A_1^h \to A_2^h \text{ dans } \operatorname{cyl}(A,h))\,. \tag{1.2}$$

Il s'agit du flux maximal dans $\operatorname{cyl}(A,h)$ entre le bord du demi-cylindre supérieur et le bord du demi-cylindre inférieur, où la notion de supérieur et d'inférieur est liée à la direction de $\vec{v}$. La figure 4 illustre ces définitions. Par le théorème du flux maximal - coupure minimale, $\tau(A,h)$ est

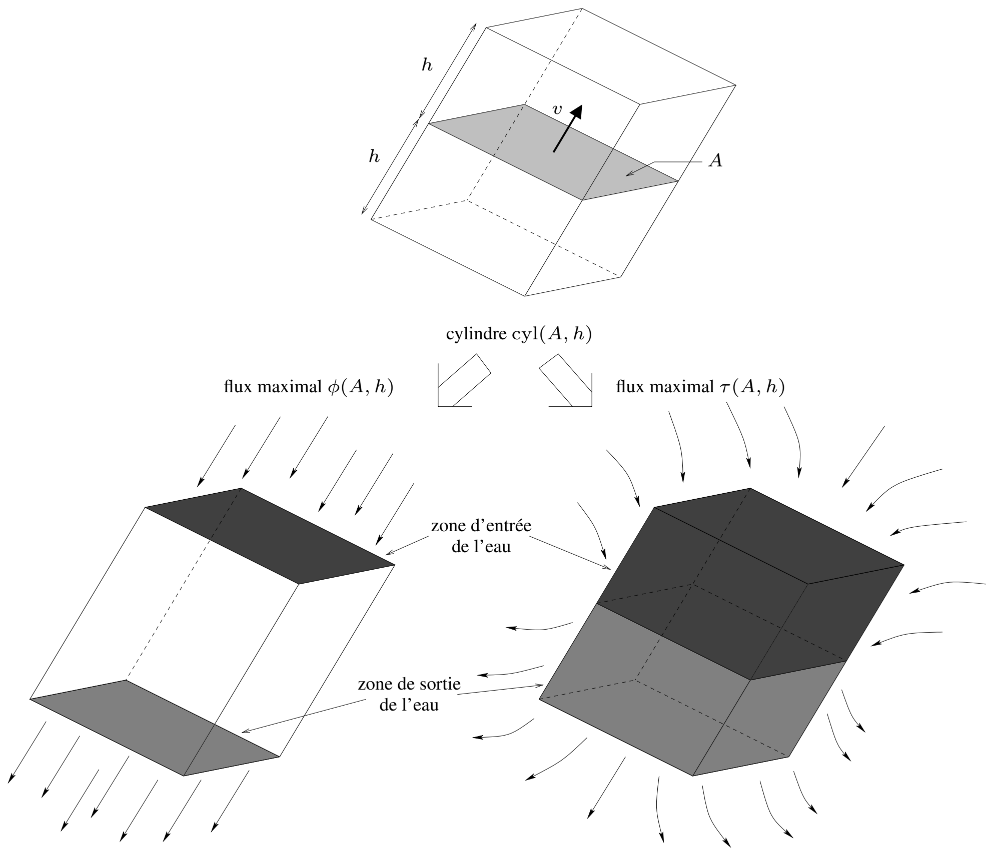

FIG. 4. Le cylindre $\operatorname{cyl}(A,h)$ et les flux maximaux $\phi(A,h)$ et $\tau(A,h)$.

égal à la capacité minimale d'une surface séparant A_1^h de A_2^h dans $\operatorname{cyl}(A,h)$. D'après la définition même des ensembles A_i^h, cette surface a une trace sur les parois verticales de $\operatorname{cyl}(A,h)$ qui est très proche du bord ∂A de A. Cette famille de variables est donc quasiment sous-additive comme nous allons l'expliquer dans la partie 2.2.1.

2.1.4. *Flux maximal dans un domaine de* $\mathbb{R}^d$. Il est plus simple d'étudier en premier lieu le flux maximal $\phi(nA,h(n))$ dans un cylindre $\operatorname{cyl}(nA,h(n))$, et donc aussi le flux maximal $\tau(nA,h(n))$ dans ce cylindre, car les cylindres présentent de bonnes propriétés de symétrie et de recollement le long de leurs faces. Néanmoins, l'étude du flux maximal en percolation de premier passage ne se limite pas à l'étude de flux maximaux dans des cylindres. Si l'on interprète notre système comme une strate de roche poreuse dans le sol, supposer qu'elle est d'épaisseur uniforme est une approximation qui a ses limites, et comprendre le comportement du flux maximal à travers cette strate de roche en fonction justement de la déformation de la strate par rapport à un cylindre d'épaisseur constante est un enjeu réel du modèle. C'est pourquoi nous allons définir le flux maximal à travers un domaine de $\mathbb{R}^d$ entre une zone d'entrée de l'eau et une zone de sortie situées sur son bord. Nous serons amenés à faire des hypothèses de régularité sur le domaine considéré, son bord et les domaines d'entrée et de sortie de l'eau, mais les domaines que nous considérerons resteront somme toute très généraux. Il n'y a pas de raison qu'une direction soit privilégiée par la forme du domaine ou une direction hypothétique de circulation du flux, c'est pourquoi nous allons faire tendre vers l'infini toutes les dimensions du domaine à la même vitesse sans le déformer, ce qui revient plus simplement à fixer le domaine et à considérer à l'intérieur un réseau carré de pas $1/n$ et faire tendre n vers l'infini. C'est donc dans ce cadre que nous allons utiliser le graphe $(\mathbb{Z}^d, \mathbb{E}^d)$ renormalisé. Idéalement, nous aimerions que notre modèle soit invariant par rotation. Néanmoins, nous travaillons avec le réseau $\mathbb{Z}^d$ pour simplifier notre étude.

Nous avons motivé l'étude du flux maximal dans un domaine de $\mathbb{R}^d$, introduisons à présent les définitions dont nous nous servirons. Pour $d \geq 2$, notons $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$ le graphe de sommets $\mathbb{Z}^d_n = \mathbb{Z}^d/n$ et d'arêtes $\mathbb{E}^d_n$ reliant les sommets les plus proches voisins pour la distance euclidienne. Lorsque nous nous placerons dans ce graphe renormalisé à la place du graphe $(\mathbb{Z}^d, \mathbb{E}^d)$ comme nous le faisons maintenant, nous le préciserons clairement. Nous considérons le modèle de percolation de premier passage sur ce graphe, et donc la famille de capacités $(t(e), e \in \mathbb{E}^d_n)$ i.i.d. à valeurs dans $\mathbb{R}^+$, de fonction de répartition F. Soit Ω un ouvert connexe (donc connexe par arcs) borné de $\mathbb{R}^d$, dont la frontière Γ est de classe $\mathcal{C}^1$ par morceaux. Soient Γ^1 et Γ^2 deux sous-ensembles ouverts disjoints de Γ. Nous voulons définir le flux maximal de Γ^1 à Γ^2 dans Ω pour le modèle de percolation de premier passage sur $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$. Nous devons donc définir une discrétisation $(\Omega_n, \Gamma_n, \Gamma^1_n, \Gamma^2_n)$ de $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$, ce que nous faisons comme suit :

$$\left\{ \begin{array}{l} \Omega_n = \{x \in \mathbb{Z}^d_n \,|\, d_\infty(x, \Omega) < 1/n\}\,, \\ \Gamma_n \;=\; \{x \in \Omega_n \,|\, \exists y \notin \Omega_n\,,\, \langle x, y\rangle \in \mathbb{E}^d_n\}\,, \\ \Gamma^i_n \;=\; \{x \in \Gamma_n \,|\, d_\infty(x, \Gamma^i) < 1/n\,,\, d_\infty(x, \Gamma^j) \geq 1/n\} \text{ pour } i = 1, 2 \text{ et } j = 3 - i\,, \end{array} \right.$$

où $\langle x, y\rangle$ désigne l'arête d'extrémités x et y, et d_∞ désigne la distance L^∞ :

$$d_\infty(x, y) \;=\; \sup_{i=1,\dots,d} |x_i - y_i|$$

pour $x = (x_1, ..., x_d)$ et $y = (y_1, ..., y_d) \in \mathbb{R}^d$. La figure 5 illustre ces définitions.

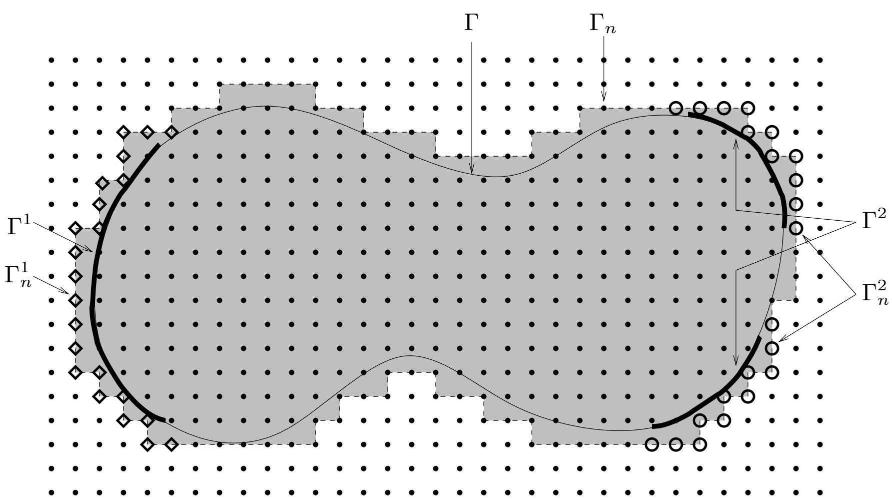

FIG. 5. Le système $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ et son approximation $(\Omega_n, \Gamma_n, \Gamma^1_n, \Gamma^2_n)$.

Nous notons alors

$$\phi_n \;=\; \phi(\Gamma^1_n \to \Gamma^2_n \text{ dans } \Omega_n)\,.$$

L'ensemble de cette thèse permet de montrer sous certaines hypothèses sur le système et la loi des capacités des arêtes un résultat de loi des grands nombres pour le flux maximal renormalisé ϕ_n/n^{d-1} quand n tend vers l'infini, assorti des vitesses des grandes déviations par au-dessus et par en dessous de cette variable. Les cylindres constituent les briques élémentaires qui s'assemblent pour former le flux maximal ϕ_n, il est donc indispensable de bien comprendre le comportement asymptotique des flux maximaux dans des cylindres pour mener à bien l'étude du flux maximal ϕ_n dans un domaine de $\mathbb{R}^d$.

2.2. État de l'art. Nous allons récapituler les résultats connus en dimension $d \geq 2$ concernant le flux maximal dans le modèle de percolation de premier passage dans le graphe $(\mathbb{Z}^d, \mathbb{E}^d)$. Nous énonçons ici les résultats spécifiques au flux maximal, dans le cas de la dimension deux nous ne citerons pas tous les travaux effectués sur la distance en percolation de premier passage même s'ils ont souvent une interprétation en termes de flux via la dualité, nous renvoyons le lecteur par exemple à [**40**] et [**8**]. Nous ne nous intéressons pas au cas de flux déterministes, i.e., au cas où les capacités des arêtes ne sont pas aléatoires. C'est un sujet qui a été étudié par exemple par Rockafellar [**51**] et qui soulève des questions très intéressantes, comme la description d'un algorithme efficace pour déterminer la position d'un ensemble de coupure minimal dans le graphe.

Nous nous plaçons donc dans le modèle de percolation de premier passage sur le graphe $(\mathbb{Z}^d, \mathbb{E}^d)$. Les résultats rassemblés ici essaient tous de décrire le comportement des flux maximaux à travers un cylindre quand les dimensions du cylindre tendent vers l'infini. Ceci correspond à faire tendre le pas du réseau vers 0 dans un cylindre fixé, à la même vitesse dans toutes les directions ou pas. C'est une approche logique dans la mesure où nous voulons comprendre le comportement d'une roche poreuse dont nous avons modélisé la porosité par de petits tuyaux de taille microscopique par rapport à la taille de la roche. L'idée intuitive derrière les résultats que nous allons énoncer est la suivante : sous certaines conditions sur la taille du système étudié et la loi des capacités des arêtes, le flux maximal à travers un système est asymptotiquement une fonction linéaire de la surface par laquelle l'eau peut rentrer et sortir. Le coefficient de linéarité asymptotique est une constante non aléatoire, qui dépend de la loi des capacités et du système considéré. Puisqu'il va être question des surfaces des domaines d'entrée et de sortie de l'eau, nous allons avoir besoin de la définition suivante : nous notons $\mathcal{H}^{d-1}$ la mesure de Hausdorff $(d-1)$-dimensionnelle sur les sous-ensembles de $\mathbb{R}^d$, i.e., pour tout $A \subset \mathbb{R}^d$, on définit

$$\mathcal{H}^{d-1}(A) \,=\, \lim_{\delta \to 0} \inf\{\alpha_{d-1} \sum_{i \in I} (\mathrm{diam}\, A_i)^{d-1} \,|\, A \subset \cup_{i \in I} A_i \,,\; \mathrm{diam}(A_i) < \delta \text{ et } I \text{ dénombrable}\} \,,$$

où $\mathrm{diam}\, A_i$ désigne le diamètre de A_i pour la distance euclidienne, et α_{d-1} est le volume de la boule unité dans $\mathbb{R}^{d-1}$.

Les résultats qui suivent sont globalement présentés par difficulté croissante, ce qui correspond à peu près à une présentation chronologique. Néanmoins, par souci de clarté, nous avons regroupé les résultats qui se complètent, d'où une certaine dépendance entre les résultats des différents paragraphes.

2.2.1. *Loi des grands nombres pour la variable τ.* Le premier résultat que nous présentons est une loi des grands nombres sur la variable τ, définie en (1.2) dans des cylindres dont on fait tendre la taille vers l'infini, et ce dans n'importe quelle dimension $d \geq 2$. Nous rappelons que F désigne la fonction de répartition de la capacité des arêtes, qui est positive, donc $F(0)$ désigne simplement la probabilité que la capacité d'une arête soit nulle. Nous notons toujours $p_c(d)$ le paramètre critique pour la percolation de Bernoulli par arêtes en dimension d. Voici un énoncé de cette loi des grands nombres :

THÉORÈME 2. *On suppose que la loi des capacités des arêtes admet un moment d'ordre* 1*, i.e.,*

$$\int_{[0,+\infty[} x \, dF(x) \,<\, \infty \,.$$

Pour toute fonction de hauteur $h : \mathbb{N} \to \mathbb{R}^+$ satisfaisant $\lim_{n\to\infty} h(n) = +\infty$, pour tout hyperrectangle non dégénéré A, la limite

$$\nu(\vec{v}) \,=\, \lim_{n\to\infty} \frac{\mathbb{E}[\tau(nA, h(n))]}{\mathcal{H}^{d-1}(nA)}$$

existe et dépend de F, de d et de la direction du vecteur unitaire $\vec{v}$ orthogonal à A mais pas de h ou de A lui-même. La constante $\nu(\vec{v})$ est strictement positive si $F(0) < 1 - p_c(d)$, et nulle si $F(0) \geq 1 - p_c(d)$. De plus, s'il existe un réel M tel que toutes les coordonnées de $M\vec{v}$ soient rationnelles, si l'origine du graphe 0 est inclue dans A, et toujours sous une condition de moment d'ordre 1 pour la loi des capacités, on a

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\vec{v}) \qquad \textit{p.s. et dans } L^1 \,.$$

Sans aucune restriction sur $\vec{v}$ et A, si $F(0) < 1 - p_c(d)$ et si la loi des capacités des arêtes admet un moment exponentiel, i.e.,

$$\exists \gamma > 0 \quad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) \,<\, \infty \,,$$

alors

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\vec{v}) \qquad \textit{p.s.}$$

La démonstration de ce théorème est basée sur la sous-additivité de la variable τ. Intéressons-nous d'abord à la convergence p.s. et dans L^1 de la suite $(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA))$ lorsque $0 \in A$ et qu'il existe M tel que $M\vec{v}$ ait toutes ses coordonnées rationnelles (on dit alors que la direction est rationnelle). Dans le cas de cylindres droits, c'est-à-dire si A est de la forme $\prod_{i=1}^{d-1} [a_i, b_i] \times \{c\}$ pour des a_i, b_i et c réels, la famille $\tau(A, h)$ est sous-additive pour tout h fixé : si un hyperrectangle A admet une partition en hyperrectangles A_i pour $i = 1, ..., n$ d'intérieurs disjoints, alors on a

$$\tau(A, h) \,\leq\, \sum_{i=1}^n \tau(A_i, h) \,.$$

En effet, si pour tout i on note $\mathcal{E}_i$ une surface de coupure entre le demi-cylindre inférieur et le demi-cylindre supérieur dans $\mathrm{cyl}(A_i, h)$, ces surfaces se recollent sur les frontières communes des A_i de telle sorte que $\cup_{i=1}^n \mathcal{E}_i$ est une surface de coupure dans $\mathrm{cyl}(A, h)$ entre le demi-cylindre inférieur et le demi-cylindre supérieur (voir la figure 6 qui illustre le recollement de surfaces de coupure entre deux cylindres).

Il y a cependant quelques précautions à prendre. Dans le cas de cylindres inclinés, la famille $\tau(A, h)$ n'est en fait pas vraiment sous-additive, car des problèmes de recollement de surfaces séparantes peuvent apparaître. Voici l'origine du défaut de sous-additivité : si nous considérons un hyperrectangle A (incliné a priori cette fois) et une partition de A en hyperrectangles A_i pour $i = 1, ..., n$ d'intérieurs disjoints, des arêtes du graphe peuvent avoir une extrémité dans la moitié supérieure d'un cylindre $\mathrm{cyl}(A_i, h)$ et l'autre dans la partie inférieure d'un cylindre $\mathrm{cyl}(A_j, h)$ pour $j \neq i$. Si pour tout i on note $\mathcal{E}_i$ un ensemble de coupure séparant les bords des deux demi-cylindres dans $\mathrm{cyl}(A_i, h)$, ce type d'arête n'apparaîtra dans aucun des $\mathcal{E}_i$ mais il faudra les rajouter à $\cup_{i=1}^n \mathcal{E}_i$ pour obtenir un ensemble d'arêtes séparant les bords des deux demi-cylindres dans $\mathrm{cyl}(A, h)$. Heureusement, ces arêtes sont localisées dans un très petit voisinage des bords ∂A_i des hyperrectangles de la partition, donc nous pouvons en contrôler le nombre. Si nous notons

$$\mathcal{G}(A) \,=\, \{z \in \mathbb{R}^d \,|\, d(z, \partial(A)) \leq \zeta\}$$

pour un ζ fixé strictement supérieur à $2d$, et $G(A)$ l'ensemble des arêtes incluses dans $\mathcal{G}(A)$, alors pour tout h et A fixés la famille $\tau'(A, h) = \tau(A, h) + V(G(A))$ indexée par A est sous-additive. Cette famille est ergodique, et en comparant $\tau(A, h)$ à la capacité d'une surface de coupure formée

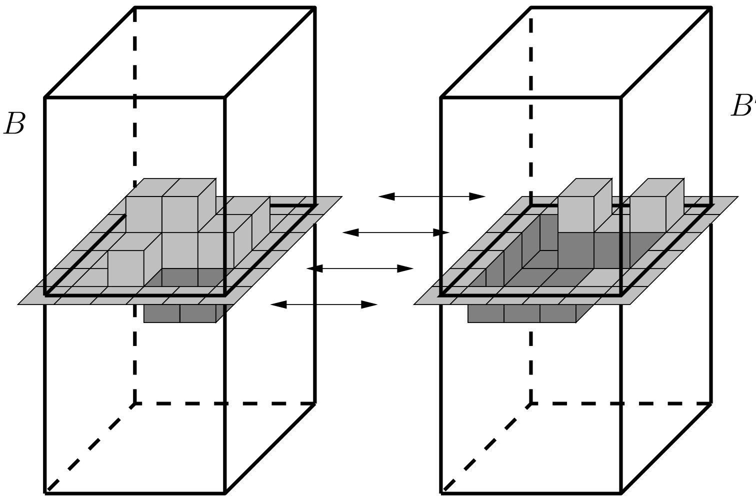

FIG. 6. Sous-additivité de τ dans des cylindres droits.

d'une couche plate d'arêtes, on obtient immédiatement que $\mathbb{E}(\tau'(A,h))/\mathcal{H}^{d-1}(A)$ est borné dès que $\mathbb{E}(t(e)) < \infty$. Lorsque $0 \in A$, tous les hyperrectangles nA, $n \in \mathbb{N}$, sont inclus dans le même hyperplan $\mathcal{P}(\vec{v})$ contenant l'origine. Si de plus la direction considérée est rationnelle, le graphe est invariant par toute une famille de translations dont les vecteurs sont inclus dans ce plan $\mathcal{P}(\vec{v})$. On peut alors directement appliquer les théorèmes ergodiques sous-additifs à paramètres multiples, plus précisément le théorème 1 de [**43**] et 1.1 de [**53**], pour obtenir pour tout couple (A,h) la convergence presque sûre et dans L^1 de la suite $(\tau'(nA,h)/\mathcal{H}^{d-1}(nA))$ quand n tend vers l'infini. De plus, le théorème 1 de [**43**] nous permet aussi d'affirmer que la limite p.s. est la même lorsque l'on considère deux hyperrectangles A et A' contenant 0 et admettant un même vecteur orthogonal unitaire $\vec{v}$ orienté dans une direction rationnelle. Par la loi du $0-1$ de Kolmogorov, puisque la limite presque sûre de cette suite est invariante par les translations à coordonnées entières du graphe, on en déduit qu'elle est égale à une constante p.s. Puisque la convergence L^1 implique la converge p.s. d'une sous-suite vers la même limite, on en déduit qu'il existe une constante $\nu_h(\vec{v})$ telle que

$$\lim_{n\to\infty} \frac{\tau'(nA,h)}{\mathcal{H}^{d-1}(nA)} \,=\, \nu_h(\vec{v}) \qquad \text{p.s. et dans } L^1\,.$$

Le cardinal de l'ensemble déterministe d'arêtes $G(nA)$ est majoré par une constante (dépendant de A) multipliée par n^{d-2}. On en déduit immédiatement que $V(G(nA))/\mathcal{H}^{d-1}(nA)$ converge p.s. et dans L^1 vers 0, et que donc

$$\lim_{n\to\infty} \frac{\tau(nA,h)}{\mathcal{H}^{d-1}(nA)} \,=\, \nu_h(\vec{v}) \qquad \text{p.s. et dans } L^1\,.$$

Cette constante $\nu_h(\vec{v})$ dépend bien de d, F et $\vec{v}$, mais pas de la forme de A lui-même. De plus, la proposition 1.1 de [**53**] établit que

$$\nu_h(\vec{v}) \,=\, \inf_n \frac{\mathbb{E}(\tau'([0,n]^{d-1},h))}{n^{d-1}}\,.$$

Notons que $h=\infty$ est un cas particulier de ce qui précède ; le flux maximal $\tau(nA,\infty)$ est alors défini via le théorème du flux maximal - coupure minimale comme la capacité minimale d'un ensemble de coupure qui sépare les bords des deux demi-cylindres infinis dans $\mathrm{cyl}(nA,\infty)$, ou de façon équivalente par la limite décroissante suivante :

$$\tau(nA,\infty) \,=\, \lim_{h\nearrow\infty} \tau(nA,h)\,.$$

On en déduit, par une simple interversion d'infimum, que

$$\lim_{h \nearrow \infty} \nu_h(\vec{v}) \,=\, \nu_\infty(\vec{v})\,.$$

Finalement, si $h : \mathbb{N} \rightarrow \mathbb{R}^+$ est une fonction de hauteur telle que $\lim_{n\rightarrow\infty} h(n) = +\infty$, pour tout h fixé il existe n_0 tel que pour tout $n \geq n_0$ on a

$$\tau(nA, \infty) \,\leq\, \tau(nA, h(n)) \,\leq\, \tau(nA, h)\,,$$

et on en déduit que $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ converge p.s. et dans L^1 vers $\nu_\infty(\vec{v})$. On note alors plus simplement $\nu(\vec{v}) = \nu_\infty(\vec{v})$, et $\nu = \nu((0, ..., 0, 1))$.

Malheureusement, dans le cas de directions non rationnelles, ou quand A ne contient pas l'origine du graphe, nous ne pouvons pas appliquer directement les théorèmes sous-additifs. En effet, les théorèmes sous-additifs à paramètres multiples classiques (voir par exemple [**1**], [**43**], [**44**] et [**53**]) sont énoncés dans le cadre suivant : la famille de variables sous-additive est indexée par des sous-ensembles d'un même espace, qui est dans notre cas un hyperplan, et la loi des variables doit être invariante par un groupe de transformations laissant stable cet espace. Ces deux conditions ne sont pas réunies si $0 \notin A$ ou si la direction considérée n'est pas rationnelle. Néanmoins, nous pouvons quand même montrer simplement la convergence de l'espérance de la variable τ renormalisée, même pour des directions non rationnelles ou si $0 \notin A$, et l'indépendance de la limite, toujours notée $\nu(\vec{v})$, par rapport à la forme précise de A une fois $\vec{v}$ fixé et par rapport à la fonction de hauteur h. La démonstration est présentée en détail dans le chapitre 5. L'idée est la suivante. Nous considérons deux hyperrectangles non dégénérés A et A' orthogonaux à un même vecteur unitaire $\vec{v}$, deux fonctions de hauteur h et h' et deux entiers N et n tels que N est très grand devant n. Nous suivons alors la même démarche que celle des démonstrations des théorèmes sous-additifs en recouvrant NA par des translatés $(T_i, i \in I)$ de nA', sauf une très petite surface. Ces $(T_i, i \in I)$ ne sont pas des images de nA' par des translations de vecteurs à coordonnées entières, donc $\tau(T_i, h'(n))$ n'est pas égal en loi à $\tau(nA', h'(n))$. Cependant, nous pouvons à nouveau translater les T_i sur une très petite distance en les décollant de l'hyperplan dans lequel se trouve NA pour obtenir des T'_i qui sont des translatés par des vecteurs à coordonnées entières de nA', donc tels que $\tau(T'_i, h'(n))$ soit bien égal en loi à $\tau(nA', h'(n))$. Nous pouvons alors comparer $\mathbb{E}[\tau(NA, h(N))]$ avec $\sum_{i\in I} \mathbb{E}[\tau(T'_i, h'(n))]$, ce qui nous permet d'obtenir la convergence de $\mathbb{E}[\tau(NA, h(N))]/\mathcal{H}^{d-1}(NA)$ quand N tend vers l'infini, et l'égalité de la limite pour les deux hyperrectangles A et A' admettant un même vecteur orthogonal $\vec{v}$, et pour les deux fonctions de hauteur h et h'. Nous utilisons ici de façon essentielle le fait que notre espace $\mathbb{R}^d$ possède une dimension de plus que le sous-espace qui contient A et A'.

Pour prouver la convergence p.s. de $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ vers $\nu(\vec{v})$ dans toutes les directions sous des hypothèses supplémentaires sur F, nous utilisons un résultat de concentration de la mesure. Puisque la suite des espérances de ces variables converge vers $\nu(\vec{v})$, si les variables restent suffisamment proches de leur espérance on peut en déduire la convergence p.s. recherchée. Ceci est fait en détail dans le chapitre 5. Pour obtenir le résultat de concentration, nous utilisons les travaux de Zhang [**59**] qui seront présentés dans le paragraphe 2.2.4. Les hypothèses demandées sur F ne sont pas optimales, et nous conjecturons qu'elles peuvent être considérablement allégées. Néanmoins, il faut noter que la convergence de l'espérance de la variable τ renormalisée, éventuellement couplée à des résultats de concentration, est suffisante pour obtenir tous les résultats présentés dans cette thèse. Nous ne nous restreindrons donc jamais au cas des directions rationnelles ou au cas $0 \in A$.

La convergence de τ renormalisé dans le cas des cylindres droits avait été montrée par Kesten dans [**41**] dans le cas de la dimension trois, dans le cas plus général où les dimensions de la base du cylindre ne tendent pas vers l'infini nécessairement à la même vitesse. La démonstration dans le cas des cylindres inclinés n'avait à notre connaissance pas été rédigée clairement auparavant, mais nous considérons cependant que ce résultat était connu, dans la mesure où il se base sur les mêmes propriétés de sous-additivité. Boivin a d'ailleurs déjà prouvé des résultats très similaires dans [**11**], nous y reviendrons au paragraphe 2.4.

En ce qui concerne la preuve du résultat énoncé dans le théorème 2 sur la stricte positivité de $\nu(\vec{v})$, voici l'idée intuitive qui est derrière : pour que le flux puisse circuler sur des distances infinies, il faut que la percolation définie sur le graphe par $t'(e) = \mathbb{1}_{\{t(e)>0\}}$ pour toute arête e soit sur-critique, i.e., que nous supposions que $F(0) < 1-p_c(d)$. Nous présentons ici un raisonnement possible pour prouver ce résultat. Nous pouvons facilement prouver que $\nu(\vec{v})$ satisfait l'inégalité triangulaire faible (voir le chapitre 5 de la thèse pour un énoncé et une preuve de cette propriété). On en déduit que pour tout vecteur unitaire $\vec{v}$, $\nu(\vec{v}) = 0$ si et seulement si $\nu((0,...,0,1)) = 0$ (résultat prouvé également dans le chapitre 5 de la thèse). On est donc ramené à étudier uniquement le cas des cylindres droits. Zhang a prouvé dans [**58**] que sous la condition de moment d'ordre 1 pour les capacités des arêtes, le flux entre demi-cylindres dans un cylindre droit, renormalisé par la surface de la base du cylindre, converge presque sûrement et dans L^1 vers 0 lorsque $F(0) = 1 - p_c(d)$. La démonstration est faite en dimension 3 mais Zhang indique qu'elle se généralise à la dimension $d \geq 3$. Nous reviendrons sur ce théorème ultérieurement, car il est en fait plus général que ce que nous en disons ici. C'est un résultat qui mérite d'être présenté dans son intégralité car l'étude du cas critique $F(0) = 1 - p_c(d)$ constitue une véritable avancée. Le résultat était par ailleurs connu en dimension 2 via l'étude des temps d'atteinte. Par couplage, nous en déduisons que $\nu((0,...,0,1)) = 0$ dès que $F(0) \geq 1 - p_c(d)$, ce qui généralise le résultat de Kesten dans [**41**] qui prouvait ceci en dimension 3 sous une condition de moment plus forte dès que $F(0) > 1 - p_c(d)$. Dans le cas $F(0) < 1 - p_c(d)$, la positivité de $\nu((0,...,0,1))$ découle par exemple du premier théorème présenté dans le chapitre 4 de la thèse, qui dit en substance que la probabilité que le flux renormalisé entre le sommet et la base d'un cylindre droit soit très petit décroît exponentiellement vite avec la surface de la base du cylindre. Puisque le flux entre demi-cylindres est supérieur ou égal au flux entre le sommet et la base d'un cylindre, on en déduit le résultat voulu. C'est une généralisation d'un résultat de Chayes et Chayes [**20**] obtenu dans le cas où les capacités des arêtes suivent une loi de Bernoulli.

Il est à noter qu'ici nous considérons un cylindre dont une direction est privilégiée car elle est donnée par la direction dans laquelle circule le flux, cependant toutes les autres directions sont équivalentes. Cela signifie que nous faisons tendre vers l'infini tous les côtés de la base du cylindre à la même vitesse, et que nous autorisons seulement la hauteur du cylindre à tendre vers l'infini à une vitesse différente. Nous garderons cette approche tout au long de la thèse, sauf dans le chapitre 5 ou nous étudierons des flux maximaux dans des ensembles plus généraux que des cylindres, donc n'ayant a priori aucune direction privilégiée : nous ferons alors grandir les dimensions du domaine vers l'infini de façon isotrope. Nous ne nous sommes pas intéressés au cas où les différents côtés du cylindre grandissent à des vitesses deux à deux potentiellement distinctes. C'est par ailleurs une question très intéressante, et les résultats de Kesten [**41**] et Zhang [**58**], [**59**] sur la variable ϕ (définie en (1.1)) que nous allons présenter dans les paragraphes suivants se placent dans ce cadre. Dans le premier de ces deux articles, un résultat de convergence pour la variable τ dans des cylindres droits dont les dimensions ne tendent pas vers l'infini à la même vitesse est également démontré. Comme nous le verrons à l'énoncé des résultats obtenus sur la variable ϕ par Kesten et Zhang dans le cas de vitesses d'expansion du cylindre différentes dans chaque direction, le problème semble dans ce contexte plus difficile, et la condition à imposer sur la fonction de hauteur du cylindre étudié est alors moins évidente, et probablement pas encore optimale.

2.2.2. *Loi des grands nombres pour la variable ϕ dans des cylindres plats.* Par "cylindres plats" nous entendons ici que la fonction de hauteur satisfait la condition :

$$\lim_{n\to\infty} \frac{h(n)}{n} = 0 .$$

Si h est une telle fonction de hauteur, pour tout hyperrectangle non dégénéré A, les variables $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$ et $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ ont le même comportement asymptotique. En effet, il est d'une part évident que $\tau(nA,h(n)) \geq \phi(nA,h(n))$. Réciproquement, si $\mathcal{E}(n)$ est un ensemble d'arêtes qui sépare la base du sommet de $\mathrm{cyl}(nA,h(n))$, et $G'(nA)$ est l'ensemble des arêtes qui se trouvent dans un voisinage de taille $\zeta \geq 2d$ des parois verticales $\mathrm{cyl}(\partial(nA),h(n))$ du cylindre, alors $\mathcal{E}(n) \cup G'(nA)$ sépare les deux demi-cylindres dans $\mathrm{cyl}(nA,h(n))$, donc

$$\tau(nA,h(n)) \;\leq\; \phi(nA,h(n)) + V(G'(nA))\,.$$

Puisque le cardinal de l'ensemble $G'(nA)$ est de l'ordre de $n^{d-2}h(n)$, donc négligeable devant n^{d-1} d'après l'hypothèse faite sur la fonction h, on en déduit que les résultats de convergence obtenus pour $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$ restent vrais pour $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$, i.e., on a le théorème suivant :

THÉORÈME 3. *On suppose que F admet un moment d'ordre 1, i.e.,*

$$\int_{[0,+\infty[} x\,dF(x) \;<\; \infty\,.$$

Alors pour toute fonction de hauteur $h:\mathbb{N}\rightarrow\mathbb{R}^+$ satisfaisant

$$\lim_{n\rightarrow\infty} h(n) \;=\; +\infty \quad \textit{et} \quad \lim_{n\rightarrow\infty} \frac{h(n)}{n} \;=\; 0\,,$$

pour tout hyperrectangle non dégénéré A de vecteur normal unitaire $\vec{v}$, on a

$$\lim_{n\rightarrow\infty} \frac{\mathbb{E}[\phi(nA,h(n))]}{\mathcal{H}^{d-1}(nA)} \;=\; \nu(\vec{v})\,.$$

De plus, s'il existe un réel M tel que toutes les coordonnées de $M\vec{v}$ soient rationnelles, si l'origine du graphe 0 est inclue dans A, et toujours sous une condition de moment d'ordre 1 pour la loi des capacités, on a

$$\lim_{n\rightarrow\infty} \frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \;=\; \nu(\vec{v}) \qquad \textit{p.s. et dans } L^1\,.$$

Sans aucune restriction sur $\vec{v}$ et A, si $F(0) < 1-p_c(d)$ et si la loi des capacités des arêtes admet un moment exponentiel, i.e.,

$$\exists\gamma>0 \quad \int_{[0,+\infty[} e^{\gamma x}\,dF(x) \;<\; \infty\,,$$

alors

$$\lim_{n\rightarrow\infty} \frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \;=\; \nu(\vec{v}) \qquad \textit{p.s.}$$

Les suites $(\phi(nA,h(n))/\mathcal{H}^{d-1}(nA), n\in\mathbb{N})$ et $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA), n\in\mathbb{N})$ sont en fait exponentiellement équivalentes sous la condition d'existence d'un moment exponentiel pour la loi des capacités. Ce résultat sera montré dans le chapitre 5 de la thèse en utilisant précisément la relation entre $\tau(nA,h(n))$ et $\phi(nA,h(n))$ via l'ensemble $G'(nA)$ décrit ci-dessus.

2.2.3. *Loi des grands nombres pour la variable ϕ dans des cylindres droits, premier résultat.* Nous présentons maintenant le résultat de l'article qui a été le fondement de cette thèse. Il s'agit de l'article de Kesten [**41**]. En substance, Kesten établit qu'en dimension trois, sous certaines conditions sur F et h, le flux maximal entre le sommet et la base d'un cylindre droit a le même comportement asymptotique que le flux maximal entre le bord du demi-cylindre supérieur et le bord du demi-cylindre inférieur, c'est-à-dire qu'une fois renormalisé par la surface de la base du cylindre il converge presque sûrement vers $\nu((0,0,1))$. Nous allons noter $D(k,l,m)$ le cylindre

$$D(k,l,m) \;=\; [0,k]\times[0,l]\times[0,m]\,,$$

de sommet $T(k,l,m) = [0,k] \times [0,l] \times \{m\}$ et de base $B(k,l,m) = [0,k] \times [0,l] \times \{0\}$, et $\phi(D(k,l,m))$ le flux maximal entre $T(k,l,m)$ et $B(k,l,m)$ dans $D(k,l,m)$. Voici le théorème tel qu'il a été énoncé par Kesten en 1987 :

THÉORÈME 4 (Kesten). *On se place en dimension trois. Si $F(0) < p_0$ pour un certain $p_0 \geq 1/27$ fixé, et si F admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) \,<\, \infty\,,$$

si $m = m(k,l)$ tend vers l'infini quand k et l tendent vers l'infini avec $k \geq l$ de telle sorte qu'il existe un $\delta > 0$ tel que

$$\lim_{k,l\to\infty} k^{-1+\delta} \log m(k,l) \,=\, 0\,,$$

alors

$$\lim_{k,\,l\to\infty} \frac{\phi(D(k,l,m))}{kl} \,=\, \nu((0,0,1)) \qquad \textit{p.s. et dans } L^1\,.$$

De plus, si $F(0) > 1 - p_c(d)$ et F admet un moment d'ordre 6, il existe une constante $C = C(F) < \infty$ telle que si $m = m(k,l)$ tend vers l'infini quand k et l tendent vers l'infini avec $k \geq l$ de telle sorte que

$$\liminf_{k,l\to\infty} \frac{m(k,l)}{\log(kl)} \,>\, C\,,$$

alors pour tous k,l suffisamment grands on a

$$\phi(D(k,l,m)) \,=\, 0 \qquad \textit{p.s.}$$

Kesten souligne lui-même dans [**41**] que les hypothèses requises dans ce théorème ne sont probablement pas optimales. Il conjecture que le premier résultat doit rester vrai dès que $F(0) < 1 - p_c(d)$, que la loi des capacités des arêtes admet un moment d'ordre deux, et que

$$\lim_{k,\,l\to\infty} (kl)^{-1} \log m(k,l) \,=\, 0\,.$$

Il indique également que dans le cas où le rapport k/l reste loin de 0 et $+\infty$, la condition sur la hauteur m de la boîte peut être améliorée, sans toutefois obtenir la condition espérée. Il présente comme problèmes ouverts la généralisation de ce théorème en dimension plus grande que trois et l'étude des déviations inférieures dans le cas $F(0) < p_0$ pour déterminer entre autres si elles sont bien d'ordre surfacique, ainsi que l'étude du cas critique $F(0) = 1 - p_c(d)$. Nous verrons que Zhang a répondu à certaines de ces questions dans [**58**] et [**59**] et que nous répondons à d'autres dans cette thèse.

Compte tenu de l'importance du théorème 4 pour l'ensemble de notre travail, nous souhaitons donner ici un aperçu des idées utilisées dans sa preuve pour le cas $F(0) < p_0$ (le cas sur-critique $F(0) > 1 - p_c(d)$ sera généralisé par Zhang et nous présenterons son travail dans le paragraphe suivant). Tout d'abord Kesten utilise son hypothèse $F(0) < p_0$ pour contrôler avec grande probabilité le nombre d'arêtes dans un ensemble de coupure de capacité presque minimale séparant le sommet de la base du cylindre. Grâce à ce contrôle, il contraint son ensemble de coupure à rester dans un sous-cylindre de hauteur proportionnelle à k à l'intérieur du cylindre initial, quitte à rajouter quelques arêtes à l'ensemble en question (et donc augmenter un peu sa capacité). Quitte à faire rétrécir un peu la base du cylindre, il contraint également cet ensemble de coupure à avoir peu d'arêtes qui touchent les parois verticales du cylindre. Via la dualité arête - plaquette, il considère son ensemble de coupure comme une surface constituée de plaquettes. Il obtient un contrôle sur le nombre de plaquettes qui intersectent les parois verticales du cylindre, donc en fait sur la longueur des courbes qui constituent l'intersection de la surface de coupure avec les parois du cylindre. Il déduit de ceci, et du contrôle sur la hauteur de la sous-boîte dans laquelle se trouve l'ensemble de coupure, une majoration du nombre de conditions aux bords possibles, c'est-à-dire d'intersections possibles de la surface de coupure avec les parois verticales du cylindre. Par ailleurs, il montre qu'un ensemble de coupure qui a des conditions aux bords données peut se recoller avec un ensemble de coupure dans une boîte voisine qui aurait des conditions aux bords symétriques du premier par rapport à la face commune des deux boîtes. Finalement, le graphe étant invariant par une telle symétrie, la probabilité pour un ensemble de coupure d'avoir l'une ou l'autre de ces conditions aux bords est la même. En considérant un ensemble de coupure de conditions aux bords de probabilité maximale par rapport à toutes les conditions aux bords possibles, Kesten se ramène à un objet quasiment sous-additif (via l'utilisation de symétries), qu'il peut donc comparer à la variable τ. Grâce au contrôle obtenu précédemment sur le nombre de conditions aux bords possibles, cette comparaison lui apporte suffisamment d'informations pour en déduire des propriétés sur ϕ lui-même dans un cylindre de dimensions (k_0, l_0, m_0) données. Plus précisément, si on note g la fonction qui à (k,l,m) associe $\mathbb{P}[\phi(D(k,l,m)) \leq (\nu-\varepsilon)kl]$ pour un $\varepsilon > 0$ fixé, Kesten montre ainsi que $g(k_0,l_0,m_0)$ est petit. Il prouve également que g satisfait une inégalité fonctionnelle, qui implique un contrôle de la valeur de $g(k,l,m)$ par la valeur de $g(k_0,l_0,m_0)$ pour tout triplet (k,l,m) dans un certain domaine fixé par (k_0,l_0,m_0). En itérant l'usage de l'inégalité fonctionnelle plusieurs fois à partir du triplet (k_0,l_0,m_0), Kesten obtient que $g(k,l,m)$ est petit pour tout triplet (k,l,m) suffisamment grand satisfaisant les conditions demandées dans le théorème 4. Puisque ϕ est majorée par τ dans un même cylindre, Kesten en déduit la loi des grands nombres annoncée.

Dans cet article, Kesten obtient donc une majoration de la probabilité que $\phi(D(k,l,m))/kl$ soit anormalement petit. Comme il l'annonce lui-même, il n'a pas la bonne vitesse des déviations inférieures pour cette variable, puisqu'il n'obtient pas des déviations surfaciques (voir chapitre 5 de la thèse). Nous avons essayé d'améliorer sa preuve pour la réutiliser dans l'étude de ces déviations inférieures, mais nous n'y sommes pas parvenu.

2.2.4. *Cas critique et contrôle du cardinal d'un ensemble de coupure minimal.* Comme nous l'avons déjà évoqué brièvement, Zhang a étudié dans [**58**] le comportement du flux maximal dans un cylindre dans le cas critique, i.e., $F(0) = 1 - p_c(d)$, où $p_c(d)$ désigne toujours le paramètre critique pour la percolation de Bernoulli par arêtes en dimension d. Nous conservons les notations introduites précédemment, et nous notons $\tau(D(k,l,\infty))$ le flux maximal entre le bord du demi-cylindre supérieur $[0,k]\times[0,l]\times]0,+\infty[$ et le bord du demi-cylindre inférieur $[0,k]\times[0,l]\times]-\infty,0[$ dans $D(k,l,\infty)$. Zhang prouve le résultat suivant :

THÉORÈME 5 (Zhang). *Soit $d=3$. Supposons que $F(0) = 1-p_c(3)$ et que la loi des capacités des arêtes admette un moment d'ordre un, i.e.,*

$$\int_{[0,+\infty[} x\, dF(x) \,<\, \infty\,.$$

Alors pour tout $l>0$ on a

$$\lim_{k,m\to\infty} \frac{\phi(D(k,l,m))}{kl} \,=\, 0\,,$$

pour tout $k>0$ on a

$$\lim_{l,m\to\infty} \frac{\phi(D(k,l,m))}{kl} \,=\, 0\,,$$

et

$$\lim_{k,l,m\to\infty} \frac{\phi(D(k,l,m))}{kl} \,=\, 0\,,$$

où k,l,m tendent vers l'infini dans la dernière équation sans aucune condition sur leur vitesse respective. De plus,

$$\lim_{k,l\to\infty} \frac{\tau(D(k,l,\infty))}{kl} \,=\, 0 \qquad \textit{p.s. et dans } L^1\,.$$

Une première remarque importante à faire est que la preuve fonctionne quelle que soit la dimension $d \geq 3$, comme Zhang le souligne lui-même dans [**58**]. Zhang a choisi d'énoncer le résultat en dimension 3 probablement car dans ce cadre il répond à une des questions posées par Kesten dans [**41**]. Par ailleurs, la démonstration de ce théorème repose sur des résultats complexes. Voici comment Zhang a présenté l'esprit de cette preuve pour la variable $\phi(D(k,l,m))$ dans [**58**] : dans le cas où les capacités des arêtes suivent une loi de Bernoulli, le flux maximal entre le sommet et la base du cylindre $D(k,l,m)$ est égal au nombre maximal de chemins ouverts disjoints entre cette base et ce sommet à l'intérieur du cylindre. Si l'arête initiale d'un tel chemin est fixée en un point précis du sommet du cylindre, le théorème 1.1 de [**7**] établit que la probabilité d'existence de ce chemin tend vers zéro quand m tend vers l'infini. Ainsi, si le nombre de tels chemins possédait une certaine propriété de stationnarité, un théorème ergodique standard permettrait d'en déduire le résultat cherché. Cependant, une telle propriété n'a pas pu être prouvée pour le nombre de chemins décrits ci-dessus, c'est pourquoi il faut utiliser un processus intermédiaire qui est lui stationnaire, et de la convergence duquel on peut déduire la convergence vers 0 du flux maximal $\phi(D(k,l,m))$ dans le cylindre. Soulignons ici le fait que la démonstration du théorème 5 utilise le théorème 1.1 de [**7**] qui est un résultat de percolation difficile, ainsi que le théorème ergodique de Birkhoff. L'étude du cas critique est effectivement très complexe. Ce résultat de Zhang permet donc de généraliser le théorème de convergence de $\phi(D(k,l,m))$ prouvé par Kesten au cas critique, et nous l'avons déjà utilisé pour montrer que $\nu((0,...,0,1)) = 0$ dans le cas $F(0) = 1 - p_c(d)$. Par couplage, on déduit immédiatement de ce théorème les mêmes résultats de convergence pour toute fonction de répartition des capacités F vérifiant $F(0) \geq 1 - p_c(d)$.

En 2007, donc pendant la réalisation de cette thèse, Zhang a réussi dans [**59**] à obtenir un contrôle sur le nombre d'arêtes dans un ensemble de coupure minimal en dimension $d \geq 2$ sous une condition beaucoup plus faible et pertinente que celle de Kesten sur l'atome de la loi en 0 : il s'est ramené à la condition $F(0) < 1 - p_c(d)$. Il énonce son résultat pour deux types d'ensembles de coupure, un ensemble qui sépare un cylindre de l'infini, et un ensemble qui sépare le sommet de la base d'un cylindre à l'intérieur de celui-ci. Il explique lui-même que son résultat peut être généralisé à d'autres cas d'ensembles de coupure, et nous nous servirons de ces généralisations possibles dans les chapitres 5 et 7 de cette thèse. Nous allons cependant nous contenter ici d'énoncer les résultats présentés par Zhang dans son article. Nous avons besoin pour cela de quelques notations. Définissons le cylindre $D(\vec{k},m)$ pour $\vec{k} \in \mathbb{R}_+^{d-1}$ par

$$D(\vec{k},m) \,=\, \prod_{i=1}^{d-1} [0,k_i] \times [0,m] \,.$$

Nous supposons ici que les coordonnées de $\vec{k}$ sont ordonnées par ordre croissant, i.e.,

$$0 \,\leq\, k_1 \,\leq\, ... \,\leq\, k_{d-1} \,.$$

Soit $\phi(\vec{k},m)$ le flux maximal entre le sommet $\prod_{i=1}^{d-1}[0,k_i] \times \{m\}$ et la base $\prod_{i=1}^{d-1}[0,k_i] \times \{0\}$ de ce cylindre, et soit $\mathcal{E}_\phi$ un ensemble de coupure de capacité minimale correspondant à ce flux, de nombre d'arêtes minimal - s'il y en a plusieurs, on en sélectionne un par un algorithme déterministe. Soit $\sigma(\vec{k},m)$ le flux maximal entre le cylindre $D(\vec{k},m)$ et l'infini dans $\mathbb{R}^d \smallsetminus D(\vec{k},m)$. On étend ici la définition du flux maximal entre deux ensembles au cas où l'un de ces ensembles est à l'infini, en considérant l'ensemble de coupure minimale correspondant via une généralisation aux graphes infinis du théorème du flux maximal - coupure minimale comme celle présentée par Garet dans [**30**], section 6. De même, soit $\mathcal{E}_\sigma$ un ensemble de coupure de capacité minimale correspondant à ce flux, de nombre d'arêtes minimal, sélectionné par un algorithme déterministe s'il y en a plusieurs. On note $\mathrm{card}(G)$ le cardinal d'un ensemble G. Zhang a montré le théorème suivant :

THÉORÈME 6 (Zhang). *On suppose que $F(0) < 1 - p_c(d)$ et que F admet un moment exponentiel :*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) \, < \, \infty \, .$$

Alors il existe des constantes $\beta = \beta(F, d)$, $l_0 = l_0(F, d)$ et $C_i = C_i(F, \beta, d)$, $i = 1, 2$, telles que
- pour tout $l_0 \leq m \leq k_1$, pour tout $n \geq \beta \prod_{i=1}^{d-1} k_i$, on a

$$\mathbb{P}(\mathrm{card}(\mathcal{E}_\sigma) \geq n) \, \leq \, C_1 \exp(-C_2 n) \, ,$$

- pour tout $l_0 \leq k_1$, pour toute hauteur m telle que $\log m \leq \prod_{i=1}^{d-1} k_i$, pour tout $n \geq \beta \prod_{i=1}^{d-1} k_i$, on a

$$\mathbb{P}(\mathrm{card}(\mathcal{E}_\phi) \geq n) \, \leq \, C_1 \exp(-C_2 n) \, .$$

Ce théorème est très important pour l'ensemble de notre travail, car sans lui plusieurs de nos théorèmes n'auraient été prouvés que sous la condition $F(0) < p_0$ proposée par Kesten dans [**41**] : le résultat sur les déviations inférieures des flux maximaux dans des cylindres (chapitre 5) et la loi des grands nombres pour le flux maximal renormalisé dans un domaine de $\mathbb{R}^d$ (chapitre 7). Ce théorème, en permettant de substituer la condition $F(0) < 1 - p_c(d)$ à la condition $F(0) < p_0$ dans le théorème 4, comble le fossé existant auparavant entre le régime $F(0) < p_0$ (donc $F(0)$ très petit), et le régime $F(0) \geq 1 - p_c(d)$. Sous une condition de moment exponentiel pour la loi des capacités, Zhang généralise dans [**59**] le théorème 4 de Kesten en établissant le résultat suivant :

THÉORÈME 7 (Zhang). *On suppose que F admet un moment exponentiel, i.e.*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) \, < \, \infty \, .$$

Si $m(\vec{k})$ tend vers l'infini quand les k_i tendent vers l'infini de telle sorte qu'il existe $\delta \in]0, 1]$ tel que

$$\log m \leq \max_{1 \leq i \leq d-1} (k_i^{1-\delta})$$

alors

$$\lim_{k_1, \dots, k_{d-1}, m \to \infty} \frac{\phi(\vec{k}, m)}{\prod_{i=1}^{d-1} k_i} \, = \, \nu((0, \dots, 0, 1)) \qquad \textit{p.s. et dans } L^1 \, .$$

De plus, cette limite est strictement positive si et seulement si $F(0) < 1 - p_c(d)$.

Nous verrons dans le chapitre 5 de la thèse que dans le cas où les côtés de la base du cylindre tendent vers l'infini à la même vitesse, nous obtenons comme conséquence de l'étude des déviations inférieures de $\phi(nA, h(n))$ une amélioration de la condition nécessaire sur la fonction de hauteur du cylindre pour avoir ce résultat de loi des grands nombres : sous les mêmes hypothèses sur F, si

$$\lim_{n \to \infty} \frac{\log h(n)}{n^{d-1}} \, = \, 0 \, ,$$

alors pour tout hyperrectangle A non dégénéré de la forme $A = \prod_{i=1}^{d-1} [a_i, b_i] \times \{c\}$ pour des réels a_i, b_i et c, $\phi(nA, h(n)) / \mathcal{H}^{d-1}(nA)$ converge presque sûrement vers $\nu((0, \dots, 0, 1))$ quand n tend vers l'infini. L'hypothèse de hauteur est ici pertinente, puisque si la loi des capacités des arêtes admet un atome en zéro, dans un cylindre dont la fonction de hauteur serait de l'ordre de $\exp(kn^{d-1})$ pour une constante k grande, la limite du flux maximal entre le sommet et la base du cylindre quand ses dimensions tendent vers l'infini est nulle presque sûrement (ceci fait l'objet d'une remarque plus détaillée dans le chapitre 3 de la thèse).

La démonstration que Zhang donne du théorème 7 dans [**59**] dans le cas $F(0) < 1 - p_c(d)$ (le cas $F(0) \geq 1 - p_c(d)$ ayant déjà été traité dans [**58**]) suit dans ses grandes lignes celle que Kesten avait donnée du théorème 4, à principalement trois différences près. D'une part il utilise le théorème 6 pour évaluer le nombre d'arêtes dans un ensemble de coupure minimale à la place d'une estimée utilisant la propriété que $F(0) < p_0$, c'est ainsi qu'il améliore l'hypothèse sur $F(0)$. Ensuite, une fois ce contrôle établi, il utilise un résultat de concentration de la mesure pour concentrer la loi de $\phi(\vec{k}, m)$ (qui ne dépend donc que d'un nombre donné d'arêtes avec grande probabilité) autour de son espérance. Ainsi, grâce à une utilisation simple du lemme de Borel-Cantelli, il n'a plus qu'à prouver la convergence de $\mathbb{E}(\phi(\vec{k}, m))/\prod_{i=1}^{d-1} k_i$, ce qui est moins complexe. Nous utiliserons des résultats de concentration similaires dans les chapitres 5 et 6 de la thèse. Finalement, il introduit une variable sous-additive en définissant l'espérance conditionnelle de $\phi(\vec{k}, m)$ sachant une certaine condition au bord sur l'ensemble de coupure correspondant, ce en quoi il suit les idées de Kesten en décrivant les conditions aux bords possibles pour les ensembles de coupure, mais il utilise cette description de façon plus efficace en mettant en évidence la famille sous-additive sous-jacente de façon claire, ce qui simplifie la démonstration.

Intéressons-nous à présent à la démonstration que Zhang a faite du théorème 6, qui est un résultat majeur pour l'étude du flux maximal en percolation de premier passage. Zhang détaille complètement sa démonstration pour l'étude de $\mathrm{card}(\mathcal{E}_\sigma)$, et ensuite en déduit le résultat sur $\mathrm{card}(\mathcal{E}_\phi)$, nous allons donc suivre sa démarche et essayer de présenter son étude de $\mathrm{card}(\mathcal{E}_\sigma)$. Il considère $\mathcal{E}_\sigma$ un ensemble de coupure minimal pour le flux entre $D(\vec{k}, m)$ et l'infini dans $\mathbb{R}^d \smallsetminus D(\vec{k}, m)$, de nombre d'arêtes minimal, sélectionné par un algorithme déterministe si plusieurs tels ensembles de coupure existent. Il contrôle le nombre d'arêtes de $\mathcal{E}_\sigma$ qui ont une capacité à valeurs dans $]0, \varepsilon]$ en choisissant un ε suffisamment petit. Il contrôle également le nombre d'arêtes de $\mathcal{E}_\sigma$ qui ont une capacité supérieure ou égale à ε grâce à un contrôle de la capacité totale de l'ensemble de coupure, chacune de ces arêtes contribuant pour une valeur supérieure ou égale à ε à cette capacité totale. La partie difficile de la preuve est de contrôler le nombre d'arêtes de $\mathcal{E}_\sigma$ de capacité nulle. Zhang considère alors une nouvelle famille de capacités sur les arêtes, $(t'(e), e \in \mathbb{E}^d)$, où $t'(e) = t(e)$ pour toutes les arêtes sauf celles de $\mathcal{E}_\sigma$ pour lesquelles il définit $t'(e) = 0$. Zhang contrôle donc le nombre d'arêtes dont la capacité a été modifiée par cette transformation. Pour cette nouvelle famille de capacités $(t'(e), e \in \mathbb{E}^d)$, $\mathcal{E}_\sigma$ est de capacité nulle. C'est donc aussi le cas pour la famille de capacités $(\widetilde{t}(e), e \in \mathbb{E}^d)$ définie par $\widetilde{t}(e) = \mathbb{1}_{\{t'(e)>0\}}$. S'il existe un ensemble de coupure de capacité nulle pour cette famille de capacités dont on arrive à contrôler le nombre d'arêtes, alors on sait que cet ensemble sera un ensemble de coupure de capacité minimale dont on contrôle le nombre d'arêtes dans le modèle initial, ce qui achèvera la démonstration. Zhang s'est donc ramené à étudier le nombre d'arêtes dans un ensemble de coupure de capacité nulle dans le modèle de percolation classique de paramètre $p > p_c(d)$, conditionné à l'existence d'un ensemble de coupure de capacité nulle. La capacité des arêtes à l'intérieur de $D(\vec{k}, m)$ pour cette percolation n'ayant aucune importance, Zhang les considère comme ouvertes, et il regarde le bord de la composante connexe ouverte de $D(\vec{k}, m)$. Ce bord peut être très enchevêtré. Pour le lisser, Zhang utilise un changement d'échelle : il définit des blocs de taille mésoscopique (c'est-à-dire grande devant 1 mais petite devant la taille du système $D(\vec{k}, m)$) et regarde tous ceux qui contiennent une partie du bord de la composante connexe ouverte de $D(\vec{k}, m)$. Parmi ceux-ci, il garde les blocs qui sont la frontière extérieure de cet ensemble de blocs, ainsi que certains blocs plus à l'intérieur qu'il définit proprement, de telle sorte qu'il est sûr de trouver un ensemble de coupure de capacité nulle à l'intérieur des blocs qu'il a conservé ; les blocs "intérieurs" qu'il a dû garder correspondent aux replis de l'ensemble de coupure vu à l'échelle mésoscopique, cette échelle ne permet pas d'éliminer tous les replis mais simplement comme on va le voir d'en contrôler la longueur. Finalement, Zhang montre qu'en régime de percolation sur-critique, tous ces blocs vérifient une propriété qui arrive avec très petite probabilité - en un sens ils contiennent tous des frontières d'arêtes fermées de taille mésoscopique - , donc avec grande probabilité il n'y a pas trop de tels blocs. Ainsi Zhang réussit à contrôler le nombre de blocs dans lesquels un ensemble de coupure minimale est inclus, et donc le nombre d'arêtes qui composent cet ensemble de coupure.

2.3. Flux maximal entre un ensemble convexe borné et l'infini en dimension deux. Nous présentons dans ce paragraphe un résultat prouvé par Garet dans [**30**] en 2006. Ce résultat est antérieur à l'article de Zhang [**59**], que nous avons néanmoins choisi de présenter avant dans la mesure où il répond directement à une question importante posée par Kesten dans [**41**]. Le théorème 6 fait également le lien entre le flux maximal $\phi(\vec{k},m)$ entre le sommet et la base du cylindre $D(\vec{k},m)$ et le flux maximal $\sigma(\vec{k},m)$ entre $D(\vec{k},m)$ et l'infini dans $\mathbb{R}^d \smallsetminus D(\vec{k},m)$. Kesten avait travaillé sur la première de ces deux variables, Garet s'est quant à lui intéressé à la deuxième. Il a montré dans le cas de la dimension deux que le flux maximal $\sigma(nA)$ dans $\mathbb{R}^2 \smallsetminus nA$ entre l'infini et nA où A est un ensemble convexe borné de $\mathbb{R}^2$ contenant 0 dans son intérieur, renormalisé par n, converge quand n tend vers l'infini vers une constante non aléatoire, qui est donnée sous forme d'intégrale. Pour A un ensemble de $\mathbb{R}^2$, nous allons noter $\partial^* A$ l'ensemble des points x de la frontière ∂A de A en lesquels A admet un unique vecteur normal extérieur unitaire $\vec{v}_A(x)$ défini au sens mesure (voir [**19**]). Si A est un convexe, l'ensemble $\partial^* A$ est aussi égal à l'ensemble des points x de ∂A en lesquels A admet un unique vecteur normal extérieur unitaire au sens classique, et ce vecteur est $\vec{v}_A(x)$. Voici le théorème exact :

THÉORÈME 8 (Garet). *Soit $d=2$. On suppose que $F(0) < 1 - p_c(2) = 1/2$ et que F admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,\infty[} e^{\gamma x}\, dF(x) \,<\, \infty\,.$$

Alors pour tout ensemble convexe borné $A \subset \mathbb{R}^2$ contenant 0 dans son intérieur, on a

$$\lim_{n\to\infty} \frac{\sigma(nA)}{n} \,=\, \int_{\partial^* A} \nu(\vec{v}_A(x))d\mathcal{H}^1(x) \,=\, \mathcal{I}(A) \,>0 \qquad p.s.$$

De plus, pour tout $\varepsilon > 0$, il existe des constantes C_1, $C_2 > 0$ dépendant de ε et F telles que

$$\forall n \geq 0 \qquad \mathbb{P}\left[\frac{\sigma(nA)}{n\mathcal{I}(A)} \notin]1-\varepsilon, 1+\varepsilon[\right] \,\leq\, C_1 \exp(-C_2 n)\,.$$

Une première remarque à faire est que puisque ce résultat est prouvé en dimension 2, via la dualité et l'invariance du graphe par une rotation d'angle $\pi/2$, la constante $\nu(\vec{v})$ définie en terme de flux est égale à la constante $\mu(\vec{v})$ qui est la limite du temps d'atteinte entre deux points 0 et x renormalisé par $\|x\|_2$, quand x tend vers l'infini de telle sorte que la droite $(0x)$ est dirigée par $\vec{v}$. Nous allons ici utiliser librement les termes de flux maximaux et de temps d'atteinte à la fois pour toujours utiliser celui qui s'adapte le mieux à la situation.

Pour prouver le théorème 8, Garet commence par approcher l'ensemble convexe A à la fois par l'intérieur et par l'extérieur par des polygones de telle sorte que $\mathcal{I}(A)$ soit très peu changé. Ainsi, il se ramène à prouver le théorème 8 uniquement pour des polygones. Si A est un polygone, $\mathcal{I}(A)$ est plus simplement égal à la somme sur les faces de A de la longueur de la face multipliée par $\nu(\vec{v})$ où $\vec{v}$ est orthogonal à la face. Garet considère une partition de $\mathbb{R}^2$ par les demi-droites $D_i = [0, s_i[$ où s_i parcourt l'ensemble des sommets de A. Sur chacune de ces demi-droites, il choisit tout d'abord le point $x_i = n(1+\eta)s_i$ pour un $\eta > 0$: c'est un point en dehors de nA, mais très proche de $\partial(nA)$. Avec une grande probabilité, il trouve un chemin entre x_i et x_{i+1} qui ne pénètre pas dans nA et qui ait un temps d'atteinte très proche de $n\|s_{i+1} - s_i\|_2 \nu(\vec{v}_i)$ pour n assez grand, où $\vec{v}_i$ désigne un vecteur unitaire orthogonal à $(s_i s_{i+1})$. Il peut alors majorer la capacité d'un ensemble de coupure pour $\sigma(nA)$ par la somme des temps d'atteinte de ces chemins, et donc obtenir les déviations supérieures recherchées. Pour étudier la probabilité que $\sigma(nA)$ soit plus petit que $n\mathcal{I}(A)$, Garet considère un ensemble de coupure de capacité minimale, vue comme un chemin autour de nA de temps de passage minimal via la dualité. En se fixant un point de départ sur ce chemin, Garet définit pour tout i le point y_i qui est le dernier point d'intersection du chemin avec la demi-droite D_i. Si B est le polygone formé pas les y_i, alors Garet montre que $\mathcal{I}(B) \geq n\mathcal{I}(A)$. Si $\sigma(nA)$ est plus petit que $n\mathcal{I}(A)$, donc que $\mathcal{I}(B)$, cela signifie qu'il existe au moins un i tel que le temps d'atteinte entre y_i et y_{i+1} soit plus petit que $\|y_{i+1} - y_i\|_2 \nu(\vec{u}_i)$ où $\vec{u}_i$ désigne un vecteur unitaire orthogonal à $(y_i y_{i+1})$. Finalement, en utilisant le fait que les demi-droites D_i s'éloignent assez rapidement les unes des autres quand on s'éloigne de nA, Garet contrôle la probabilité qu'un tel couple de points $(y_i, y_{i+1}) \in D_i \times D_{i+1} \smallsetminus (nA)$ existe, ce qui achève la démonstration.

Notons au passage que l'objectif de Garet dans cet article était la démonstration d'une loi des grands nombres, et non l'étude des grandes déviations de $\sigma(nA)$, il n'a donc pas cherché à obtenir la bonne vitesse des grandes déviations par au-dessus. Une autre remarque, qui est d'importance, est qu'il conjecture dans [**30**] que le théorème reste vrai en dimension d. En fait, le passage à une dimension plus grande est délicat pour la raison suivante : si on considère une surface de coupure qui sépare un polygone convexe nA de l'infini en dimension $d \geq 3$, que l'on intersecte cette surface avec les morceaux d'hyperplans P_i qui sont définis par l'origine du réseau et les arêtes du polygone A, la trace de la surface sur ces portions d'hyperplans est en fait très compliquée. En dimension deux, cette trace est constituée de points, mais dès la dimension trois apparaissent des formes beaucoup plus difficilement manipulables. Nous nous appuierons sur la même spécificité de la dimension deux dans nos travaux du chapitre 6, qui ne sont eux non plus pas transposables facilement à une dimension $d \geq 3$. Le chapitre 7 de la thèse présentera des techniques pour l'étude du flux maximal dans un domaine de $\mathbb{R}^d$ qui devraient pouvoir s'adapter pour généraliser la loi des grands nombres de Garet à une dimension $d \geq 3$.

Nous voulons encore souligner le fait que derrière le théorème 8 se cache un principe variationnel. En effet, au cours de la démonstration, Garet prouve en utilisant l'inégalité triangulaire sur $\nu = \mu$ que pour tous polygones A et B tels que $A \subset B$ et A convexe, on a $\mathcal{I}(A) \leq \mathcal{I}(B)$. Cette propriété est d'ailleurs essentielle dans la preuve. Via l'approximation polyédrale présentée par Garet, on en déduit que pour tout ensemble convexe borné $A \subset \mathbb{R}^2$, $\mathcal{I}(A)$ est aussi égal à l'infimum de $\mathcal{I}(B)$ sur tous les polygones B contenant A, c'est-à-dire tous les polygones B qui séparent A de l'infini. Si cette formulation a l'air quelque peu artificielle ici, elle prendra tout son sens à l'énoncé des résultats qui seront prouvés dans les chapitres 6 et 7 de la thèse.

2.4. Autres axes de recherche. Nous évoquons dans ce paragraphe d'autres travaux portant sur des problèmes de flux maximaux dans des graphes. Les directions prises par la recherche dans ce domaine sont multiples, c'est pourquoi nous ne pouvons pas être exhaustifs. Nous souhaitons néanmoins évoquer ici les approches de Boivin [**11**] et Aldous [**3**], [**4**]. Boivin s'est intéressé dans [**11**] à une généralisation du modèle de percolation de premier passage dans $\mathbb{Z}^3$ dans laquelle l'hypothèse pour la famille des capacités des arêtes d'être i.i.d. est relâchée puisqu'il considère une famille stationnaire et ergodique. Il étudie la capacité minimale de surfaces de plaquettes dont le bord est fixé le long d'une courbe $\mathcal{C}$ incluse dans un plan $\mathcal{P}$, comme par exemple une surface de coupure pour le flux maximal τ, et il montre que la convergence p.s. de cette capacité minimale renormalisée a lieu et est uniforme par rapport à la direction de $\mathcal{P}$ sous une condition de moment sur la loi des capacités qui dépend de la courbure de $\mathcal{C}$: moment d'ordre strictement plus grand que $3/2$ si $\mathcal{C}$ est un cercle, et strictement plus grand que 2 si $\mathcal{C}$ est le bord d'un rectangle.

Aldous a étudié récemment dans [**3**] un autre problème de flux maximal. Il considère le tore $N \times N$, des capacités aléatoires $(c(e))$ i.i.d. sur l'ensemble des arêtes du tore. Il suppose que le réseau transmet un flux a_N entre toute paire de sites ; le flux moyen par arête étant alors de l'ordre de $N^3 a_N$, il considère un flux renormalisé $\rho = N^3 a_N$ qu'il fixe égal à 1. Si on note $f(e)$ la quantité de flux qui traverse l'arête e suivant le courant f réalisant le flux ρ, le coût de f est défini comme étant $\sum_e f(e)^2 c(e)$. Le coût C_N aléatoire associé au flux renormalisé $\rho = 1$ est alors l'infimum du coût des courants qui réalisent le flux. Aldous prouve que si les capacités des arêtes sont bornées, et si la quantité maximale de flux qui peut traverser une arête est elle aussi bornée uniformément sur les arêtes, alors la limite de $\mathbb{E}[C_N]/N^2$ quand N tend vers l'infini existe p.s. et est une constante. Dans [**4**], pour une même définition du coût mais un graphe différent, il présente un algorithme pour calculer cette limite. Le problème majeur auquel il est confronté dans l'étude de ce modèle est qu'il n'y a pas de sous-additivité sous-jacente, il doit donc utiliser d'autres techniques. Il s'inspire de la méthode de la cavité ("cavity method" en anglais) pour surmonter cette difficulté.

3. Contributions

Cette thèse a été entièrement dévolue à l'étude du flux maximal dans le modèle de percolation de premier passage dans le graphe $(\mathbb{Z}^d, \mathbb{E}^d)$ en dimension $d \geq 2$. Nous nous sommes donné trois objectifs : étudier les grandes déviations par au-dessus et par en dessous pour le flux maximal dans des cas où la loi des grands nombres pour ce flux était connue, prouver si possible les principes de grande déviation correspondant pour mieux décrire le comportement du flux, et étendre la loi des grands nombres au flux maximal dans des cylindres inclinés ou dans des sous-ensembles de $\mathbb{R}^d$ plus généraux. Nous présentons ici les principales contributions de cette thèse à l'étude du flux maximal en percolation de premier passage. Nous donnerons pour chaque chapitre l'énoncé précis des principaux résultats démontrés, puis nous essaierons de présenter succinctement les idées utilisées dans les preuves.

3.1. Partie 1 : Déviations supérieures pour les flux maximaux dans des cylindres.

3.1.1. *Chapitre 2 : la variable ϕ dans des cylindres droits.* On considère un cylindre droit $(B(n))$ de la forme $[0,n[^{d-1}\times[0,h(n)[$ pour une fonction de hauteur h qui tend vers l'infini quand n tend vers l'infini. Grâce aux travaux de Kesten [**41**] et de Zhang [**59**], on sait que le flux maximal entre le sommet $[0,n[^{d-1}\times\{h(n)\}$ et la base $[0,n[^{d-1}\times\{0\}$ du cylindre, noté $\phi(B(n))$, renormalisé par n^{d-1} converge presque sûrement vers $\nu = \nu((0,...,0,1))$. Nous cherchons à évaluer la probabilité que le flux soit anormalement grand. Nous prouvons dans le chapitre 2 que cette probabilité décroît exponentiellement vite avec le volume du cylindre. De plus, nous démontrons un principe de grande déviation pour ce flux renormalisé, ce qui signifie en un certain sens que nous arrivons à déterminer la probabilité que le flux renormalisé appartienne à un intervalle inclus dans $]\nu, +\infty[$. Voici l'énoncé du théorème obtenu :

THÉORÈME 9. *Soit $\phi(B(n))$ le flux maximal entre le sommet $[0,n[^{d-1}\times\{h(n)\}$ et la base $[0,n[^{d-1}\times\{0\}$ du cylindre $B(n) = [0,n[^{d-1}\times[0,h(n)[$. On suppose que la fonction de hauteur $h:\mathbb{N}\to\mathbb{N}$ satisfait la condition*

$$\lim_{n\to\infty} \frac{h(n)}{\log n} \,=\, +\infty\,.$$

Alors pour tout $\lambda\in\mathbb{R}^+$, la limite

$$\psi(\lambda) \,=\, \lim_{n\to\infty} \frac{-1}{n^{d-1}h(n)} \log \mathbb{P}[\phi(B(n)) \geq \lambda n^{d-1}]$$

existe dans $\mathbb{R}^+\cup\{+\infty\}$ et est indépendante de h. De plus, ψ est convexe sur $\mathbb{R}^+$, finie et continue sur l'ensemble $\{\lambda \,|\, F([\lambda,+\infty[) > 0\}$. Si F admet un moment d'ordre 1, i.e.,

$$\int_{[0,+\infty[} x\, dF(x) \,<\, \infty\,,$$

alors ψ est nulle sur $[0,\nu]$. Si F admet un moment exponentiel, i.e.,

$$\exists\,\theta>0 \qquad \int_{[0,+\infty[} e^{\theta x} dF(x) \,<\, \infty\,,$$

alors ψ est strictement positive sur $]\nu,+\infty[$ et de plus la suite

$$\left(\frac{\phi(B(n))}{n^{d-1}}\right)_{n\in\mathbb{N}}$$

satisfait un principe de grande déviation de vitesse $n^{d-1}h(n)$ et gouverné par la bonne fonction de taux ψ.

L'allure de la fonction de taux ψ est donnée par la figure 7.

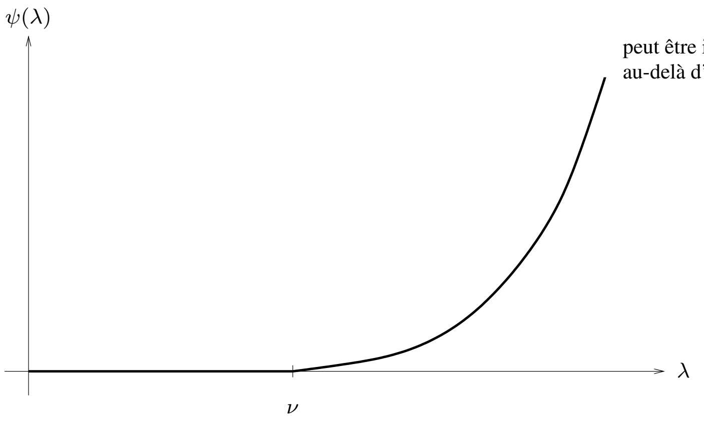

FIG. 7. Allure de la fonction de taux ψ.

Ce théorème a été énoncé et prouvé dans le cas de cylindres à base carrée uniquement pour simplifier les notations, mais il demeure valide pour des cylindres droits de base rectangulaire, et la fonction ψ est indépendante du rectangle $\prod_{i=1}^{d-1}[na_i, nb_i] \times \{nc\}$, $a_i, b_i, c \in \mathbb{R}$, qui forme la base du cylindre. De même, il reste valide dans les cylindres fermés $[0,n]^{d-1} \times [0, h(n)]$. La positivité de ψ sur $]\nu, +\infty[$ ne requiert en fait pas de condition sur h autre que $\lim_{n\to\infty} h(n) = +\infty$, ceci sera clairement énoncé dans le chapitre 3.

Essayons de comprendre pourquoi ce résultat est naturel. Il est évident que si toutes les arêtes du cylindre ont une capacité anormalement grande, alors elles laisseront circuler un flux anormalement grand. Le nombre d'arêtes dans le cylindre est de l'ordre du volume $n^{d-1}h(n)$ du cylindre, donc cette probabilité est au moins de l'ordre de $\exp(-cn^{d-1}h(n))$ pour une constante c positive, dès que les arêtes n'ont pas une capacité constante. Réciproquement, pour que le flux renormalisé soit anormalement grand, on peut envisager deux causes : soit il y a quelques chemins d'arêtes de capacité énorme de l'ordre de n^{d-1} entre le sommet et la base du cylindre, soit il y a de l'ordre de n^{d-1} chemins d'arêtes de capacité légèrement anormalement élevée. Sous une condition sur la queue de la distribution des capacités des arêtes (existence d'un moment exponentiel), on s'attend à ce que le deuxième phénomène soit le plus probable. Chaque chemin entre la base et le sommet du cylindre étant composé d'au moins $h(n)$ arêtes, il faudrait alors qu'une proportion strictement positive des arêtes du cylindre aient une capacité anormalement grande pour permettre d'augmenter le flux renormalisé. Ce raisonnement intuitif nous laisse penser que la probabilité que le flux renormalisé soit anormalement grand ne peut pas être plus grande que $c'_1 \exp(-c'_2 n^{d-1}h(n))$ pour des constantes c'_i données, et donc que les déviations supérieures sont d'ordre volumique. Ce résultat est prouvé ici, lorsque nous montrons que la fonction de taux du principe de grande déviation satisfait par le flux renormalisé est strictement positive sur $]\nu, +\infty[$.

La démonstration de cette positivité est basée sur le théorème de Cramér dans $\mathbb{R}$, appliqué à la variable τ définie dans des cylindres droits de taille mésoscopique, c'est-à-dire de taille grande devant 1 mais petite devant la taille n du système étudié. Pour prouver le principe de grande déviation lui-même, nous devons décrire comment l'eau circule à travers le cylindre, en particulier la façon dont elle entre et sort du cylindre. Si on impose des conditions sur la façon dont l'eau entre et sort par les extrémités du cylindre, on peut superposer verticalement des cylindres en garantissant que l'eau les traverse tous sans que le flux maximal ne diminue trop du fait de cet empilement. Grâce à cela, en utilisant l'invariance du graphe par la symétrie d'hyperplan $\mathbb{R}^{d-1} \times \{h(n)\}$ et une discrétisation des valeurs prises par le flux maximal, nous montrons l'existence de la fonction de taux du principe de grandes déviations. Cette preuve d'existence est la partie difficile de la démonstration, puisque les techniques utilisées ensuite pour prouver le principe de grande déviation lui-même sont classiques.

3.1.2. *Chapitre 3 : une généralisation de l'étude des déviations supérieures des flux maximaux dans des cylindres.* Le but de ce chapitre est d'étendre les résultats obtenus dans le chapitre 2 au cas de la variable τ, et au cas de la variable ϕ dans des cylindres penchés. Nous considérons le cylindre $\mathrm{cyl}(nA, h(n))$ de base un hyperrectangle A non dégénéré de vecteur normal unitaire $\vec{v}$ et de fonction de hauteur h. Nous savons que $\mathbb{E}[\tau(nA, h(n))]/\mathcal{H}^{d-1}(nA)$ converge vers la constante $\nu(\vec{v})$ quand n tend vers l'infini, et que $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ converge p.s. vers la même limite sous certaines conditions sur F, ou sur $\vec{v}$ et A. Contrairement à la variable $\phi(B(n))$ du chapitre précédent, nous montrons que $\tau(nA, h(n))$ n'a pas forcément des déviations supérieures d'ordre volumique dès que la loi des capacités des arêtes a un moment exponentiel. Nous prouvons le résultat suivant :

THÉORÈME 10. *Soit A un hyperrectangle non dégénéré, $\vec{v}$ un des deux vecteurs unitaires orthogonal à A, et $h : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction vérifiant $\lim_{n\rightarrow\infty} h(n) = +\infty$.*
- Si la capacité des arêtes est bornée, alors pour tout $\lambda > \nu(\vec{v})$ on a

$$\lim_{n\rightarrow\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)\min(n, h(n))} \log \mathbb{P}[\tau(nA, h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)] \;>\; 0\,;$$

- Si la capacité des arêtes suit la loi exponentielle de paramètre 1, alors il existe $n_0(d, A, h)$, et pour tout $\lambda > \nu(\vec{v})$ il existe une constante $D = D(\lambda, d)$ telle que pour tout $n \geq n_0$ on a

$$\mathbb{P}[\tau(nA, h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)] \;\geq\; \exp(-D\mathcal{H}^{d-1}(nA))\,;$$

- Si la loi des capacités des arêtes admet des moments exponentiels de tous ordres, i.e.,

$$\forall \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) \;<\; +\infty\,,$$

alors pour tout $\lambda > \nu(\vec{v})$ on a

$$\lim_{n\rightarrow\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}[\tau(nA, h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)] \;=\; -\infty\,.$$

Nous montrons également que la probabilité que le flux maximal $\phi(nA, h(n))$ dans le même cylindre $\mathrm{cyl}(nA, h(n))$ soit plus grand que $\nu(\vec{v})\mathcal{H}^{d-1}(nA)$ décroît exponentiellement vite avec le volume du cylindre :

THÉORÈME 11. *Soit A un hyperrectangle non dégénéré, $\vec{v}$ un des deux vecteurs unitaires orthogonal à A, et $h : \mathbb{N} \rightarrow \mathbb{R}^+$ une fonction vérifiant $\lim_{n\rightarrow\infty} h(n) = +\infty$. Si la loi des capacités des arêtes admet un moment exponentiel, i.e.,*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) \;<\; +\infty\,,$$

alors pour tout $\lambda > \nu(\vec{v})$ on a

$$\liminf_{n\rightarrow\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)h(n)} \log \mathbb{P}[\phi(nA, h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)] \;>\; 0\,.$$

Une remarque sur la portée de ce deuxième théorème s'impose : dans le cas où $h(n)$ est négligeable devant n ou dans le cas des cylindres droits, $\nu(\vec{v})$ étant la limite presque sûre de $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ sous certaines conditions, ce résultat montre que les grandes déviations par au-dessus pour le flux maximal $\phi(nA, h(n))$ renormalisé sont d'ordre volumique. Nous soulignons ici le fait que pour les autres fonctions de hauteur h, $\nu(\vec{v})$ n'est a priori pas la limite du flux renormalisé, donc nous n'avons ici qu'une information partielle. Des compléments seront apportés à cette étude dans le cas de la dimension deux (voir le chapitre 6) et dans le cas où $h(n)$ est proportionnel à n (c'est un cas particulier de l'étude du chapitre 7). Par ailleurs ce résultat, même partiel, nous sera utile dans le chapitre 7.

Les démonstrations des deux théorèmes énoncés ici sont basées sur une adaptation de la technique utilisée dans le chapitre 2 pour montrer le résultat analogue dans le cas des cylindres droits pour $\phi(B(n))$ (découpage en sous-cylindres de taille mésoscopique et utilisation du théorème de Cramér dans $\mathbb{R}$). La principale différence est que le défaut de sous-additivité pour les flux maximaux dans le cas des cylindres penchés crée quelques complications techniques.

Prenons un instant pour essayer de comprendre la différence de comportement entre les variables $\phi(nA, h(n))$ et $\tau(nA, h(n))$. Dans le cas du flux maximal $\tau(nA, h(n))$, la distance entre les zones d'entrée et de sortie de l'eau à la surface du cylindre étant nulle, il suffit d'un nombre constant d'arêtes (et non $h(n)$ arêtes) pour former un chemin qui véhicule du flux entre le demi-cylindre inférieur et le demi-cylindre supérieur. Si la queue de distribution de la loi des capacités des arêtes est trop lourde, par exemple dans le cas où la capacité des arêtes suit la loi exponentielle, nous en déduisons que la probabilité que $\tau(nA, h(n))$ soit anormalement grand est au moins de l'ordre de $\exp(-cn^{d-1})$ pour une constante c. Par ailleurs, puisqu'il existe un ensemble de coupure contenant de l'ordre de n^{d-1} arêtes (imaginons une surface de coupure plate), nous savons que la probabilité que le flux soit anormalement grand décroît au moins exponentiellement vite avec n^{d-1}, donc nous obtenons que les déviations supérieures sont exactement d'ordre surfacique. Par contre, si les capacités des arêtes sont bornées, il n'y a pas assez de chemins très courts entre les deux demi-cylindres pour influencer $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$, donc les déviations supérieures ne sont pas d'ordre surfacique. Néanmoins, en ce qui concerne le flux $\tau(nA, h(n))$, l'eau peut entrer et sortir du cylindre par ses faces verticales $\mathrm{cyl}(\partial(nA), h(n))$. Ceci implique qu'il suffit que les arêtes qui se trouvent à une distance plus petite que Cn de nA pour une constante C aient une capacité anormalement grande pour que $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ soit anormalement grand. Nous savons donc que la probabilité que $\tau(nA, h(n))$ soit anormalement grand est au moins d'ordre $\exp(-c'n^{d-1} \min(n, h(n)))$ pour une constante c'. La vitesse de décroissance que nous obtenons pour les déviations supérieures de $\tau(nA, h(n))$ dans le cas des capacités bornées est donc la bonne, et elle diffère de celle pour $\phi(nA, h(n)$ dès que $h(n)/n$ n'est pas borné. Heuristiquement, nous pouvons lier ceci au fait qu'une surface de coupure pour $\tau(nA, h(n))$ a ses bords localisés le long de $\partial(nA)$, et donc elle n'explore qu'une partie du cylindre de hauteur de l'ordre de n, alors qu'une surface de coupure pour $\phi(nA, h(n))$ n'est pas localisée du tout. Ceci n'est qu'une heuristique car il est en fait difficile d'étudier la localisation d'un ensemble de coupure minimal, nous avons essayé vainement de le faire en utilisant les résultats de Dobrushin [**25**] et Gielis et Grimmett [**31**]. Finalement, nous étudions le cas où la loi des capacités des arêtes admet des moments exponentiels de tous ordres pour montrer que des régimes intermédiaires peuvent exister.

Les techniques utilisées pour montrer l'existence de la fonction ψ dans le cas de la variable $\phi(B(n))$ dans des cylindres droits ne peuvent pas être adaptées facilement à $\tau(nA, h(n))$ car les surfaces par lesquelles l'eau entre et sort sur le bord du cylindre n'ont alors pas du tout les mêmes propriétés de symétrie par rapport au graphe, les cylindres ne s'empilent donc pas convenablement pour transmettre du flux. Ces techniques ne peuvent pas non plus être adaptées à des cylindres inclinés, car alors le graphe n'est pas symétrique par rapport aux hyperplans définis par les faces du cylindre. Aucun principe de grandes déviations par au-dessus n'a donc pu être montré dans ce chapitre pour généraliser celui obtenu au chapitre précédent.

3.2. Partie 2 : Déviations inférieures pour les flux maximaux dans des cylindres.

3.2.1. *Chapitre 4 : un résultat partiel.* Nous souhaitons évaluer la probabilité que le flux maximal $\phi(B(n))$ entre le sommet et la base d'un cylindre droit $B(n) = [0, n]^{d-1} \times [0, h(n)]$ soit anormalement petit. Nous prouvons ici un résultat partiel, à savoir que sous certaines conditions sur F, la probabilité que le flux $\phi(B(n))$ dans le cylindre $B(n)$ soit plus petit que εn^{d-1} pour un ε assez petit décroît exponentiellement vite avec n^{d-1} quand n tend vers l'infini. Plus précisément, nous montrons le résultat suivant :

THÉORÈME 12. *On suppose que $F(0) < 1 - p_c(d)$. Alors il existe une constante $\varepsilon_0 > 0$, dépendant seulement de d et F, et une constante $C > 0$, dépendant seulement de d, telles que pour toute fonction $h : \mathbb{N} \rightarrow \mathbb{N}$ vérifiant*

$$\lim_{n\rightarrow\infty} \frac{\log h(n)}{n^{d-1}} \,=\, 0$$

on a

$$\forall \varepsilon < \varepsilon_0 \qquad \liminf_{n\rightarrow\infty} \frac{-1}{n^{d-1}} \log \mathbb{P}[\phi(B(n)) \leq \varepsilon n^{d-1}] \,\geq\, C \,>\, 0\,.$$

Voici l'idée intuitive de ce théorème : nous savons que s'il existe une couche plate horizontale d'épaisseur constante dans le cylindre à l'intérieur de laquelle les arêtes ont toutes une capacité anormalement petite, le flux maximal $\phi(B(n))$ sera anormalement petit. Nous en déduisons que la probabilité que $\phi(B(n))$ soit anormalement petit est supérieure ou égale à $\exp(-cn^{d-1})$ pour une constante $c > 0$. Nous souhaitons montrer que les grandes déviations par en dessous pour $\phi(B(n))/n^{d-1}$ sont d'ordre surfacique, c'est-à-dire que la vitesse décrite ci-dessus est la bonne. Le résultat que nous montrons ici est incomplet (voir le chapitre 5 pour le résultat complet), mais il conforte notre intuition que les déviations étudiées sont bien d'ordre surfacique. De plus, il montre que la constante $\nu((0, ..., 0, 1))$ est strictement positive dès que $F(0) < 1 - p_c(d)$, puisque $\tau(B(n)) \geq \phi(B(n))$. Par ailleurs, la condition $F(0) < 1 - p_c(d)$ est pertinente, puisque nous savons que si $F(0) \geq 1 - p_c(d)$ alors $\nu = \nu((0, ..., 0, 1)) = 0$.

Nous utilisons dans la preuve du théorème 12 une technique de changement d'échelle : nous regardons des blocs de taille mésoscopique dans le système, dans lesquels nous considérons un événement arrivant avec très grande probabilité. Cet événement doit être choisi avec soin pour que sa réalisation dans un grand nombre de blocs assure la circulation d'un flux important à travers le cylindre. Le théorème 12 est une généralisation d'un résultat de Chayes et Chayes [**20**], obtenu pour des capacités de loi de Bernoulli et dans des cylindres dont la fonction de hauteur est soumise à une contrainte plus forte.

3.2.2. *Chapitre 5 : le résultat complet.* Tout le chapitre 5 de la thèse présente un travail réalisé en collaboration avec Raphaël Rossignol, actuellement maître de conférences à l'université Paris XI.

Nous prouvons dans ce chapitre que les déviations inférieures pour les flux maximaux renormalisés $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ et $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$, dans les cas où nous savons déjà que les espérances de ces variables renormalisées convergent, sont bien d'ordre surfacique comme nous l'avions prédit au chapitre précédent. Nous montrons de plus que ces variables satisfont des principes de grande déviation par en dessous. Il est difficile de résumer tout ceci en un seul théorème, c'est pourquoi nous donnons ici plusieurs énoncés. Le premier concerne la variable $\tau(nA, h(n))$ dans des cylindres quelconques :

THÉORÈME 13. *On suppose que $F(0) < 1 - p_c(d)$ et que F admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) < \infty\,.$$

Alors pour tout $\varepsilon > 0$ il existe une constante $C_1(\varepsilon, F, d) > 0$ telle que pour toute fonction de hauteur $h : \mathbb{N} \rightarrow \mathbb{R}^+$ vérifiant $\lim_{n\rightarrow\infty} h(n) = +\infty$, pour tout vecteur unitaire $\vec{v}$, pour tout hyperrectangle A non dégénéré orthogonal à $\vec{v}$, on a

$$\liminf_{n\rightarrow\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}[\tau(nA, h(n)) \leq (\nu(\vec{v}) - \varepsilon)\mathcal{H}^{d-1}(nA)] \,\geq\, C_1(\varepsilon, F, d) \,>\, 0\,.$$

De plus si on suppose que $F(0) < 1 - p_c(d)$ et que F admet des moments exponentiels de tous ordres :

$$\forall \gamma > 0 \,, \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) < \infty \,,$$

pour tous tels h, $\vec{v}$ et A, la suite

$$\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right)_{n\in\mathbb{N}}$$

satisfait un principe de grande déviation de vitesse $\mathcal{H}^{d-1}(nA)$ gouverné par la bonne fonction de taux $\mathcal{J}_{\vec{v}}$ qui dépend uniquement de $\vec{v}$, F et d et non de A ou h. Cette fonction est convexe sur $\mathbb{R}^+$, infinie sur $[0, \delta\|\vec{v}\|_1[\cup]\nu(\vec{v}), +\infty[$ où $\delta = \inf\{\lambda \,|\, \mathbb{P}(t(e) \le \lambda) > 0\}$, nulle en $\nu(\vec{v})$, et si de plus $\delta\|\vec{v}\|_1 < \nu(\vec{v})$ cette fonction est finie sur $]\delta\|\vec{v}\|_1, \nu(\vec{v})[$, continue et strictement décroissante sur $[\delta\|\vec{v}\|_1, \nu(\vec{v})]$ et strictement positive sur $[\delta\|\vec{v}\|_1, \nu(\vec{v})[$.

L'expression de $\mathcal{J}_{\vec{v}}$, en termes de limite d'une suite, est donnée dans le chapitre 5, et la figure 8 présente l'allure de cette fonction.

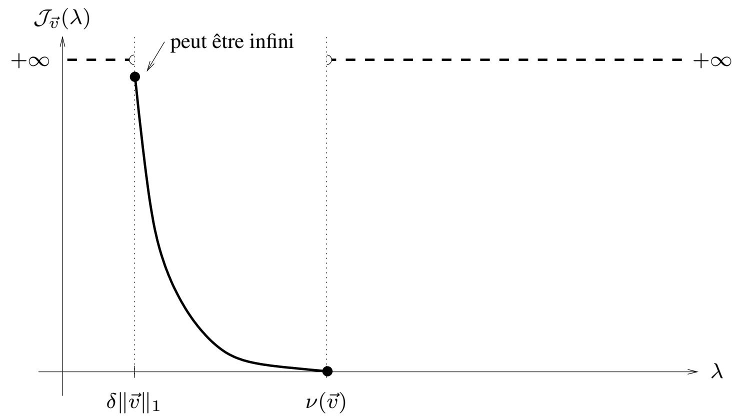

FIG. 8. Allure de la fonction de taux $\mathcal{J}_{\vec{v}}$.

Le deuxième résultat que nous présentons est un corollaire du premier, il concerne la variable $\phi(nA, h(n))$ dans des cylindres plats :

COROLLAIRE 3.1. *On suppose que $F(0) < 1 - p_c(d)$ et que F admet un moment exponentiel :*

$$\exists \, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) < \infty \,.$$

Alors pour tout $\varepsilon > 0$ il existe une constante $C_2(\varepsilon, F, d) > 0$ telle que pour toute fonction $h : \mathbb{N} \to \mathbb{R}^+$ vérifiant $\lim_{n\to\infty} h(n) = +\infty$ et

$$\lim_{n\to\infty} \frac{h(n)}{n} \;=\; 0 \,,$$

pour tout vecteur unitaire $\vec{v}$, pour tout hyperrectangle A non dégénéré orthogonal à $\vec{v}$, on a

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}[\phi(nA, h(n)) \le (\nu(\vec{v}) - \varepsilon)\mathcal{H}^{d-1}(nA)] \;\ge\; C_2(\varepsilon, F, d) \;>\; 0 \,.$$

De plus si on suppose que $F(0) < 1 - p_c(d)$ et que F admet des moments exponentiels de tous ordres :

$$\forall \gamma > 0 \,, \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) < \infty \,,$$

pour tous tels h, $\vec{v}$ et A, la suite

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right)_{n\in\mathbb{N}}$$

satisfait un principe de grande déviation de vitesse $\mathcal{H}^{d-1}(nA)$ *gouverné par la bonne fonction de taux* $\mathcal{J}_{\vec{v}}$ *(la même que dans le théorème 13).*

Le troisième théorème concerne la variable $\phi(nA, h(n))$ dans des cylindres droits :

THÉORÈME 14. *On suppose que* $F(0) < 1 - p_c(d)$ *et que* F *admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) < \infty\,.$$

Alors pour toute fonction $h : \mathbb{N} \to \mathbb{R}^+$ *telle que* $\lim_{n\to\infty} h(n) = +\infty$ *et*

$$\lim_{n\to\infty} \frac{\log h(n)}{n^{d-1}} \;=\; 0\,,$$

pour tout hyperrectangle A *non dégénéré de la forme* $\prod_{i=1}^{d-1}[a_i, b_i] \times \{c\}$ *pour des réels* a_i, b_i, c*, la suite*

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right)_{n\in\mathbb{N}}$$

satisfait un principe de grande déviation de vitesse $\mathcal{H}^{d-1}(nA)$ *gouverné par la bonne fonction de taux* $\mathcal{J}_{\vec{v}}$ *avec* $\vec{v} = (0, ..., 0, 1)$ *(la fonction* $\mathcal{J}_{\vec{v}}$ *est la même que dans le théorème 13). En particulier, cela implique que pour tout* $\varepsilon > 0$ *il existe une constante* $C_3(\varepsilon, F, d) > 0$ *telle que*

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}[\phi(nA, h(n)) \leq (\nu(\vec{v}) - \varepsilon)\mathcal{H}^{d-1}(nA)] \;\geq\; C_3(\varepsilon, F, d) \;>\; 0\,.$$

En fait, l'idée intuitive qui nous fait comprendre que les grandes déviations par en dessous sont d'ordre surfacique nous donne aussi la clef pour les étudier : il n'est pas nécessaire de comprendre comment le courant circule à travers tout le cylindre, il suffit de regarder comment se comporte la couche des arêtes qui limitent le flux maximal, c'est-à-dire un ensemble de coupure de capacité minimale dans le cylindre, qu'il s'agisse de l'étude de $\phi(nA, h(n))$ ou $\tau(nA, h(n))$. En effet, ce sont ces arêtes qui vont jouer un rôle déterminant dans les déviations inférieures pour le flux maximal. Grâce au travail de Zhang [**59**], sous une condition de moment exponentielle pour la loi des capacités des arêtes, nous avons un contrôle sur le nombre d'arêtes dans un tel ensemble de coupure : avec très grande probabilité, ce nombre est de l'ordre de $\mathcal{H}^{d-1}(nA)$, la surface de la base du cylindre. Nous pouvons alors utiliser un résultat de concentration de la mesure tiré de [**54**] pour obtenir une inégalité de concentration pour nos flux maximaux $\phi(nA, h(n))$ et $\tau(nA, h(n))$ autour de leur espérance. Dans les cas où nous savons que $\mathbb{E}(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA))$ (respectivement $\mathbb{E}(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)))$ converge, nous en déduisons que les grandes déviations par en dessous sont bien d'ordre surfacique comme prévu. C'est le cas des cylindres droits et des cylindres inclinés plats (i.e. $h(n)$ négligeable devant n) pour la variable $\phi(nA, h(n))$, et de tous les cylindres pour la variable $\tau(nA, h(n))$.

Tout l'enjeu de la preuve des principes de grandes déviations est de prouver l'existence de la fonction de taux $\mathcal{J}_{\vec{v}}$, comme dans le cas du principe de grande déviation par au-dessus pour $\phi(B(n))$ dans les cylindres droits. Grâce à la quasi sous-additivité de $\tau(nA, h(n))$, nous montrons que la limite

$$\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA) \right)$$

existe et ne dépend que de $\vec{v}$ et de λ, par une technique analogue à celle utilisée pour montrer la convergence $\mathbb{E}(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ vers une limite ne dépendant que de $\vec{v}$,. La valeur de cette limite sera, à une régularisation près, celle de $\mathcal{J}_{\vec{v}}(\lambda)$. Le terme en $1/\sqrt{n}$, qui paraît très artificiel, est un terme correctif essentiel qui permet d'obtenir une quantité sous-additive, car la famille $\tau(nA, h(n))$ est presque sous-additive mais pas tout à fait. Le principe de grande déviation satisfait par $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ est ensuite obtenu par des méthodes classiques, en utilisant les estimées de déviations supérieures prouvées dans le chapitre 3. Le principe de grande déviation satisfait par $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ dans le cas où $h(n)$ est négligeable devant n est un corollaire du précédent, car les deux suites en question sont alors exponentiellement tendues.

Dans le cas de $\phi(nA, h(n))$ dans des cylindres droits, l'enjeu de la preuve du principe de grande déviation est de montrer que $\mathcal{J}_{\vec{v}}(\lambda)$ est aussi égal à la limite :

$$\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left(\phi(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\right) .$$

Quitte à réduire un peu les dimensions du cylindre, grâce au contrôle de Zhang sur le nombre d'arêtes dans un ensemble de coupure, on peut supposer qu'un ensemble de coupure de capacité minimale ne contient pas trop d'arêtes qui intersectent le bord du cylindre, c'est-à-dire qu'on contrôle la longueur de sa trace sur le bord du cylindre. Nous imposons une condition aux bords fixée parmi l'ensemble des traces possibles et nous utilisons l'invariance du graphe par les symétries d'hyperplans parallèles aux hyperplans des coordonnées pour créer artificiellement une variable proche de $\phi(nA, h(n))$ qui soit sous-additive, comme dans les travaux de Kesten [**41**] et de Zhang [**59**]. Nous pouvons alors comparer le comportement de $\tau(nA, h(n))$ avec celui de cette variable auxiliaire. Nous contrôlons finalement le nombre de traces possibles de l'ensemble de coupure sur le bord du cylindre, ce qui permet de comparer la variable auxiliaire à $\phi(nA, h(n))$, et donc de conclure. La démonstration du principe de grande déviation par en dessous pour $(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$ est alors la même que pour $(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$.

Ce principe de grande déviation pour $\phi(nA, h(n))$ dans des cylindres droits, couplé avec les estimées de déviations supérieures pour $\phi(nA, h(n))$ dans des cylindres droits et avec le lemme de Borel-Cantelli, nous redonne le résultat de convergence de $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ vers $\nu = \nu((0, ..., 0, 1))$ énoncé dans [**59**], à la différence près que nous ne considérons que le cas où tous les côtés de la base de notre cylindre tendent vers l'infini à la même vitesse. Dans ce cas, comme nous l'avions annoncé en présentant les résultats de Kesten [**41**] et Zhang [**59**], nous avons amélioré la condition imposée à la fonction h, puisque nous demandons seulement que

$$\lim_{n\to+\infty} \frac{\log h(n)}{n^{d-1}} \,=\, 0\,.$$

Cette condition de hauteur est pertinente, comme nous l'expliquerons dans une remarque du chapitre 5.

3.3. Partie 3 - chapitre 6 : Cas de la dimension deux. Le résultat de loi des grands nombres et le principe de grande déviation par en dessous présentés ici sont le fruit d'une nouvelle collaboration avec Raphaël Rossignol.

Nous nous plaçons dans toute cette partie dans le cas $d = 2$. Comme nous l'avons expliqué précédemment, le flux maximal est en général mieux compris en dimension deux. Ceci s'explique par le fait que, via la dualité, le flux maximal correspond en fait à un temps d'atteinte dans le graphe dual, si on considère les variables aléatoires associées aux arêtes duales comme des temps de passage. Cependant, les objets d'étude "naturels" dans les deux cas ne sont pas toujours les mêmes. En particulier, il est naturel suivant notre approche de regarder le flux maximal entre le sommet et la base d'un cylindre incliné, mais le problème correspondant en termes de temps de passage n'avait pas été étudié. Nous nous concentrons donc sur cette étude dans ce chapitre. Nous obtenons en fait un résultat triple. Nous prouvons d'abord une loi des grands nombres pour $\phi(nA, h(n))/nl(A)$, où $l(A) > 0$ est la longueur du segment A, sous certaines conditions (abstraites) sur la fonction hauteur h. Puis, dans un cas où cette loi des grands nombres est vérifiée, nous prouvons que les déviations inférieures du flux maximal sont d'ordre n et nous montrons le principe de grande déviation correspondant. Finalement, nous montrons que les déviations supérieures sont d'ordre $nh(n)$. Nous utilisons la notation $\nu_\theta = \nu((\cos\theta, \sin\theta))$ pour tout angle θ. Voici les théorèmes exacts démontrés dans le chapitre 6 :

THÉORÈME 15. *On suppose que* $F(0) < 1 - p_c(2) = 1/2$ *et que* F *admet un moment d'ordre* $2+\varepsilon$ *pour un* $\varepsilon > 0$ *:*

$$\exists\, \varepsilon > 0 \qquad \int_{[0,+\infty[} x^{2+\varepsilon}\, dF(x) < \infty\,.$$

Pour toute fonction $h : \mathbb{N} \to \mathbb{R}^+$ *telle que* $\lim_{n\to\infty} h(n) = +\infty$ *et* $\lim_{n\to\infty} \log h(n)/n = 0$*, pour tout segment* A *de longueur* $l(A) > 0$ *orthogonal au vecteur de coordonnées* $(\cos\theta, \sin\theta)$ *pour* $\theta \in [0,\pi[$*, on définit*

$$\overline{\mathcal{D}} \;=\; \limsup_{n\to\infty} \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)}{nl(A)}\right)\right]$$

et

$$\underline{\mathcal{D}} \;=\; \liminf_{n\to\infty} \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)}{nl(A)}\right)\right]\,.$$

Alors on a

$$\limsup_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} \;=\; \inf\left\{\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \,|\, \widetilde{\theta} \in \overline{\mathcal{D}}\right\} \qquad \text{p.s.}$$

et

$$\liminf_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} \;=\; \inf\left\{\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \,|\, \widetilde{\theta} \in \underline{\mathcal{D}}\right\} \qquad \text{p.s.}$$

On obtient donc une condition nécessaire et suffisante (à savoir l'égalité des valeurs de la limite supérieure et de la limite inférieure) pour que $\phi(nA, h(n))/nl(A)$ converge p.s. quand n tend vers l'infini, et on connaît une expression de sa limite sous la forme d'un infimum. Lorsque cette limite existe, nous la noterons $\eta_{\theta,h}$.

COROLLAIRE 3.2. *Sous les hypothèses du théorème 15, s'il existe* $\alpha \in [0, \pi/2]$ *tel que*

$$\lim_{n\to\infty} \frac{2h(n)}{nl(A)} \;=\; \tan\alpha \;\in\; [0,+\infty]\,,$$

alors

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} \;=\; \inf\left\{\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \,|\, \widetilde{\theta} \in [\theta-\alpha, \theta+\alpha]\right\} \qquad \text{p.s.}\,.$$

THÉORÈME 16. *On suppose que* $F(0) < 1 - p_c(2) = 1/2$ *et que* F *admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) < \infty\,.$$

Soit A *un segment de longueur* $l(A) > 0$*, orthogonal au vecteur* $(\cos\theta, \sin\theta)$ *et* $h : \mathbb{N} \to \mathbb{R}^+$ *une fonction de hauteur vérifiant* $\lim_{n\to\infty} h(n) = +\infty$ *et* $\lim_{n\to\infty} \log h(n)/n = 0$ *telle que*

$$\lim_{n\to\infty} \frac{2h(n)}{nl(A)} \;=\; \tan\alpha$$

existe dans $[0,+\infty]$*. Alors la suite*

$$\left(\frac{\phi(nA, h(n))}{nl(A)}\right)_{n\in\mathbb{N}}$$

satisfait un principe de grande déviation de vitesse $nl(A)$*, gouverné par la bonne fonction de taux* $\mathcal{K}$*. En définissant*

$$\delta_{\theta,h} \;=\; \inf\{\lambda \,|\, \mathbb{P}(t(e) \leq \lambda) > 0\} \times \inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} \frac{|\cos\theta| + |\sin\theta|}{\cos(\widetilde{\theta}-\theta)}\,,$$

nous pouvons énoncer les propriétés suivantes vérifiées par la fonction $\mathcal{K}$ *: elle est infinie sur* $[0, \delta_{\theta,h}[\cup]\eta_{\theta,h}, +\infty[$*, finie sur* $]\delta_{\theta,h}, \eta_{\theta,h}]$*, strictement positive sur* $[\delta_{\theta,h}, \eta_{\theta,h}[$ *si* $\delta_{\theta,h} < \eta_{\theta,h}$*, nulle* *en $\eta_{\theta,h}$ et strictement décroissante là où elle est finie, i.e., si $\mathcal{K}(\lambda) < \infty$, alors pour tout $\varepsilon > 0$ on a $\mathcal{K}(\lambda - \varepsilon) > \mathcal{K}(\lambda)$.*

La fonction $\mathcal{K}$ (qui dépend de θ et h) est définie dans le chapitre 6 à l'aide d'une optimisation sur les fonctions $\mathcal{I}_{\vec{v}}$ définies dans le chapitre 5. L'allure de $\mathcal{K}$ est donnée dans la figure 9, son comportement est moins bien connu que celui de ψ et $\mathcal{J}_{\vec{v}}$, en particulier nous ne savons pas si elle est convexe ou continue.

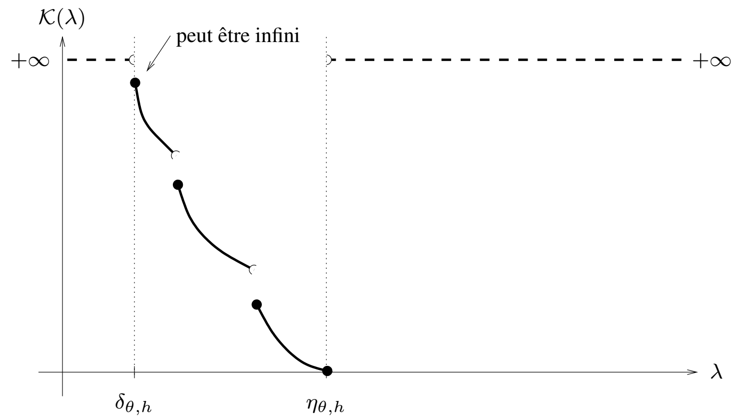

FIG. 9. Allure de la fonction de taux $\mathcal{K}$.

THÉORÈME 17. *On suppose que $F(0) < 1 - p_c(2) = 1/2$ et que F admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, dF(x) < \infty\,.$$

Soit A un segment de longueur $l(A) > 0$, orthogonal au vecteur $(\cos\theta, \sin\theta)$ et $h : \mathbb{N} \to \mathbb{R}^+$ une fonction de hauteur vérifiant $\lim_{n\to\infty} h(n) = +\infty$ et $\lim_{n\to\infty} \log h(n)/n = 0$ telle que

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} \;=\; \eta_{\theta,h}$$

existe p.s. Alors pour tout $\lambda > \eta_{\theta,h}$, on a

$$\liminf_{n\to\infty} \frac{-1}{nl(A)h(n)} \log \mathbb{P}[\phi(nA, h(n)) \geq \lambda nl(A)] \;>\; 0\,.$$

Bien qu'il ne soit pas satisfaisant de n'obtenir des résultats qu'en dimension deux, cette étude a le mérite de montrer le lien et les différences entre le comportement de $\phi(nA, h(n))$ et le comportement de $\tau(nA, h(n))$ en fonction du comportement asymptotique de la hauteur $h(n)$ du cylindre par rapport à sa base : même comportement quand $h(n)/n$ tend vers 0, mais comportements manifestement différents dans des cas très simples, par exemple pour des capacités déterministes $t(e) = 1$ et des cylindres carrés $h(n) = n$ inclinés dans la direction $\theta = \pi/3$. De plus, la forme de $\eta_{\theta,h}$ est la même que celle de la limite $\mathcal{I}(A)$ du théorème 8 de Garet [**30**] vu en termes de principe variationnel comme nous l'avons expliqué dans la partie 2.3. Ici, l'expression sous la forme d'un infimum est indispensable.

La démonstration des théorèmes 15 et 16 est très spécifique à la dimension deux, car elle s'appuie de façon essentielle sur la remarque suivante : un ensemble de coupure minimal (i.e. sans arête inutile) pour $\phi(nA, h(n))$ en dimension deux s'appuie sur chacune des deux faces verticales $\mathrm{cyl}(\partial(nA), h(n))$ du cylindre exactement une fois. Une propriété similaire avait déjà été utilisée par Garet dans [**30**]. Cela vient du passage au graphe dual : un ensemble de coupure correspond alors à un chemin entre les deux faces verticales de la boîte duale, et quitte à enlever des arêtes inutiles on sait qu'on peut supposer que ce chemin ne touche les faces verticales qu'en ses deux extrémités. Nous pouvons donc décrire les bords d'un ensemble de coupure par deux points ou encore un point (sur un des bords) et une direction (la direction dans laquelle est le deuxième point vu du premier), et nous notons génériquement κ une telle condition aux bords. Un ensemble de coupure dans le cylindre avec condition aux bords κ a grossièrement le même comportement que la variable τ dans un cylindre qui serait incliné dans la direction donnée par κ. Par ailleurs, plus la direction donnée par κ est différente de l'orientation du cylindre initial, plus un ensemble de coupure de conditions aux bords κ va devoir contenir d'arêtes : si le cylindre initial a une base de longueur $nl(A)$ et est orienté dans la direction donnée par θ, un ensemble de coupure de condition aux bords κ définissant une direction $\widetilde{\theta}$ contiendra au moins $nl(A)/\cos(\widetilde{\theta}-\theta)$ arêtes. L'ensemble de coupure de capacité minimale typique pour $\phi(nA, h(n))$ va donc être orienté dans une direction $\widetilde{\theta}$ qui minimise le rapport du flux asymptotique par unité de surface dans la direction $\widetilde{\theta}$, à savoir $\nu((\cos\widetilde{\theta}, \sin\widetilde{\theta}))$, divisé par $\cos(\widetilde{\theta}-\theta)$. Bien sûr, cette optimisation ne peut se faire que parmi les conditions aux bords possibles dans le cylindre. Celles-ci sont parfaitement décrites grâce au rapport $h(n)/n$. Suivant le comportement asymptotique de $h(n)/n$, plusieurs comportements possibles de $\phi(nA, h(n))/(nl(A))$ sont observables : si $h(n)/n$ tend vers 0, les seules conditions aux bords possibles asymptotiquement sont celles qui sont orientées dans la même direction que le cylindre (i.e. $\widetilde{\theta}=\theta$), donc $\phi(nA, h(n))/(nl(A))$ a le même comportement asymptotique que la variable $\tau(nA, h(n))/(nl(A))$. Si $h(n)/n$ tend vers l'infini, toutes les conditions aux bords sont possibles dans le cylindre, et l'ensemble de coupure va s'orienter dans la direction optimale pour le problème d'optimisation décrit ci-dessus. Pour des régimes intermédiaires de $h(n)/n$, on peut voir apparaître toutes les limites possibles comprises entre les deux cas extrêmes énoncés précédemment. Si $h(n)/n$ ne converge pas quand n tend vers l'infini, alors $\phi(nA, h(n))/n$ peut ne pas converger non plus dès que le minimum du problème d'optimisation décrit ci-dessus n'est pas stable quand n varie.

L'idée intuitive de la démonstration de la loi des grands nombres est expliquée ci-dessus, mais il faut aussi utiliser un argument plus technique pour la mener à bien. Grâce à un résultat de concentration tiré de [**13**] et un argument de type Borel-Cantelli, nous montrons qu'il suffit de prouver la convergence de $\mathbb{E}(\phi(nA, h(n)))/(nl(A))$ pour obtenir la convergence de $\phi(nA, h(n))/(nl(A))$. Ensuite, nous ramenons l'étude de $\mathbb{E}(\phi(nA, h(n)))$ à celle du minimum sur les conditions aux bords possibles κ de $\mathbb{E}(\phi^\kappa(nA, h(n)))$, où $\phi^\kappa(nA, h(n))$ désigne la capacité minimale d'un ensemble de coupure dans le cylindre dont les bords sont donnés par la condition κ. C'est la variable $\phi^\kappa(nA, h(n))$ que nous pouvons comparer avec la capacité d'un ensemble de coupure qui sépare le demi-cylindre inférieur du demi-cylindre supérieur dans un cylindre orienté dans la direction donnée par la condition aux bords κ, pour montrer que le comportement de ces deux variables est grossièrement le même. La démonstration du théorème 16 utilise les mêmes techniques que la précédente ainsi que des méthodes classiques de grandes déviations.

La démonstration du théorème 17 est quant à elle calquée sur celles de la partie 1 de la thèse. La seule adaptation à faire ici est la prise en compte du fait qu'il existe une direction privilégiée pour les ensembles de coupure dans le cylindre, i.e., la direction $\widetilde{\theta}$ qui optimise le rapport $\nu((\cos\widetilde{\theta}, \sin\widetilde{\theta}))/\cos(\widetilde{\theta}-\theta)$ dans la loi des grands nombres. C'est donc dans cette direction que nous divisons notre cylindre en différentes couches pour obtenir des sous-cylindres de taille mésoscopique. Cependant, nous ne sommes pas capables d'adapter les techniques utilisées dans les cylindres droits - même dans le cas de la dimension deux - pour obtenir un principe de grande déviation par au-dessus pour $\phi(nA, h(n))$ dans des cylindres inclinés.

3.4. Partie 4 - chapitre 7 : Flux maximal dans un domaine de $\mathbb{R}^d$. Le travail présenté dans toute cette section résulte d'une collaboration avec Raphaël Cerf.

Nous nous plaçons à nouveau dans le cas général $d \geq 2$. Nous souhaitons à présent étudier le flux maximal dans un ouvert connexe borné Ω de $\mathbb{R}^d$, de bord Γ de classe $\mathcal{C}^1$ par morceaux, entre des sous-ensembles ouverts disjoints de Γ notés Γ^1 et Γ^2. Nous avons expliqué précédemment pourquoi nous considérions un domaine Ω fixé et le graphe renormalisé $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$ de pas $1/n$ dans cette étude, et nous avons défini une approximation discrète du système, que nous avons notée $(\Omega_n, \Gamma_n, \Gamma^1_n, \Gamma^2_n)$. Soit donc ϕ_n le flux maximal entre Γ^1_n et Γ^2_n dans Ω_n pour le modèle standard de percolation de premier passage dans $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$. Nous prouvons dans ce chapitre que sous certaines conditions sur le système $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ et sur F, le flux renormalisé ϕ_n/n^{d-1} converge presque sûrement vers une constante strictement positive qui dépend du système et de F. De plus, ses déviations inférieures sont d'ordre surfacique et ses déviations supérieures sont d'ordre volumique. Nous devons introduire quelques définitions pour énoncer les théorèmes. Si $\mathcal{F} \subset \mathbb{R}^d$, le périmètre de $\mathcal{F}$ dans Ω est par définition égal à

$$\sup \left\{ \int_{\mathcal{F}} \operatorname{div} f(x) d\mathcal{L}^d(x) \,,\, f \in \mathcal{C}^\infty_c(\Omega, B(0,1)) \right\}$$

où $\mathcal{C}^\infty_c(\Omega, B(0,1))$ désigne l'ensemble des fonction de classe $\mathcal{C}^\infty$ et à support compact inclus dans Ω et à valeurs dans $B(0,1)$, la boule de centre 0 et de rayon 1 dans $\mathbb{R}^d$. Pour tout ensemble $\mathcal{F} \subset \mathbb{R}^d$, on note $\partial \mathcal{F}$ la frontière de $\mathcal{F}$, et $\partial^* \mathcal{F}$ sa frontière réduite (voir [**19**] pour une définition précise de ces termes). En tout point x de $\partial^* \mathcal{F}$, l'ensemble $\mathcal{F}$ admet un vecteur extérieur orthogonal unitaire $\vec{v}_{\mathcal{F}}(x)$ au sens donné dans la théorie de la mesure (voir toujours [**19**]). Pour tout $\mathcal{F} \subset \mathbb{R}^d$ de périmètre fini dans Ω, on définit

$$\begin{aligned} \mathcal{I}_\Omega(\mathcal{F}) \,=\, & \int_{\partial^* \mathcal{F} \cap \Omega} \nu(\vec{v}_{\mathcal{F}}(x)) d\mathcal{H}^{d-1}(x) + \int_{\Gamma^2 \cap \partial^* (\mathcal{F} \cap \Omega)} \nu(\vec{v}_{\mathcal{F}}(x)) d\mathcal{H}^{d-1}(x) \\ & + \int_{\Gamma^1 \cap \partial^* (\Omega \smallsetminus \mathcal{F})} \nu(\vec{v}_\Omega(x)) d\mathcal{H}^{d-1}(x) \,. \end{aligned}$$

Si le périmètre de $\mathcal{F}$ dans Ω est infini, on pose $\mathcal{I}_\Omega(\mathcal{F}) = +\infty$. On dit qu'un ensemble est polyédral si son bord est inclus dans une union finie d'hyperplans. On dit que deux hypersurfaces S_1 et S_2 de classe $\mathcal{C}^1$ par morceaux sont transverses si, en tout point x de leur intersections, aucun des vecteurs orthogonaux aux hypersufaces $\mathcal{C}^1$ dans lesquelles est inclus S_1 en x n'est colinéaire à l'un des vecteurs orthogonaux aux hypersufaces $\mathcal{C}^1$ dans lesquelles est inclus S_2 en x. On note $\overset{\circ}{\mathcal{F}}$ l'intérieur de $\mathcal{F} \subset \mathbb{R}^d$. Nous pouvons maintenant énoncer précisément les résultats du chapitre 7 :

THÉORÈME 18. *On suppose que $F(0) < 1 - p_c(d)$ et que F admet un moment exponentiel, i.e.,*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) < \infty \,.$$

On définit la constante $\phi_\Omega < \infty$ par

$$\phi_\Omega \,=\, \inf \left\{ \mathcal{I}_\Omega(\mathcal{F}) \,|\, \mathcal{F} \subset \mathbb{R}^d \right\} \,.$$

Alors pour tout $\lambda < \phi_\Omega$ on a

$$\limsup_{n \to \infty} \frac{1}{n^{d-1}} \log \mathbb{P}[\phi_n \leq \lambda n^{d-1}] \,<\, 0 \,.$$

THÉORÈME 19. *On suppose que $F(0) < 1 - p_c(d)$ et que F admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) < \infty \,.$$

On suppose que la distance euclidienne entre Γ^1 et Γ^2 est strictement positive. On définit la constante $\widetilde{\phi_\Omega} < \infty$ par

$$\widetilde{\phi_\Omega} \,=\, \inf \left\{ \mathcal{I}_\Omega(P) \,\middle|\, \begin{array}{c} P \subset \mathbb{R}^d \,,\, P \text{ est polyédral} \,,\, \partial P \text{ est transverse à } \Gamma \\ \overline{\Gamma^1} \subset \overset{\circ}{P} \,,\, \overline{\Gamma^2} \subset \overset{\circ}{\widehat{\mathbb{R}^d \smallsetminus P}} \end{array} \right\} .$$

Alors pour tout $\lambda > \widetilde{\phi_\Omega}$ on a

$$\limsup_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}[\phi_n \geq \lambda n^{d-1}] \,<\, 0 \,.$$

THÉORÈME 20. *On suppose que Ω est un domaine de Lipschitz, et que Γ est inclus dans une union finie d'hypersurfaces de classe $\mathcal{C}^1$ qui sont deux à deux transverses. On suppose aussi que Γ^1 et Γ^2 sont des ouverts dans Γ dont le bord relatif dans Γ est de mesure $\mathcal{H}^{d-1}$ nulle, et que la distance euclidienne entre Γ^1 est Γ^2 est strictement positive. Alors*

$$\phi_\Omega \,=\, \widetilde{\phi_\Omega} \,.$$

Si de plus $F(0) < 1 - p_c(d)$ et F admet un moment d'ordre 1 :

$$\int_{[0,+\infty[} x \, dF(x) \,<\, \infty \,,$$

cette constante est strictement positive.

On déduit immédiatement des trois théorèmes précédents le corollaire suivant :

COROLLAIRE 3.3. *On suppose que $F(0) < 1 - p_c(d)$ et que F admet un moment exponentiel :*

$$\exists\, \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, dF(x) < \infty \,.$$

Si le système $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ satisfait toutes les hypothèses demandées dans le théorème 20, alors

$$\lim_{n\to\infty} \frac{\phi_n}{n^{d-1}} \,=\, \phi_\Omega \qquad \textit{p.s.} \,,$$

où

$$\phi_\Omega \,=\, \inf \{ \mathcal{I}_\Omega(F) \,|\, F \subset \Omega \} \,\subset]0, \infty[\,.$$

Soulignons le fait que les résultats qui sont obtenus ici sont donc valables dans des cylindres inclinés, c'est-à-dire que nous avons montré la convergence p.s. de la variable $\phi(nA, h(n))$ dans un cylindre incliné dans le modèle de percolation de premier passage sur le graphe $(\mathbb{Z}^d, \mathbb{E}^d)$ dans le cas particulier où $h(n)/n$ est constant. Notre approche a été ici d'utiliser des outils géométriques pour ramener notre étude à celle - déjà en partie menée - de flux maximaux dans des cylindres inclinés.

Les constantes ϕ_Ω et $\widetilde{\phi_\Omega}$ ont des expressions très similaires, et sous les conditions du théorème 20 elles sont égales. En fait, pour tout $\mathcal{F} \subset \mathbb{R}^d$, l'ensemble

$$(\partial \mathcal{F} \cap \Omega) \cup (\Gamma^2 \cap \partial(\mathcal{F} \cap \Omega)) \cup (\Gamma^1 \cap \partial(\Omega \smallsetminus \mathcal{F}))$$

est une surface de coupure continue entre Γ^1 et Γ^2 dans $\overline{\Omega}$, dans le sens où tout chemin reliant un point de Γ^1 à Γ^2 qui est inclus dans $\overline{\Omega}$ intersecte nécessairement cet ensemble. Dans le cas où

$$\overline{\Gamma^1} \,\subset\, \overset{\circ}{\mathcal{F}} \qquad \text{et} \qquad \overline{\Gamma^2} \,\subset\, \overset{\circ}{\widehat{\mathbb{R}^d \smallsetminus \mathcal{F}}}$$

cet ensemble est plus simplement égal à $\partial \mathcal{F} \cap \Omega$. Par ailleurs, $\nu(\vec{v})$ est la valeur moyenne du flux qui peut traverser une surface unitaire dans la direction $\vec{v}$ par seconde, donc $\mathcal{I}_\Omega(\mathcal{F})$ peut s'interpréter comme la capacité de la surface de coupure continue décrite ci-dessus (et définie par $\mathcal{F}$). Les constantes ϕ_Ω et $\widetilde{\phi_\Omega}$ sont alors les capacités minimales que peuvent avoir des surfaces de coupures qui appartiennent à certaines familles de telles surfaces. Nous retrouvons ici la même expression pour $\mathcal{I}_\Omega(\mathcal{F})$ que dans le résultat de Garet [**30**], à ceci près que notre problème présente des bords $\Gamma, \Gamma^1, \Gamma^2$ dont nous devons tenir compte, et nous obtenons un principe variationnel en dimension d.

Intéressons-nous dans un premier temps à la démonstration du théorème 18. Pour arriver à nos fins, nous allons tout d'abord essayer de localiser un ensemble de coupure de capacité minimale dans Ω. Nous utilisons le contrôle de Zhang [**59**] sur le nombre d'arêtes dans un ensemble de coupure minimal pour plonger la surface de coupure minimale dans un ensemble compact de surfaces, duquel on peut extraire un recouvrement fini. Nous n'avons donc qu'un nombre fini de voisinages de surfaces à observer pour être sûrs d'y trouver notre surface de coupure minimale. Ensuite, nous utilisons le théorème de recouvrement de Vitali pour recouvrir chacune de ces surfaces par des boules suffisamment petites pour que la surface soit localement presque plate. Dans chacune des boules qui recouvrent la surface dont est proche notre surface de coupure, celle-ci est donc presque plate aussi, et après avoir réussi à aplanir ses irrégularités, nous pouvons voir notre surface de coupure intersectée avec la boule comme une surface de coupure dans un petit cylindre incliné dans cette boule. Si le flux ϕ_n est plus petit que $\phi_\Omega n^{d-1}$, cela implique que le flux maximal dans un des petits cylindres obtenus est anormalement petit. Nous pouvons donc finalement utiliser les estimées de déviations inférieures pour $\tau(A,h)$ dans un cylindre incliné $\mathrm{cyl}(A,h)$, obtenues dans le chapitre 5, pour conclure la preuve du théorème 18.

Regardons à présent le théorème 19. Nous remarquons tout de suite que la condition

$$d(\Gamma^1,\Gamma^2) > 0$$

est pertinente pour obtenir des déviations supérieures d'ordre volumique, puisque nous montrons dans le chapitre 2 que les déviations supérieures de la variable τ, pour laquelle cette condition n'est pas satisfaite, ne sont pas nécessairement de cet ordre. Le plan de la démonstration est le suivant. Nous montrons d'abord que $\widetilde{\phi_\Omega}$ est bien fini, en construisant un ensemble polyédral ayant les propriétés demandées dans la définition de cette constante. Nous considérons ensuite un ensemble polyédral P satisfaisant également ces propriétés et dont la capacité est quasiment minimale, c'est-à-dire que $\widetilde{\phi_\Omega}$ est très proche de $\mathcal{I}_\Omega(P)$. Nous recouvrons le bord ∂P de cet ensemble dans un voisinage de Ω par des hyperrectangles, à l'exception d'une petite partie de la surface de mesure $\mathcal{H}^{d-1}$ inférieure à ε, et nous considérons des cylindres ayant pour base ces hyperrectangles. Si ϕ_n est plus grand que $\widetilde{\phi_\Omega} n^{d-1}$, cela implique que soit il existe au moins un cylindre dans lequel circule un flux anormalement grand, soit le flux qui traverse la petite surface oubliée dans le recouvrement par les cylindres est très grand. Nous montrons ainsi que la probabilité que ϕ_n soit plus grand que $\widetilde{\phi_\Omega} n^{d-1}$ tend vers 0. Pour obtenir la vitesse de décroissance annoncée pour les déviations supérieures, il faut d'une part utiliser les estimées de déviations supérieures pour $\phi(A,h)$ par rapport à $\nu(\vec{v})$ dans les cylindres inclinés obtenues au chapitre 3, et d'autre part effectuer un travail supplémentaire d'optimisation en ce qui concerne le choix des arêtes qui servent à relier les cylindres entre eux pour former un ensemble de coupure dans Ω_n tout entier.

Le théorème 20 est de nature purement géométrique. Il s'agit de montrer que l'infimum des capacités des surfaces de coupure peut être approché par des surfaces polyédrales, transverses au bord de Ω, et qui ne se collent pas à Γ^1 et Γ^2. Cette étude peut paraître artificielle, mais elle est nécessaire car les objets limites naturels ne sont pas les mêmes dans l'étude des déviations inférieures et dans celle des déviations supérieures. L'approximation polyédrale était d'ailleurs aussi utilisée par Garet dans [**30**], mais elle est ici beaucoup plus difficile à montrer car nous nous plaçons en dimension $d \geq 2$, et dans un sous-ensemble Ω de $\mathbb{R}^d$ qui a des conditions aux bords, les Γ^i pour $i = 1, 2$, et ce sont ces conditions au bords qui rendent l'étude géométrique si difficile. Il s'y ajoute finalement la démonstration de la stricte positivité de ϕ_Ω, qui utilise de la géométrie différentielle.

4. Questions ouvertes

Cette thèse est loin de répondre à toutes les questions que l'on peut se poser sur le comportement du flux maximal dans le modèle de percolation de premier passage. Qu'en est-il de cylindres, ou plus généralement de domaines, dont les dimensions tendent vers l'infini de façon anisotrope ? Peut-on montrer un principe de grande déviation par au-dessus pour le flux maximal ϕ dans des cylindres inclinés, ou pour le flux maximal τ dans ces cylindres ? Qu'en est-il d'éventuels principes de grande déviation pour le flux maximal à travers un domaine de $\mathbb{R}^d$? Peut-on localiser l'ensemble de coupure correspondant à un flux maximal, et décrire le courant qui le réalise ? Quelle influence une inhomogénéité spatiale de la loi des capacités des arêtes aurait-elle ? Quel serait le comportement de ϕ_Ω sous l'effet d'une déformation du domaine Ω ? Voici quelques pistes qui méritent probablement d'être explorées à l'avenir.

Part 1

Upper large deviations for maximal flows in cylinders

CHAPTER 2

Upper large deviations for the maximal flow from the top to the bottom of a straight cylinder

We consider the standard first passage percolation in $\mathbb{Z}^d$ for $d \geq 2$ and we denote by $\phi_{n^{d-1},h(n)}$ the maximal flow through the cylinder $]0,n]^{d-1}\times]0,h(n)]$ from its bottom to its top. Kesten proved a law of large numbers for the maximal flow in dimension three: under some assumptions, $\phi_{n^{d-1},h(n)}/n^{d-1}$ converges towards a constant ν. We look now at the probability that $\phi_{n^{d-1},h(n)}/n^{d-1}$ is greater than $\nu+\varepsilon$ for some $\varepsilon > 0$, and we show under some assumptions that this probability decays exponentially fast with the volume $n^{d-1}h(n)$ of the cylinder. Moreover, we prove a large deviation principle for the sequence $(\phi_{n^{d-1},h(n)}/n^{d-1}, n \in \mathbb{N})$.

Contents

1. Definitions and main results

We will use generally the notations introduced in [**40**] and [**41**] but some changes will be done, for example to obtain independent objects. Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$ the set of all the pairs of nearest neighbours for the standard L^1 norm. With each edge e in $\mathbb{E}^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}^d)$ is independent and identically distributed, with a common distribution function F. More formally, we take the product measure $\mathbb{P}$ on $\Omega = \prod_{e\in\mathbb{E}^d}[0,\infty[$, and we write its expectation $\mathbb{E}$. We interpret $t(e)$ as the capacity of the edge e; it means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time. For a given realisation $(t(e), e \in \mathbb{E}^d)$ we denote by $\phi_{\vec{k},m} = \phi_B$ the maximal flow through the box

$$B(\vec{k},m) \,=\, \prod_{i=1}^{d-1}]0,k_i]\times]0,m]\,,$$

where $\vec{k} = (k_1, ..., k_{d-1}) \in \mathbb{Z}^{d-1}$, from its bottom

$$F_0 = \prod_{i=1}^{d-1}]0, k_i] \times \{0\}$$

to its top

$$F_m = \prod_{i=1}^{d-1}]0, k_i] \times \{m\}.$$

Let us define this quantity properly. We recall that $\mathbb{E}^d$ is the set of the edges of the graph. An edge $e \in \mathbb{E}^d$ can be written $e = \langle x, y \rangle$, where x, $y \in \mathbb{Z}^d$ are the endpoints of e. The edges of $\mathbb{E}^d$ are unoriented, hence $\langle x, y \rangle = \langle y, x \rangle$. We will say that $e = \langle x, y \rangle$ is included in a subset A of $\mathbb{R}^d$ ($e \subset A$) if the segment joining x to y (except possibly its extremities) is included in A. Now we define $\widetilde{\mathbb{E}}^d$ as the set of all the oriented edges, i.e., an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}^d$ is an ordered pair of vertices. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}^d$ by $\langle\langle x, y \rangle\rangle$, where $x, y \in \mathbb{Z}^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from x towards y. We consider now the set $\mathcal{S}$ of all pairs of functions (g, o), with $g : \mathbb{E}^d \to \mathbb{R}^+$ and $o : \mathbb{E}^d \to \widetilde{\mathbb{E}}^d$ such that $o(\langle x, y \rangle) \in \{\langle\langle x, y \rangle\rangle, \langle\langle y, x \rangle\rangle\}$, satisfying

- for each edge e in B we have

$$0 \leq g(e) \leq t(e),$$

- for each vertex v in $B \smallsetminus F_m$ (remember that $F_0 \cap B = \emptyset$) we have

$$\sum_{e \in B \,:\, o(e) = \langle\langle v, \cdot \rangle\rangle} g(e) = \sum_{e \in B \,:\, o(e) = \langle\langle \cdot, v \rangle\rangle} g(e).$$

A couple $(g, o) \in \mathcal{S}$ is a possible stream in B: $g(e)$ is the amount of fluid that goes through the edge e, and $o(e)$ gives the direction in which the fluid goes through e. The first condition on (g, o) expresses only the fact that the amount of fluid that can go through an edge is bounded by its capacity. The second one is a balance equation: it means that there is no loss of fluid in the cylinder. With each possible stream we associate the corresponding flow

$$flow(g, o) = \sum_{u \in B \smallsetminus F_m \,,\, v \in F_m \,:\, \langle u, v \rangle \in \mathbb{E}^d} g(\langle u, v \rangle) \mathbb{1}_{o(\langle u, v \rangle) = \langle\langle u, v \rangle\rangle} - g(\langle u, v \rangle) \mathbb{1}_{o(\langle u, v \rangle) = \langle\langle v, u \rangle\rangle}.$$

This is the amount of fluid that crosses the cylinder B if the fluid respects the stream (g, o). The maximal flow through the cylinder B from its bottom to its top is the supremum of this quantity over all possible choices of streams in $\mathcal{S}$

$$\phi_B = \phi_{\vec{k}, m} = \sup \{flow(g, o) \,:\, (g, o) \in \mathcal{S}\}.$$

If $\phi_B = flow(g, o)$ we say that the stream (g, o) realizes the flow ϕ_B.

Kesten proved in 1987 the following law of large numbers for the maximal flow in dimension 3 (see [**41**]):

THEOREM 1. *We consider a cylinder $B((k, l), m)$ such that $\lim_{k \geq l \to \infty} m(k, l) = \infty$ in such a way that for some $\delta > 0$ we have*

$$\lim_{k \geq l \to \infty} \frac{\ln m(k, l)}{k^{1-\delta}} = 0.$$

There exists a positive p_0 with the following property: If F satisfies $F(0) < p_0$ and $\int_{[0, \infty[} e^{\theta x} dF(x)$ is finite for some positive θ, then there exists a constant $\nu(F) < \infty$ such that

$$\lim_{k, l \to \infty} \frac{\phi_{(k,l),m}}{kl} = \nu \qquad \textit{with probability one and in } L^1.$$

Actually, the constant $\nu(F)$ is defined as the limit of another object under weaker assumptions on F (see [**41**] and (2.1) in the next section), and we rely on this definition to state the following result. We are now interested in the deviations of the rescaled flow from its typical behaviour. We will show two results in dimensions $d \geq 2$. The first one states the existence of a limit, and some of its properties.

THEOREM 2. *Let $\phi_{(n,\dots,n),h(n)}$ be the maximal flow through the cylinder $B((n,...,n),h(n))$, where the function $h : \mathbb{N} \to \mathbb{N}$ satisfies*

$$\lim_{n\to\infty} \frac{h(n)}{\ln n} = \infty\,.$$

For every λ in $\mathbb{R}^+$, the limit

$$\psi(\lambda) = \lim_{n\to\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{(n,\dots,n),h(n)} \geq \lambda n^{d-1}\right]$$

exists and is independent of h. Moreover ψ is convex on $\mathbb{R}^+$, finite and continuous on the set $\{\,\lambda \mid F([\lambda,+\infty[) > 0\,\}$. If $\int_{[0,+\infty[} x dF(x)$ is finite, then ψ vanishes on $[0,\nu]$, where ν is defined in (2.1). If $\int_{[0,+\infty[} e^{\theta x} dF(x)$ is finite for some positive θ, then ψ is positive on $]\nu,+\infty[$.

We say that a sequence $(X_n, n \in \mathbb{N})$ of random variables with values in $D \subset \mathbb{R}$ satisfies a large deviation principle with speed $v(n)$ and governed by the rate function $\mathcal{I}$ if and only if

- for any closed subset $\mathcal{F} \subset D$, we have
$$\limsup_{n\to\infty} \frac{1}{v(n)} \ln \mathbb{P}\left[X_n \in \mathcal{F}\right] \leq -\inf_{\mathcal{F}} \mathcal{I}\,,$$
- for any open subset $\mathcal{O} \subset D$, we have
$$\liminf_{n\to\infty} \frac{1}{v(n)} \ln \mathbb{P}\left[X_n \in \mathcal{O}\right] \geq -\inf_{\mathcal{O}} \mathcal{I}\,.$$

Now, with the help of the function ψ, we can state the following large deviation principle for the rescaled flow:

THEOREM 3. *Let $h : \mathbb{N} \to \mathbb{N}$ be such that*

$$\lim_{n\to\infty} \frac{h(n)}{\ln n} = \infty\,.$$

If there exists a positive θ such that

$$\int_{[0,+\infty[} e^{\theta x} dF(x) < \infty\,,$$

then the sequence

$$\left(\frac{\phi_{(n,\dots,n),h(n)}}{n^{d-1}}\right)_{n\in\mathbb{N}}$$

satisfies a large deviation principle, with speed $n^{d-1}h(n)$, and governed by the good rate function ψ.

REMARK 1. When the capacity t of an edge is bounded, we do not need the condition

$$\lim_{n\to\infty} \frac{h(n)}{\ln n} = +\infty\,,$$

the results hold under the weaker condition

$$\lim_{n\to\infty} h(n) = +\infty\,.$$

Actually, the role of the condition $\lim_{n\to\infty} h(n)/\ln n = +\infty$ is not fully understood yet. For example, when t is equal in law to the absolute value of a Gaussian variable this condition can also be replaced by $\lim_{n\to+\infty} h(n) = +\infty$. Unfortunately we could not find satisfying sufficient conditions (in particular on the moments of the law of t) to get rid of the condition $\lim_{n\to\infty} h(n)/\ln n = +\infty$.

A special aspect of the proof of theorem 2 is the use of a discrete version of the model. Indeed, we are confronted with a combinatorial problem: we need to look at boundary conditions for streams to glue together streams in different cylinders, but when the capacity of an edge takes its values in $\mathbb{R}^+$ we cannot count the number of possible boundary conditions. Our strategy is to consider a discrete approximation of the capacity of the edges and the corresponding maximal flow. We work with these objects. To handle the boundary conditions, we use a technique introduced by Chow and Zhang [**21**]. We finally compare the real maximal flow to this approximation.

2. Max-flow min-cut theorem

It is difficult to work with the expression of the maximal flow that we have seen in the previous part, this is the reason why we will use the max-flow min-cut theorem to express the maximal flow differently. First we need some definitions. A path on a graph ($\mathbb{Z}^d$ for example) from v_0 to v_n is a sequence $(v_0, e_1, v_1, ..., e_n, v_n)$ of vertices $v_0, ..., v_n$ alternating with edges $e_1, ..., e_n$ such that v_{i-1} and v_i are neighbours in the graph, joined by the edge e_i, for i in $\{1, ..., n\}$. Two paths are said disjoint if they have no common edge. A set E of edges of $B(\vec{k}, m)$ is said to separate F_0 from F_m in $B(\vec{k}, m)$ if there is no path from F_0 to F_m in $B(\vec{k}, m) \smallsetminus E$. We call E an (F_0, F_m)-cut if E separates F_0 from F_m in $B(\vec{k}, m)$ and if no proper subset of E does. With each set E of edges we associate the variable

$$V(E) \,=\, \sum_{e \in E} t(e)\,.$$

The max-flow min-cut theorem (see [**12**]) states that

$$\phi_B \,=\, \min\{\, V(E) \,|\, E \textit{ is an } (F_0, F_m) - \textit{cut in } B\,\}\,.$$

REMARK 2. In the special case where $t(e)$ belongs to $\{0, 1\}$, let us consider the graph obtained from the initial graph (not necessarily $\mathbb{Z}^d$) by removing all the edges e with $t(e) = 0$. Menger's theorem (see [**12**]) states that the minimal number of edges in $B(\vec{k}, m)$ that have to be removed from this graph to disconnect F_0 from F_m is exactly the maximal number of disjoint paths that connect F_0 to F_m. By the max-flow min-cut theorem, it follows immediately that the maximal flow in the initial graph through B from F_0 to F_m is exactly the maximal number of disjoint open paths from F_0 to F_m, where a path is open if and only if the capacity of all its edges is one. Such a set of ϕ_B disjoint open paths from F_0 to F_m corresponds obviously to a stream (g,o):

- $g(e) \,=\, \begin{cases} 1 & if\ e\ belongs\ to\ one\ of\ these\ paths \\ 0 & otherwise\,, \end{cases}$
- $o(e) \,=\, \begin{cases} \langle\langle x, y\rangle\rangle & if\ e = \langle x, y\rangle\ is\ crossed\ from\ x\ to\ y\ by\ one\ of\ these\ paths \\ \langle\langle y, x\rangle\rangle & if\ e = \langle x, y\rangle\ is\ crossed\ from\ y\ to\ x\ by\ one\ of\ these\ paths \\ \hat{o}(e) & otherwise\,, \end{cases}$

where $\hat{o}$ is some determined orientation ($\hat{o}(\langle x, y\rangle) \in \{\langle\langle x, y\rangle\rangle, \langle\langle y, x\rangle\rangle\}$) which does not matter. The stream (g, o) realizes the maximal flow ϕ_B (whatever $\hat{o}$).

We come back to the general case. We will also need the definition of a cut over a hyper-rectangle. Let $S = \prod_{i=1}^{d-1}]k_i, l_i]$ be a hyper-rectangle, with $k_i \leq l_i$, k_i, l_i in $\mathbb{Z}$. We say that a set E of edges in $S \times \mathbb{R}$ separates $-\infty$ from $+\infty$ over S if there exists no path in $(S \times \mathbb{R}) \smallsetminus E$ from $S \times \{-N\}$ to $S \times \{+N\}$ for some $N > 0$. Similarly, we call E a cut over S if E separates $-\infty$ from $+\infty$ over S, but no proper subset of E does. Let $\partial^{in}(S \times \mathbb{R})$ be the inner vertex boundary of the cylinder $S \times \mathbb{R}$

$$\partial^{in}(S \times \mathbb{R}) \,=\, \{\, x \in S \times \mathbb{R} \,|\, \exists y \notin S \times \mathbb{R}\,,\ \langle x, y\rangle \in \mathbb{E}^d\,\}\,.$$

We define the corresponding set of edges

$$\mathbb{E}(\partial^{in}(S \times \mathbb{R})) \,=\, \{\langle x, y\rangle \,|\, x, y \in \partial^{in}(S \times \mathbb{R})\}\,.$$

We say that an edge e is vertical if $e = \langle x, x + (0, ..., 0, 1)\rangle$; e is said horizontal otherwise. We denote by $(*)$ the condition on E

$$(*) \qquad E \cap \mathbb{E}(\partial^{in}(S \times \mathbb{R})) \subset \{e \in \mathbb{E}^d \,|\, e \textit{ is vertical}\,,\ e \subset \mathbb{R}^{d-1} \times [0,1]\}\,,$$

which means in a way to say that the boundary of E is fixed on the perimeter of the rectangle $S \times \{0\}$. We define the variable τ by

$$\tau(S) \,=\, \inf\{\, V(E) \,|\, E \textit{ is a cut over } S \textit{ and } E \textit{ satisfies } (*)\,\}\,.$$

For simplicity, we denote by $\tau_{k^{d-1}}$ the variable $\tau(]0,k]^{d-1})$. If S_1, S_2 are two disjoint hyper-rectangles having a common side (so $S_1 \cup S_2$ is an hyper-rectangle too), then we have

$$\tau(S_1 \cup S_2) \,\leq\, \tau(S_1) + \tau(S_2)\,.$$

Indeed if E_1 (respectively E_2) is a cut over S_1 (respectively S_2) satisfying $(*)$ for S_1 (respectively S_2) then E_1 and E_2 are both pinned at the boundary between $S_1 \times \{0\}$ and $S_2 \times \{0\}$ because they both satisfy $(*)$, so they can be glued together and $E_1 \cup E_2$ separates $-\infty$ from $+\infty$ over $S_1 \cup S_2$. By a subadditive argument (see [**1**]), the following limit exists almost surely

$$\nu(F) \,=\, \lim_{k\to\infty} \frac{\tau_{k^{d-1}}}{k^{d-1}}\,, \tag{2.1}$$

where we know that $\nu(F)$ is a constant almost surely thanks to Kolmogorov's $0-1$ law. We will denote it by ν when no doubt about F is possible. This is the "ν" in theorems 1 and 2.

3. Proof of Theorem 2

We take $h : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} h(n) \,=\, +\infty\,.$$

We will see during the proof where we need the stronger condition

$$\lim_{n\to\infty} \frac{h(n)}{\ln n} \,=\, +\infty\,.$$

We will need to describe how the fluid goes in and out of a cylinder in order to glue together two cylinders without loosing any flow. The problem is that we need too much information to describe this precisely. The feature of the proof is to consider a discrete approximation of the capacity of the edges (see section 3.1), to work with this discrete model (sections 3.2, 3.3 and 3.4) and then to compare it to the original one (section 3.5). The method used to prove the existence of the limit was developed in [**21**]. We study then the properties of ψ as in [**19**].

3.1. Discrete version. Let $k \in \mathbb{N}$ (we will choose it later). We associate with $(t(e), e \in \mathbb{Z}^d)$ a new family of independent and identically distributed variables $(t^k(e), e \in \mathbb{Z}^d)$ by setting

$$\forall e \in \mathbb{E}^d \qquad t^k(e) \,=\, \lfloor kt(e) \rfloor \times \frac{1}{k}\,,$$

and we denote by ϕ^k the maximal flow corresponding to these new variables.

Let us consider for a brief moment the graph G obtained by replacing each edge e by p edges $\widetilde{e}_1, ..., \widetilde{e}_p$, where $p = kt^k(e)$. In this new graph the capacity of each edge is simply one. The remark 2 also holds for G: the maximal flow ϕ_B^G for G from F_0 to $F_{h(n)}$ in $B = B((n, ..., n), h(n))$ is exactly the maximal number of disjoint paths connecting F_0 to $F_{h(n)}$ in G. We have seen that we can associate with each such family of ϕ_B disjoint paths in B a stream $(\widetilde{g}, \widetilde{o})$ in G that realizes ϕ_B^G. Actually, we can always reduce to the case where $\widetilde{o}(\widetilde{e}_1) = \widetilde{o}(\widetilde{e}_2)$ if the edges $\widetilde{e}_1$ and $\widetilde{e}_2$ are replacing in G the same edge $\langle x, y\rangle \in \mathbb{E}^d$. Indeed, if for such edges $\widetilde{e}_1$ and $\widetilde{e}_2$ we have $\widetilde{g}(\widetilde{e}_1) = \widetilde{g}(\widetilde{e}_2) = 1$ and $\widetilde{o}(\widetilde{e}_1) \neq \widetilde{o}(\widetilde{e}_2)$, we know that there exists a path l_1 (respectively l_2) from F_0 to $F_{h(n)}$ going through $\widetilde{e}_1$ (respectively $\widetilde{e}_2$) and crossing this edge from x to y (respectively from y to x). We can create two new disjoint paths in G, l_a which is equal to l_1 from F_0 to x and to l_2 from x to

$F_{h(n)}$, and l_b which is equal to l_2 from F_0 to y and to l_1 from y to $F_{h(n)}$, that can replace l_1 and l_2 in the set of ϕ_B disjoint paths (see figure 1). The corresponding stream $(\widetilde{g}', \widetilde{o}')$ is equal to $(\widetilde{g}, \widetilde{o})$ except in $\widetilde{e}_1$ and $\widetilde{e}_2$ where we have $\widetilde{g}'(\widetilde{e}_1) = \widetilde{g}'(\widetilde{e}_2) = 0$ and $\widetilde{o}'(\widetilde{e}_1) = \widetilde{o}'(\widetilde{e}_2) = \hat{o}(\langle x, y\rangle)$. Then we

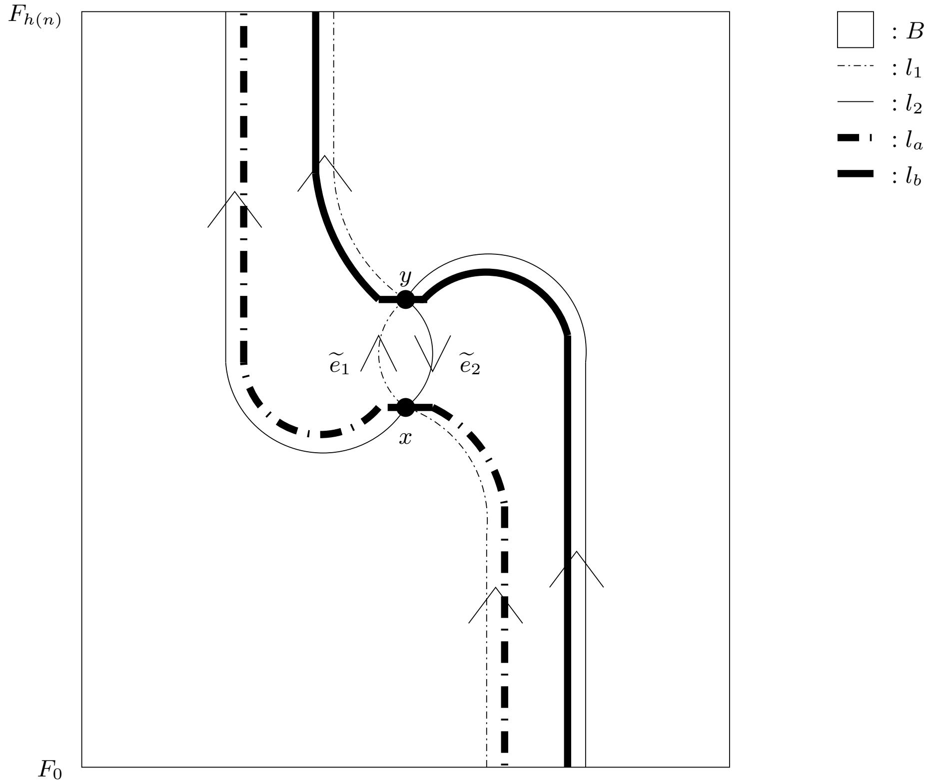

FIGURE 1. Description of paths in G

just have to deal with the case $\widetilde{g}(\widetilde{e}_1) = 1$ and $\widetilde{g}(\widetilde{e}_2) = 0$. The definition of $\hat{o}$ is arbitrary, we can change the orientation of $\hat{o}(\widetilde{e}_2)$ to have $\widetilde{o}(\widetilde{e}_2) = \hat{o}(\widetilde{e}_2) = \widetilde{o}(\widetilde{e}_1)$. We obtain thus a stream such that $\widetilde{o}(\widetilde{e}_1) = \widetilde{o}(\widetilde{e}_2)$ if the edges $\widetilde{e}_1$ and $\widetilde{e}_2$ are replacing in G the same edge $\langle x, y\rangle \in \mathbb{E}^d$. Moreover we can assume that each path of a family of ϕ_B disjoint open paths has only its first vertex in F_0 and its last vertex in $F_{h(n)}$, otherwise we can restrict the path to obtain such a path. Thanks to a good choice of $\hat{o}$, we can thus suppose that if $\widetilde{e} \subset B$ has one endpoint x in $F_{h(n)}$ (respectively F_0) and one endpoint y not in $F_{h(n)}$ (respectively F_0) then $\widetilde{o}(\widetilde{e}) = \langle\langle y, x\rangle\rangle$ (respectively $\widetilde{o}(\widetilde{e}) = \langle\langle x, y\rangle\rangle$), and if $\widetilde{e}$ has both endpoints in $F_{h(n)}$ then $\widetilde{g}(\widetilde{e}) = 0$.

Coming back to the graph $\mathbb{Z}^d$, we remark that the maximal flow ϕ_B^k between F_0 and $F_{h(n)}$ in B is equal to ϕ_B^G/k, and it can be realized by the stream (g, o) defined as follows. Let e be an edge of $\mathbb{E}^d$. If there is no edge in G associated with e, we set $g(e) = 0$ and $o(e) = \hat{o}(e)$. Otherwise, we define

$$g(e) \,=\, \sum_{\widetilde{e} \sim e} \frac{\widetilde{g}(\widetilde{e})}{k}$$

where the sum is over the edges $\widetilde{e}$ that replace e in G, and $o(e) = \widetilde{o}(\widetilde{e})$ for some edge $\widetilde{e}$ associated with e (recall that if $\widetilde{e}_1 \sim e$ and $\widetilde{e}_2 \sim e$ then $\widetilde{o}(\widetilde{e}_1) = \widetilde{o}(\widetilde{e}_2)$). We will call such a stream, built from the graph G, a discrete stream. A discrete stream has three particular properties: g takes its values in $\mathbb{N}/k$, $o(\langle x, y\rangle) = \langle\langle x, y\rangle\rangle$ as soon as we have $x \in F_0$ and $y \in B$ ($y \notin F_0$) or $y \in F_{h(n)}$ and $x \in B \setminus F_{h(n)}$, and $g(e) = 0$ if e has both endpoints in $F_{h(n)}$.

Let λ be in $\mathbb{R}^+$. For a discrete stream (g, o) and $h \in \mathbb{Z}$ we define the truncated projection of g on the vertical edges that intersect the hyper-plane $\{(x_1, ..., x_d) \in \mathbb{R}^d \,|\, x_d = h + 1/2\}$ by

$$\forall x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \qquad \pi_h^{\lambda,n}(g, x) \;=\; g\left(\langle (x, h), (x, h+1) \rangle\right) \wedge \left(\lfloor \lambda n^{d-1} \rfloor + 1 \right) \,.$$

Thanks to the properties of discrete streams, we can state the following lemma:

LEMMA 1 (Junction of two boxes). *We consider the cylinders*

$$B_1 \;=]0, n]^{d-1} \times]0, h(n)] \text{ and } B_2 \;=]0, n]^{d-1} \times]h(n), 2h(n)] \,.$$

If there exist a discrete stream (g_1, o_1) in B_1 and a discrete stream (g_2, o_2) in B_2 such that

$$flow(g_1, o_1) \;\geq\; \lambda n^{d-1} \qquad and \qquad flow(g_2, o_2) \;\geq\; \lambda n^{d-1}$$

and

$$\forall x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \qquad \pi_{h(n)-1}^{\lambda,n}(g_1, x) \;=\; \pi_{h(n)}^{\lambda,n}(g_2, x) \,,$$

then $\phi_{B_1 \cup B_2}^k \geq \lambda n^{d-1}$.

Proof :
To prove this lemma, we consider two cases:

(1) if for all $e = \langle (x, h(n) - 1), (x, h(n)) \rangle$ with $x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1}$ we have $g_1(e) \leq \lambda n^{d-1}$, then we can define the following discrete stream (g_{tot}, o_{tot}):
- $g_{tot}(e) \;=\; \begin{cases} g_1(e) & if\ e \subset B_1 \\ g_2(e) & if\ e \subset B_2 \\ 0 & otherwise\,, \end{cases}$
- $o_{tot}(e) \;=\; \begin{cases} o_1(e) & if\ e \subset B_1 \\ o_2(e) & if\ e \subset B_2 \\ \hat{o}(e) & otherwise\,, \end{cases}$

where $\hat{o}$ is still some arbitrarily determined orientation. We can check that (g_{tot}, o_{tot}) is a discrete stream thanks to the properties of (g_1, o_1) and (g_2, o_2), in particular if $e_1 = \langle (x, h(n)-1), (x, h(n)) \rangle$ and $e_2 = \langle (x, h(n)), (x, h(n)+1) \rangle$ with $x \in]0, n]^{d-1}$, we have $g(e_1) = g(e_2)$, $g(e) = 0$ for all others edges $e = \langle (x, h(n)), \cdot \rangle$, $o(e_1) = \langle\langle (x, h(n) - 1), (x, h(n)) \rangle\rangle$ and $o(e_2) = \langle\langle (x, h(n)), (x, h(n) + 1) \rangle\rangle$, hence the balance equation is satisfied. Moreover $flow(g_{tot}, o_{tot}) = flow(g_1, o_1) = flow(g_2, o_2)$ so $\phi_{B_1 \cup B_2}^k \geq \lambda n^{d-1}$.

(2) Suppose there exists an edge $e = \langle (x, h(n) - 1), (x, h(n)) \rangle$ such that $g_1(e) > \lambda n^{d-1}$. The discrete stream (g_1, o_1) corresponds to $k \times flow(g_1, o_1)$ disjoint paths from F_0 to $F_{h(n)}$ in B_1 for the modified graph G. The inequality $g_1(e) > \lambda n^{d-1}$ implies that at least $q = \lceil \lambda n^{d-1} k \rceil$ of these paths, that we will denote by $l_1, ..., l_q$, go out of B_1 through e. The equality $\pi_{h(n)-1}^{\lambda,n}(g_1, x) = \pi_{h(n)}^{\lambda,n}(g_2, x)$ implies that $g_2(f) > \lambda n^{d-1}$ where $f = \langle (x, h(n)), (x, h(n) + 1) \rangle$. By the same argument, we can find at least q disjoint paths $l'_1, ..., l'_q$ from $F_{h(n)}$ to $F_{2h(n)}$ in B_2 for G, all going in B_2 through the edge f. Now we can glue together these q paths $l_1, ..., l_q$ in B_1 with the q paths $l'_1, ..., l'_q$ in B_2 because e and f are adjacent. This way we obtain q disjoint paths from F_0 to $F_{2h(n)}$ in $B_1 \cup B_2$ for G, and by considering the corresponding discrete flow (g_{tot}, o_{tot}) in the initial graph we obtain $\phi_{B_1 \cup B_2}^k \geq flow(g_{tot}, o_{tot}) = \lceil \lambda n^{d-1} k \rceil / k$.

■

We define the boundary conditions of the discrete stream (g, o) in $B((n, ..., n), h(n))$ as

$$\begin{aligned} \Pi^{\lambda,n}(g) \;&=\; \left(\Pi_1^{\lambda,n}(g), \Pi_2^{\lambda,n}(g) \right) \\ &=\; \left(\left(\pi_0^{\lambda,n}(g, x), x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \right), \left(\pi_{h(n)-1}^{\lambda,n}(g, x), x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \right) \right) \,. \end{aligned}$$

The number $N^k_{\lambda,n}$ of possible boundary conditions for discrete streams satisfies

$$N^k_{\lambda,n} \,\leq\, \left(k\left(\lfloor \lambda n^{d-1}\rfloor + 1\right) + 1\right)^{2n^{d-1}} \,.$$

3.2. Existence of the limit for $\phi^{k_n}_{n^{d-1},h(n)}$. In this section, we will prove the existence of the limit appearing in the theorem 2 with ϕ^k instead of ϕ. We denote $\phi_{(n,\dots,n),h(n)}$ by $\phi_{n^{d-1},h(n)}$, and we define

$$\mu \,=\, \sup\{\, \lambda \,|\, F([0,\lambda[) < 1 \,\} \,.$$

We will prove the following result:

THEOREM 4. *For every pair $(h,(\chi_n, n \in \mathbb{N}))$ with $h : \mathbb{N} \to \mathbb{N}$ a function such that $\lim_{n\to+\infty} h(n) = +\infty$ and $(\chi_n)_{n\in\mathbb{N}}$ a non-decreasing sequence of integers tending to infinity, satisfying*

$$\lim_{n\to+\infty} \frac{\chi_n \ln n}{h(n)} \,=\, 0\,, \tag{2.2}$$

for every λ in $\mathbb{R}^+ \smallsetminus \{\mu\}$ (or in $\mathbb{R}^+$ if μ is infinite), the limit

$$\widetilde{\psi}(\lambda, h, (\chi_n)) \,=\, \lim_{n\to+\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{2\chi_n}_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right]$$

exists. Moreover $\widetilde{\psi}$ is independent of such a pair $(h,(\chi_n))$, i.e. if $(h,(\chi_n))$ and $(h',(\chi'_n))$ satisfy all the previous conditions, then $\widetilde{\psi}(\lambda,h,(\chi_n)) = \widetilde{\psi}(\lambda,h',(\chi'_n))$ for all λ in $\mathbb{R}^+ \smallsetminus \{\mu\}$ (or in $\mathbb{R}^+$). We will thus denote this limit by $\widetilde{\psi}(\lambda)$.

We now prove theorem 4 by considering different cases.

• $\lambda > \mu$: Then

$$\forall k \in \mathbb{N}\,,\ \forall n \in \mathbb{N} \qquad \mathbb{P}\left[\phi^k_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] \,=\, 0\,,$$

so for every sequence (χ_n) we have

$$\widetilde{\psi}(\lambda, h, (\chi_n)) \,=\, \lim_{n\to\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{2\chi_n}_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] \,=\, +\infty \,=\, \widetilde{\psi}(\lambda)\,.$$

• $\lambda < \mu$: We take $N, n \in \mathbb{N}$ with $n \leq N$ and let $N = nm + r$ be the Euclidean algorithm. We consider two functions h, $\widetilde{h} : \mathbb{N} \to \mathbb{N}$, with $\lim_{n\to\infty} h(n) = \lim_{n\to\infty} \widetilde{h}(n) = +\infty$, and let $\widetilde{h}(N) = h(n)\widetilde{m} + \widetilde{r}$ be the Euclidean algorithm. We take $k \in \mathbb{N}$ which will be chosen later. We want to compare $\phi^k_{N^{d-1},\widetilde{h}(N)}$ and $\phi^k_{n^{d-1},h(n)}$.

The idea is to divide $B((N,...,N),\widetilde{h}(N))$ into m^{d-1} boxes which are disjoint translates of $B((n,...,n),\widetilde{h}(N))$, then to cut again $B((n,...,n),\widetilde{h}(N))$ into $\widetilde{m}$ disjoint translates of the elementary box $B((n,...,n),h(n))$ and to use here the lemma of junction (see figure 2). We define two quantities that will allow us to deal with the edges belonging to the part of $\phi^k_{N^{d-1},\widetilde{h}(N)}$ that does not enter in any translate of $\phi^k_{n^{d-1},h(n)}$. On one hand, by the definition of μ, $\lambda < \mu$ implies that $F([0,\lambda]) < 1$, so there exists a positive η such that

$$F([0,\lambda+\eta]) \,<\, 1,\ i.e.,\ p(\eta) \,=\, \mathbb{P}[t(e) \geq \lambda + \eta] \,>\, 0\,.$$

It follows that there exists k_0 such that

$$\forall k \geq k_0 \qquad \mathbb{P}\left[t^k(e) \geq \lambda + \frac{\eta}{2}\right] \,\geq\, p(\eta) \,>\, 0\,.$$

On the other hand, if we define

$$\gamma_k \,=\, \max\{\, p \in \mathbb{N} \,|\, \mathbb{P}[t(e) \geq pk] > 0 \,\} \wedge \left(\lfloor \lambda k n^{d-1}\rfloor + 1\right)\,,$$

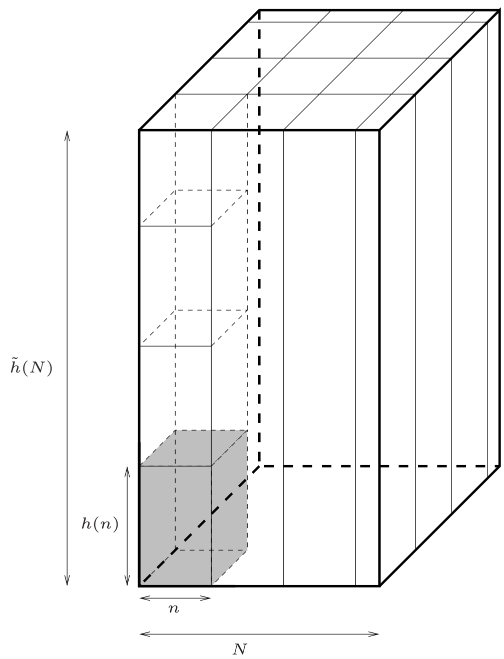

FIGURE 2. Comparison between $\phi^k_{N^{d-1},\widetilde{h}(N)}$ and $\phi^k_{n^{d-1},h(n)}$

then we have

$$p_k \,=\, \mathbb{P}\,[t(e) \geq \gamma_k k] \,>\, 0\,.$$

Let $k \geq k_0$ in $\mathbb{N}$. For $i_1, ..., i_{d-1}$ in $\{0, ..., m\}$, we define

$$B_{i_1,\dots,i_{d-1}} \,=\, \prod_{j=1}^{d-1}]i_j n, (i_j+1)n] \times]0, \widetilde{h}(N)]$$

and

$$B_{m^{d-1}+1} \,=\, B((N, ..., N), \widetilde{h}(N)) \smallsetminus \bigcup_{i_1,\dots,i_{d-1}=0}^{m-1} B_{i_1,\dots,i_{d-1}}\,.$$

REMARK 3. It is easy (and very useful) to see that if $C_i \times]0, h]$, $i = 1, 2$ are two cylinders with disjoint bases C_1, $C_2 \subset \mathbb{R}^{d-1}$ having a common side and with maximal flows ϕ_i, $i = 1, 2$, the maximal flow through $(C_1 \cup C_2) \times]0, h]$ is at least $\phi_1 + \phi_2$.

We deduce from this remark that if for every $i_1, ..., i_{d-1}$ in $\{0, ..., m-1\}$ we have $\phi^k_{B_{i_1,\dots,i_{d-1}}} \geq \lambda n^{d-1}$ and if all the vertical edges e in $B_{m^{d-1}+1}$ satisfy $t(e) \geq (\lambda + \eta)$, we have

$$\phi^k_{N^{d-1},\widetilde{h}(N)} \,\geq\, \lambda N^{d-1}\,.$$

By independence we obtain

$$\begin{aligned}
\mathbb{P}\left[\phi^k_{N^{d-1},\widetilde{h}(N)} \geq \lambda N^{d-1}\right] \,&\geq\, \prod_{i_1,\dots,i_{d-1}=0}^{m-1} \mathbb{P}\left[\phi^k_{B_{i_1,\dots,i_{d-1}}} \geq \lambda n^{d-1}\right] \times p(\eta)^{(d-1)N^{d-2}r\widetilde{h}(N)} \\
&\geq\, \mathbb{P}\left[\phi^k_{n^{d-1},\widetilde{h}(N)} \geq \lambda n^{d-1}\right]^{m^{d-1}} \times p(\eta)^{(d-1)N^{d-2}r\widetilde{h}(N)}\,.
\end{aligned}$$

We study next $\phi^k_{n^{d-1},\widetilde{h}(N)}$. We define for j in $\{0,...,(\widetilde{m}-1)\}$

$$B'_j \ =]0,n]^{d-1} \times]jh(n),(j+1)h(n)]$$

and

$$B'_{\widetilde{m}} \ = \ B((n,...,n),\widetilde{h}(N)) \smallsetminus \bigcup_{j=0}^{\widetilde{m}-1} B'_j\,.$$

The probability of the boundary conditions $\Pi \in \{0,1/k,2/k,...,(\lfloor \lambda n^{d-1} \rfloor + 1)/k\}^{2n^{d-1}}$ in B is the probability that there exists a discrete stream (g,o) in B satisfying $\Pi^{\lambda,n}(g) = \Pi$. Remember that every discrete stream (g,o) must satisfy the balance equation, so once we know that such a discrete stream exists, $flow(g,o)$ is just given by the projection of g on the vertical edges that intersect the hyper-plane $\{(x_1,...,x_d) \in \mathbb{R}^d \,|\, x_d = h(n) - 1/2\}$, so $\Pi^{\lambda,n}(g)$ contains enough information to know if $flow(g,o)$ is bigger than λn^{d-1} or not. We denote by $\Pi^k_{\lambda,n} = (\Pi^k_{\lambda,n,1}, \Pi^k_{\lambda,n,2})$ one of the boundary conditions of highest probability in $B((n,...,n),h(n))$ which corresponds to a discrete stream (g,o) such that $flow(g,o) \geq \lambda n^{d-1}$ and we define $(\Pi^k_{\lambda,n})^* = (\Pi^k_{\lambda,n,2}, \Pi^k_{\lambda,n,1})$. The model is invariant under reflections in the coordinates hyperplanes or translates of these hyperplanes, so by symmetry we have $\mathbb{P}[\Pi^k_{\lambda,n}] = \mathbb{P}[(\Pi^k_{\lambda,n})^*]$. Using the lemma of junction (lemma 1), we know that if

- we can define a discrete stream in B'_0 with boundary conditions $\Pi^k_{\lambda,n}$,
- we can define a discrete stream in B'_1 with boundary conditions $(\Pi^k_{\lambda,n})^*$,
- we can define a discrete stream in B'_2 with boundary conditions $\Pi^k_{\lambda,n}$,
- $\cdots$,
- and all the vertical edges e in $B'_{\widetilde{m}}$ satisfy $t(e) \geq \gamma_k k$,

then $\phi^k_{n^{d-1},\widetilde{h}(N)} \geq \lambda n^{d-1}$.

REMARK 4. It is not sufficient to impose here that all the vertical edges e in $B'_{\widetilde{m}}$ satisfy $t(e) \geq \lambda$, because the amount of fluid that goes out of $B'_{\widetilde{m}-1}$ at its top through one fixed edge f can exceed λ - we have no information about $\Pi^k_{\lambda,n}$ - and we cannot accept to lose fluid at the exit of f, unless it exceeds λn^{d-1}. This is the reason why we introduced γ_k.

Now by independence we obtain

$$\mathbb{P}\left[\phi^k_{n^{d-1},\widetilde{h}(N)} \geq \lambda n^{d-1}\right] \ \geq \ \mathbb{P}\left[\Pi^k_{\lambda,n}\right]^{\widetilde{m}} \times p_k^{n^{d-1}\widetilde{r}}\,, \tag{2.3}$$

whence

$$\mathbb{P}\left[\phi^k_{N^{d-1},\widetilde{h}(N)} \geq \lambda N^{d-1}\right] \ \geq \ \mathbb{P}\left[\Pi^k_{\lambda,n}\right]^{m^{d-1}\widetilde{m}} \times p_k^{n^{d-1}\widetilde{r}m^{d-1}} p(\eta)^{(d-1)N^{d-2}r\widetilde{h}(N)}\,. \tag{2.4}$$

Let $\mathbf{\Pi}$ be the set of all the boundary conditions corresponding to a discrete stream (g,o) such that $flow(g,o) \geq \lambda n^{d-1}$. We have seen that a maximal flow ϕ^k is always realized by a discrete stream, so we have

$$\begin{aligned}
\mathbb{P}\left[\phi^k_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] \ &\leq \ \mathbb{P}\left[\bigcup_{\Pi \in \mathbf{\Pi}} \Pi\right] \\
&\leq \ \sum_{\Pi \in \mathbf{\Pi}} \mathbb{P}[\Pi] \\
&\leq \ N^k_{\lambda,n} \times \mathbb{P}\left[\Pi^k_{\lambda,n}\right]\,,
\end{aligned}$$

where we remember that $N^k_{\lambda,n}$ is the number of possible boundary conditions for discrete streams.

To obtain later a result independent of k, we need to consider two sequences $(k_n, n \in \mathbb{N})$ and $(\widetilde{k}_n, n \in \mathbb{N})$ such that $\lim_{n\to\infty} k_n = \lim_{n\to\infty} \widetilde{k}_n = +\infty$. We want to get rid of $N^k_{\lambda,n}$. We remember that

$$N^{k_n}_{\lambda,n} \le \left(k_n\left(\lfloor \lambda n^{d-1}\rfloor + 1\right) + 1\right)^{2n^{d-1}} .$$

Under the condition

$$\lim_{n\to\infty} \frac{\ln(k_n n)}{h(n)} = 0 \tag{2.5}$$

we have

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \ge \lambda n^{d-1}\right] \le \limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\Pi^{k_n}_{\lambda,n}\right] . \tag{2.6}$$

Consider (2.4) again. This equation is satisfied for every $k \ge k_0$, so it is true for k_n with a fixed n not too small. We need to compare t^{k_n} with $t^{\widetilde{k}_N}$, but the relation is simple only if $\widetilde{k}_N$ is divisible by k_n. That is the reason why from now on we will consider only sequences $(k_n, n \in \mathbb{N})$ and $(\widetilde{k}_n, n \in \mathbb{N})$ such that

$$\forall n \in \mathbb{N} \qquad k_n = 2^{\chi_n} \qquad and \qquad \widetilde{k}_n = 2^{\widetilde{\chi}_n}$$

where $(\chi_n, n \in \mathbb{N})$ and $(\widetilde{\chi}_n, n \in \mathbb{N})$ are non-decreasing sequences of integers. Of course the condition $\lim_{n\to+\infty} k_n = \lim_{n\to+\infty} \widetilde{k}_n = +\infty$ implies $\lim_{n\to+\infty} \chi_n = \lim_{n\to+\infty} \widetilde{\chi}_n = +\infty$. In that case for large N we have $\widetilde{\chi}_N \ge \chi_n$, so $\widetilde{k}_N$ is divisible by k_n and then $t^{\widetilde{k}_N} \ge t^{k_n}$, whence

$$\phi^{\widetilde{k}_N}_{N^{d-1},\widetilde{h}(N)} \ge \phi^{k_n}_{N^{d-1},\widetilde{h}(N)} . \tag{2.7}$$

We use (2.4) with $k = k_n = 2^{\chi_n}$ and (2.7) to obtain for n and N large enough

$$\begin{aligned}
&\frac{1}{N^{d-1}\widetilde{h}(N)} \ln \mathbb{P}\left[\phi^{\widetilde{k}_N}_{N^{d-1},\widetilde{h}(N)} \ge \lambda N^{d-1}\right] \\
&\qquad \ge \frac{1}{N^{d-1}\widetilde{h}(N)} \ln \mathbb{P}\left[\phi^{k_n}_{N^{d-1},\widetilde{h}(N)} \ge \lambda N^{d-1}\right] \\
&\qquad \ge \frac{m^{d-1}\widetilde{m}}{N^{d-1}\widetilde{h}(N)} \ln \mathbb{P}\left[\Pi^{k_n}_{\lambda,n}\right] + \frac{n^{d-1}m^{d-1}\widetilde{r}}{N^{d-1}\widetilde{h}(N)} \ln p_{k_n} + \frac{(d-1)N^{d-2}r\widetilde{h}(N)}{N^{d-1}\widetilde{h}(N)} \ln p(\eta) .
\end{aligned}$$

We send first N to $+\infty$ and then n to $+\infty$; this gives us with the help of (2.6)

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{N^{d-1}\widetilde{h}(N)} \ln \mathbb{P}\left[\phi^{\widetilde{k}_N}_{N^{d-1},\widetilde{h}(N)} \ge \lambda N^{d-1}\right]
&\ge \limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\Pi^{k_n}_{\lambda,n}\right] \\
&\ge \limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \ge \lambda n^{d-1}\right] .
\end{aligned}$$

By considering the case $h = \widetilde{h}$ and $k_n = \widetilde{k}_n = 2^{\chi_n}$, under the condition (2.5) on h and (k_n) - i.e., the condition (2.2) on h and (χ_n) -, we obtain the existence of the limit

$$\widetilde{\psi}(\lambda, h, (\chi_n)) = \lim_{n\to\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{2^{\chi_n}}_{n^{d-1},h(n)} \ge \lambda n^{d-1}\right] .$$

For general h, $\widetilde{h}$, χ and $\widetilde{\chi}$ we obtain that $\widetilde{\psi}(\lambda)$ is independent of the pair $(h, (\chi_n))$ satisfying (2.2), so theorem 4 is proved.

REMARK 5. Thanks to this independence, we can prove some properties of $\widetilde{\psi}$ by studying the behaviour of the limit involved in theorem 4 for specific choices of pairs $(h, (\chi_n))$.

Moreover, still in the case $\lambda < \mu$, we have immediately that for n sufficiently large

$$\begin{aligned}\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] &\geq \mathbb{P}\left[\begin{array}{c} all\ the\ vertical\ edges\ e\ in\ B((n,...,n),h(n)) \\ satisfy\ t(e) \geq (\lambda+\eta)\end{array}\right] \\ &\geq p(\eta)^{n^{d-1}h(n)}\,,\end{aligned}$$

thus

$$\widetilde{\psi}(\lambda) \leq -\ln p(\eta) < +\infty\,.$$

If the capacity t of an edge is bounded by a constant M, we can simply define

$$\forall x \in \mathbb{Z}^{d-1} \cap]0,n]^{d-1} \qquad \pi_h^{\lambda,n}(g,x) = g\left(\langle (x,h),(x,h+1)\rangle\right)$$

without truncating g because g is already bounded by M. Then the number of possible boundary conditions N_n^k satisfies

$$N_n^k \leq (k(M+1))^{2n^{d-1}}$$

so we can replace the hypothesis (2.5) by

$$\lim_{n\to\infty} \frac{\ln k_n}{h(n)} = 0\,. \tag{2.8}$$

REMARK 6. We don't study the case $\lambda = \mu$ for the moment, it is more adapted to study it with the continuity of $\widetilde{\psi}$.

3.3. Convexity of $\widetilde{\psi}$. Let $\lambda_1 \leq \lambda_2 < \mu$, and $\alpha \in]0,1[$. We want to show that

$$\widetilde{\psi}(\alpha\lambda_1 + (1-\alpha)\lambda_2) \leq \alpha\widetilde{\psi}(\lambda_1) + (1-\alpha)\widetilde{\psi}(\lambda_2)\,. \tag{2.9}$$

We know that $\widetilde{\psi}$ does not depend on the couple $(h,(\chi_n))$ satisfying (2.2), so we can take $h(n) = n$ to simplify the notations and we will take an adapted (χ_n). First we fix k in $\mathbb{N}$, we will make it vary later. We fix n, m in $\mathbb{N}$, and take $N = nm$. We set $u = \lfloor \alpha m^{d-1} \rfloor$. We keep the same notations as in the previous section for $B_{i_1,...,i_{d-1}}$, $i_1,...,i_{d-1}$ in $\{0,...,m-1\}$. We use the lexicographic order to order $\{(i_1,...,i_{d-1}),\ i_j \in \{0,...,m-1\},\ 1 \leq j \leq (d-1)\}$ and use this to rename these cylinders $(B_j, 1 \leq j \leq m^{d-1})$. On the event

$$\left\{\forall j \in \{1,...,u\},\ \phi^k_{B_j} \geq \lambda_1\right\} \cap \left\{\forall j \in \{(u+1),...,m^{d-1}\},\ \phi^k_{B_j} \geq \lambda_2\right\}$$

we have (see remark 3)

$$\begin{aligned}\phi^k_{N^{d-1},N} &\geq \left(u\lambda_1 n^{d-1} + (m^{d-1}-u)\lambda_2 n^{d-1}\right) \\ &\geq N^{d-1}\left(\frac{u}{m^{d-1}}\lambda_1 + \left(1 - \frac{u}{m^{d-1}}\right)\lambda_2\right) \\ &\geq N^{d-1}\left(\alpha\lambda_1 + (1-\alpha)\lambda_2\right)\end{aligned}$$

because $\lambda_1 < \lambda_2$, so

$$\begin{aligned}\mathbb{P}\Big[\phi^k_{N^{d-1},N} \geq & N^{d-1}\left(\alpha\lambda_1 + (1-\alpha)\lambda_2\right)\Big] \\ &\geq \mathbb{P}\left[\phi^k_{n^{d-1},N} \geq \lambda_1 n^{d-1}\right]^u \times \mathbb{P}\left[\phi^k_{n^{d-1},N} \geq \lambda_2 n^{d-1}\right]^{m^{d-1}-u}\,.\end{aligned}$$

As in the previous section (see (2.3)), we have

$$\mathbb{P}\left[\phi^k_{n^{d-1},N} \geq \lambda_i n^{d-1}\right] \geq \mathbb{P}\left[\Pi^k_{\lambda_i,n}\right]^m \qquad i = 1,2\,.$$

so

$$\frac{1}{N^d}\ln\mathbb{P}\left[\,\phi^k_{N^{d-1},N}\geq N^{d-1}\left(\alpha\lambda_1+(1-\alpha)\lambda_2\right)\right]$$
$$\geq\;\frac{m^{d-1}}{N^d}\left(\frac{u}{m^{d-1}}\ln\mathbb{P}\left[\Pi^k_{\lambda_1,n}\right]+\left(1-\frac{u}{m^{d-1}}\right)\ln\mathbb{P}\left[\Pi^k_{\lambda_2,n}\right]\right)\,.$$

We make now k vary, $k_n=2^{\chi_n}$ with $(n,(\chi_n))$ satisfying the condition (2.2) (for example $\chi_n=\lfloor n^{1/2}\rfloor$), and we use the property $\phi^{2^{\chi_N}}_{N^{d-1},N}\geq\phi^{2^{\chi_n}}_{N^{d-1},N}$ for large N; we send first N to $+\infty$ and then n to $+\infty$. We proved in the previous section that

$$\limsup_{n\to\infty}\frac{1}{n^{d-1}h(n)}\ln\mathbb{P}\left[\Pi^{2^{\chi_n}}_{\lambda,n}\right]\;=\;-\widetilde{\psi}(\lambda)\,,$$

so we obtain (2.9).

3.4. Continuity of $\widetilde{\psi}$. We want to show that $\widetilde{\psi}$ is continuous on $[0,\mu]$ when μ is finite or on $[0,+\infty[$ when μ is infinite (remember that $\widetilde{\psi}$ is infinite on $]\mu,+\infty[$). The function $\widetilde{\psi}$ is convex and finite on $[0,\mu[$, so $\widetilde{\psi}$ is continuous on $]0,\mu[$. We assume then that $0<\mu$: $\widetilde{\psi}(0)=0$ and $\widetilde{\psi}$ is non-negative on $\mathbb{R}^+$, so $\widetilde{\psi}$ is right continuous at 0. We assume then that $0<\mu<+\infty$. The only point which remains to study is the left continuity of $\widetilde{\psi}$ at μ. Remember that we did not define $\widetilde{\psi}$ at μ, we will do it now. We set

$$q_\mu\;=\;\mathbb{P}[t(e)=\mu]\,.$$

Notice that q_μ can be null. We remark that

$$\mathbb{P}\left[\phi_{n^{d-1},h(n)}\geq\mu n^{d-1}\right]\;=\;\mathbb{P}\left[\begin{array}{c}all\ the\ vertical\ edges\ e\ in\ B((n,...,n),h(n))\\ satisfy\ t(e)=\mu\end{array}\right]$$
$$=\;q_\mu^{n^{d-1}h(n)}\,,$$

so

$$\lim_{n\to\infty}-\frac{1}{n^{d-1}h(n)}\ln\mathbb{P}\left[\phi_{n^{d-1},h(n)}\geq\mu n^{d-1}\right]\;=\;-\ln q_\mu$$

is finite as soon as $q_\mu>0$. Unfortunately, the existence of an atom for the law of $t(e)$ at μ does not imply the existence of an atom for the law of $t^k(e)$ at μ, so we can have $q_\mu>0$ and

$$\lim_{n\to\infty}-\frac{1}{n^{d-1}h(n)}\ln\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)}\geq\mu n^{d-1}\right]\;=\;+\infty\,.$$

This is the reason why we did not study $\widetilde{\psi}(\mu)$ previously. We define (for every pair $(h,(\chi_n))$ as in theorem 4)

$$\widetilde{\psi}(\mu)\;=\;-\ln q_\mu\,,$$

which can eventually be infinite.

Now we want to check that $\widetilde{\psi}$ is left continuous at μ (if $q_\mu=0$ we will show that $\lim\widetilde{\psi}(\lambda)=+\infty$ when $\lambda\leq\mu$ and $\lambda\to\mu$). The idea of the proof is simple: if the flow in a cylinder is big, it must be big in each horizontal section of this cylinder. We fix $\varepsilon>0$, and we take $h(n),k_n\to_{n\to+\infty}+\infty$, $k_n=2^{\chi_n}$, satisfying the condition (2.5). We define for i in $\{0,...,(h(n)-1)\}$

$$C_i\;=]0,n]^{d-1}\times]i,i+1]$$

and we denote by $t_1,...,t_{n^{d-1}}$ the capacities of the n^{d-1} vertical edges in C_0. We have

$$\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)}\geq(\mu-\varepsilon)n^{d-1}\right]\;\leq\;\mathbb{P}\left[\bigcap_{i=0}^{h(n)-1}\{\,\phi^{k_n}_{C_i}\geq(\mu-\varepsilon)n^{d-1}\,\}\right]$$

$$\leq \mathbb{P}\left[\phi^{k_n}_{n^{d-1},1} \geq (\mu-\varepsilon)n^{d-1}\right]^{h(n)} ,$$

and we know that

$$\phi^{k_n}_{n^{d-1},1} \,=\, \sum_{j=1}^{n^{d-1}} t_j^{k_n} \,\leq\, \sum_{j=1}^{n^{d-1}} t_j \,,$$

so we have

$$\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu-\varepsilon)n^{d-1}\right] \,\leq\, \mathbb{P}\left[\sum_{j=1}^{n^{d-1}} (t_j-\mu) \geq -\varepsilon n^{d-1}\right]^{h(n)} .$$

For every positive ρ we obtain

$$\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu-\varepsilon)n^{d-1}\right] \,\leq\, e^{\rho\varepsilon n^{d-1}h(n)}\mathbb{E}[e^{\rho(t-\mu)}]^{n^{d-1}h(n)} .$$

This expectation is well defined, because $(t-\mu)\leq 0$. Let $\eta>0$. Since

$$\lim_{\rho\rightarrow+\infty} \mathbb{E}[e^{\rho(t-\mu)}] \,=\, q_\mu \,,$$

then there exists ρ_0 such that

$$\forall \rho \geq \rho_0 \qquad \mathbb{E}[e^{\rho(t-\mu)}] \,\leq\, (q_\mu+\eta) .$$

It follows that

$$\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu-\varepsilon)n^{d-1}\right] \,\leq\, \rho_0\varepsilon + \ln(q_\mu+\eta) \,,$$

so

$$\widetilde{\psi}(\mu-\varepsilon) \,\geq\, -\rho_0\varepsilon - \ln(q_\mu+\eta) \,,$$

whence

$$\lim_{\varepsilon\rightarrow 0} \widetilde{\psi}(\mu-\varepsilon) \,\geq\, -\ln(q_\mu+\eta) .$$

This is true for every positive η, so

$$\lim_{\varepsilon\rightarrow 0} \widetilde{\psi}(\mu-\varepsilon) \,\geq\, \lim_{\eta\rightarrow 0} -\ln(q_\mu+\eta) \,=\, -\ln q_\mu \,=\, \widetilde{\psi}(\mu) .$$

If $q_\mu=0$, we have the desired equality. Otherwise, we remark that for every positive ε we have

$$\begin{aligned} &\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu-\varepsilon)n^{d-1}\right] \\ &\qquad \geq \mathbb{P}\left[all\ the\ vertical\ edges\ e\ in\ B((n,...,n),h(n))\ satisfy\ t^{k_n}(e) \geq (\mu-\varepsilon)\right] \\ &\qquad \geq \mathbb{P}\left[t^{k_n}(e) \geq \mu-\varepsilon\right]^{n^{d-1}h(n)} . \end{aligned}$$

Now for k_n sufficiently large we have

$$\mathbb{P}\left[t^{k_n}(e) \geq \mu-\varepsilon\right] \,\geq\, \mathbb{P}\left[t(e) \geq \mu - \frac{\varepsilon}{2}\right] \,\geq\, q_\mu \,,$$

thus

$$\forall \varepsilon>0 \qquad \widetilde{\psi}(\mu-\varepsilon) \,\leq\, -\ln q_\mu \,=\, \widetilde{\psi}(\mu) .$$

This ends the proof of the continuity of $\widetilde{\psi}$ on $[0,\mu]$ (or $[0,+\infty[$ if μ is infinite). We deduce immediately from this continuity that $\widetilde{\psi}$ is good.

3.5. Existence of the limit for $\phi_{n^{d-1},h(n)}$. We come back to the existence of the limit involving ϕ in theorem 2. We consider three cases.

• $\lambda > \mu$: Then

$$\forall n \in \mathbb{N} \qquad \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] \,=\, 0\,,$$

so the limit involved in theorem 2 exists and satisfies

$$\psi(\lambda) \,=\, \lim_{n\to\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] \,=\, +\infty \,=\, \widetilde{\psi}(\lambda)\,.$$

• $\lambda = \mu$: As we saw by studying the continuity of $\widetilde{\psi}$, we have

$$\psi(\mu) \,=\, \lim_{n\to\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \mu n^{d-1}\right] \,=\, -\ln q_\mu \,=\, \widetilde{\psi}(\mu)$$

by definition of $\widetilde{\psi}(\mu)$.

• $\lambda < \mu$: We will compare $\phi^{k_n}_{n^{d-1},h(n)}$ with $\phi_{n^{d-1},h(n)}$. We fix $k \in \mathbb{N}$. We know that $t(e) \geq t^k(e)$ so $\phi_{n^{d-1},h(n)} \geq \phi^k_{n^{d-1},h(n)}$. For a set of edges E, we denote by $V^k(E)$ the quantity $\sum_{e\in E} t^k(e)$. Thanks to the max-flow min-cut theorem we obtain

$$\phi^k_{n^{d-1},h(n)} \,=\, \min\{\, V^k(E) \,|\, E \textit{ is an } (F_0, F_{h(n)}) - \textit{cut}\,\}\,.$$

Let E_0 be an $(F_0, F_{h(n)})$-cut realizing this minimum (it may depend on k). Then

$$\begin{aligned}
\phi^k_{n^{d-1},h(n)} \,&=\, V^k(E_0)\\
&=\, \sum_{e\in E_0} t^k(e)\\
&\geq\, \sum_{e\in E_0} t(e) - \frac{|E_0|}{k}\\
&\geq\, \min\{\, V(E) \,|\, E \textit{ is an } (F_0, F_{h(n)}) - \textit{cut}\,\} - 2dn^{d-1}\frac{h(n)}{k}\\
&\geq\, \phi_{n^{d-1},h(n)} - 2dn^{d-1}\frac{h(n)}{k}\,.
\end{aligned}$$

We fix $\lambda \geq 0$, and we make now k vary. We take $k_n = 2^{\chi_n}$ (with (χ_n) a non-decreasing sequence of integers such that $\lim_{n\to+\infty} \chi_n = +\infty$) satisfying with h the condition (2.5). If the sequence (k_n) satisfies also the condition

$$\lim_{n\to\infty} \frac{h(n)}{k_n} \,=\, 0 \tag{2.10}$$

then we have for every $\lambda' < \lambda$ the existence of $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \qquad \lambda - \frac{h(n)}{k_n} \,\geq\, \lambda'\,.$$

We deduce that under the condition (2.10) we have for all $n \geq n_0$

$$\mathbb{P}\left[\frac{\phi^{k_n}_{n^{d-1},h(n)}}{n^{d-1}} \geq \lambda\right] \,\leq\, \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq \lambda\right] \,\leq\, \mathbb{P}\left[\frac{\phi^{k_n}_{n^{d-1},h(n)}}{n^{d-1}} \geq \lambda'\right]\,.$$

We conclude thanks to the hypothesis (2.5) that

$$\widetilde{\psi}(\lambda) \,\geq\, \limsup_{n\to\infty}\,(\square) \,\geq\, \liminf_{n\to\infty}\,(\square) \,\geq\, \widetilde{\psi}(\lambda')$$

where

$$(\square) \,=\, -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right]\,.$$

Sending λ' to λ, thanks to the continuity of $\widetilde{\psi}$ in $[0,\mu[$, we obtain the existence of the limit

$$\psi(\lambda,h) \,=\, \lim_{n\to\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] \,=\, \widetilde{\psi}(\lambda)\,.$$

Moreover we know that this limit is independent of h satisfying $\lim_{n\to+\infty} h(n) = +\infty$ and such that there exists a non-decreasing sequence of integers (χ_n), $\lim_{n\to+\infty}\chi_n = +\infty$ for which the pair $(h,(2^{\chi_n}))$ satisfies (2.5) and (2.10): we denote it by $\psi(\lambda)$. It is finally obvious that the existence of such a sequence (χ_n) is equivalent to the condition

$$\lim_{n\to\infty} \frac{h(n)}{\ln n} \,=\, +\infty$$

(let $\chi_n = \lfloor 2\ln h(n)/\ln 2 \rfloor$ for example). This ends the proof of the existence of the limit ψ in theorem 2, and we have $\psi = \widetilde{\psi}$ so the properties proved for $\widetilde{\psi}$ still hold for ψ.

If the capacity t of an edge is bounded, we can replace the condition (2.5) by (2.8); in that case, as soon as

$$\lim_{n\to\infty} h(n) \,=\, +\infty$$

we can find a sequence $(k_n) = (2^{\chi_n})$ satisfying (2.8) and (2.10), so the limit exists.

3.6. The function ψ vanishes on $[0,\nu(F)]$. This could be proved easily thanks to theorem 1 in dimension three and with the hypothesis on F required in theorem 1, but we prefer to prove it directly in the general case without theorem 1.

We suppose now that $\mathbb{E}[t]$ is finite. We suppose that $\nu > 0$ (otherwise there is nothing to prove), and we take $\lambda = \nu - \varepsilon$, with a positive ε. Remark 5 holds for ψ too: we know that ψ is independent of h satisfying $\lim_{n\to+\infty} h(n)/\ln n = +\infty$ so we can make a specific choice of function h and study the corresponding limit to show a general result on ψ. We take $h\to\infty$ such that

$$\lim_{n\to\infty} \frac{h(n)}{n} \,=\, 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{h(n)}{\ln n} \,=\, +\infty\,. \tag{2.11}$$

We remember that

$$\tau_{n^{d-1}} \,=\, \tau(]0,n]^{d-1}) \,=\, \inf\{\, V(E)\,|\, E \textit{ is a cut over }]0,n]^{d-1} \textit{ and } E \textit{ satisfies } (*)\,\}\,,$$

where $(*)$ is defined at the end of the section 2. We define for S a hyper-rectangle the variable

$$\tau(S,k) \,=\, \inf\{\, V(E)\,|\, E \textit{ is a cut over } S\,,\ E \textit{ satisfies } (*) \textit{ and } E\subset S\times]-k,k]\,\}\,,$$

and

$$\tau_{n^{d-1},k} \,=\, \tau(]0,n]^{d-1},k)\,.$$

We define the set of edges F as

$$F \,=\, \{\langle x,y\rangle\,|\, x\in B\,,\ y\notin B \textit{ and } \langle x,y\rangle \in \mathbb{R}^{d-1}\times[1,h(n)]\}\,.$$

This is the set of the edges through which some fluid could escape from B somewhere else than at its bottom or at its top. We denote by $|F|$ the cardinality of F, $|F| = 2(d-1)n^{d-2}h(n)$. We consider the larger cylinder

$$B' \,=]-1,n+1]^{d-1}\times]0,h(n)]\,,$$

and we define

$$\tau'_{(n+2)^{d-1},h(n)} \,=\, \tau\left(]-1,n+1]^{d-1},h(n)\right)\,.$$

We finally define the set of edges

$$F' \,=\, \{\, e\in B'\smallsetminus B\,|\, e \textit{ is vertical}\,,\ e\in\mathbb{R}^{d-1}\times[0,1]\}$$

of cardinality $|F'| = 2(d-1)(n+1)^{d-2}$ (see figure 3 in dimension two). We remark that if E is an $(F_0,F_{h(n)})$-cut in $B((n,...,n),h(n))$, the set of edges $E\cup F\cup F'$ contains a cut over

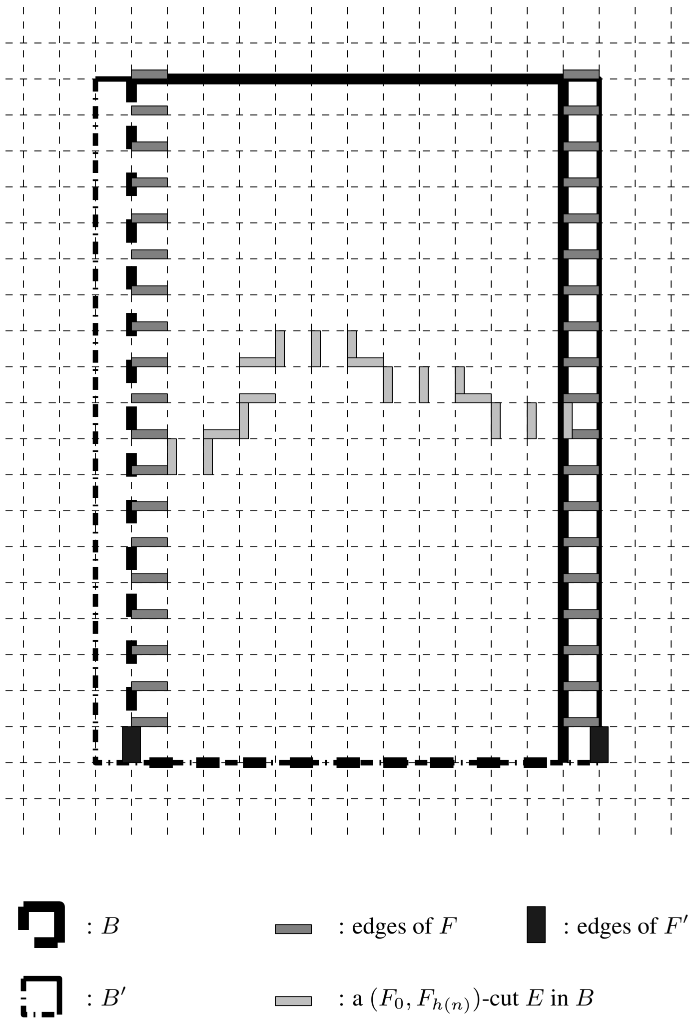

FIGURE 3. Comparison between ϕ and τ in dimension two

$]-1,n+1]^{d-1}$ satisfying the condition $(*)$ for $S=]-1,n+1]^{d-1}$, so

$$\tau'_{(n+2)^{d-1},h(n)} - \phi_{n^{d-1},h(n)} \leq \sum_{e\in F} t(e) + \sum_{e\in F'} t(e)\,.$$

We obtain for $M > \mathbb{E}[t]$

$$\begin{aligned}
\mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq \lambda\right] \;&\geq\; \mathbb{P}\left[\left\{\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq \lambda\right\} \cap \left\{\frac{\tau'_{(n+2)^{d-1},h(n)} - \phi_{n^{d-1},h(n)}}{|F|+|F'|} \leq M\right\}\right]\\
&\geq\; \mathbb{P}\left[\left\{\frac{\tau'_{(n+2)^{d-1},h(n)}}{n^{d-1}} \geq \lambda + M\frac{|F|+|F'|}{n^{d-1}}\right\}\right.\\
&\qquad\qquad \left.\cap \left\{\frac{\tau'_{(n+2)^{d-1},h(n)} - \phi_{n^{d-1},h(n)}}{|F|+|F'|} \leq M\right\}\right]\,.
\end{aligned}$$

We remark that $\tau'_{(n+2)^{d-1},h(n)}$ is equal in law to $\tau_{(n+2)^{d-1},h(n)}$, so

$$\mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq \lambda\right] \;\geq\; 1 - \Bigg(\mathbb{P}\left[\frac{\tau_{(n+2)^{d-1},h(n)}}{n^{d-1}} < \lambda + M\frac{|F|+|F'|}{n^{d-1}}\right]$$

$$+\,\mathbb{P}\left[\frac{\tau'_{(n+2)^{d-1},h(n)}-\phi_{n^{d-1},h(n)}}{|F|+|F'|}>M\right]\Bigg)$$
$$\geq\; 1-\left(\mathbb{P}\left[\frac{\tau_{(n+2)^{d-1}}}{n^{d-1}}<\nu-\frac{\varepsilon}{2}\right]+\mathbb{P}\left[\frac{1}{|F|+|F'|}\sum_{e\in F\cup F'}t(e)\geq M\right]\right)$$

for n sufficiently large, thanks to (2.11) and the fact that $\tau_{(n+2)^{d-1},h(n)} \geq \tau_{(n+2)^{d-1}}$. We know that $M > \mathbb{E}[t]$ and $\lim_{n\to\infty}(\tau_{(n+2)^{d-1}}/n^{d-1}) = \nu$ almost surely, so

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}}\geq\lambda\right]\;=\;1\,,$$

which leads to

$$\psi(\lambda)\;=\;0\,.$$

To conclude that $\psi(\nu)=0$ we need only to check that ψ is left continuous at ν, i.e., to be sure that $\nu\leq\mu$. Suppose that $\nu>\mu$, then $\mathbb{P}[t\geq\nu]=0$, so $\mathbb{E}[t]<\nu$, and we can find a positive ε such that $\mathbb{E}[t]<\nu-\varepsilon$. Now if we denote by $(\widetilde{t}_i, i=1,...,n^{d-1})$ the capacities of the vertical edges in $]0,n]^{d-1}\times]0,1]$, we have

$$\mathbb{P}\left[\frac{\tau_{n^{d-1}}}{n^{d-1}}\geq\nu-\varepsilon\right]\;\leq\;\mathbb{P}\left[\frac{\sum_{i=1}^{n^{d-1}}\widetilde{t}_i}{n^{d-1}}\geq\nu-\varepsilon\right]\xrightarrow[n\to\infty]{}0\,.$$

This is absurd because $(\tau_{n^{d-1}}/n^{d-1})$ converges toward ν almost surely. We conclude that $\nu\leq\mu$ and that $\psi(\nu)=0$.

3.7. The function ψ is positive on $]\nu(F),+\infty[$. We suppose that there exists a positive θ such that $\int_{[0,+\infty[}e^{\theta x}dF(x)$ is finite. The proof is based on the Cramér theorem in $\mathbb{R}$.

Let $\lambda=\nu+\varepsilon$, for a positive ε. We fix k, $N\in\mathbb{N}$, we will choose them later. We define

$$u\;=\;\left\lfloor\frac{h(N)}{2k}\right\rfloor\,.$$

Just as in the study of the continuity of ψ, by cutting $B((N,...,N),h(N))$ into horizontal sections of height $2k$, we have

$$\mathbb{P}\Big[\phi_{N^{d-1},h(N)}\geq(\nu+\varepsilon)N^{d-1}\Big]$$
$$\leq\;\mathbb{P}\left[\phi_{N^{d-1},2k}\geq(\nu+\varepsilon)N^{d-1}\right]^u\;=\;\mathbb{P}\left[\phi_{\mathcal{B}(k)}\geq(\nu+\varepsilon)N^{d-1}\right]^u\,,$$

where $\mathcal{B}(k)=]0,N^{d-1}]\times]-k,k]$ because $\phi_{N^{d-1},2k}$ and $\phi_{\mathcal{B}}$ are equal in law. Now $\mathbb{E}[\tau(S,k)]$ is subadditive in the sense that for disjoint hyper-rectangles S_1 and S_2 having a common side, we have

$$\tau(S_1\cup S_2,k)\;\leq\;\tau(S_1,k)+\tau(S_2,k)\,.$$

Moreover $\mathbb{E}[\tau(S,k)]$ is non-negative and finite (because $\mathbb{E}[t]<\infty$), so by a classical subadditive argument we have the existence of

$$\nu_k\;=\;\lim_{n\to\infty}\frac{\mathbb{E}[\tau_{n^{d-1},k}]}{n^{d-1}}$$

and we know that

$$\nu_k\;=\;\inf_n\frac{\mathbb{E}[\tau_{n^{d-1},k}]}{n^{d-1}}\,.$$

The sequence $(\nu_k, k \in \mathbb{N})$ is non-increasing in k and non-negative, so it converges; we denote by $\widetilde{\nu}$ its limit: $\widetilde{\nu} = \lim_{k\to\infty} \nu_k = \inf_k \nu_k$. By the same subadditive argument, we have

$$\lim_{n\to\infty} \frac{\mathbb{E}[\tau_{n^{d-1}}]}{n^{d-1}} = \nu = \inf_n \frac{\mathbb{E}[\tau_{n^{d-1}}]}{n^{d-1}} .$$

We obtain

$$\widetilde{\nu} = \inf_k \inf_n \frac{\mathbb{E}[\tau_{n^{d-1},k}]}{n^{d-1}} = \inf_n \inf_k \frac{\mathbb{E}[\tau_{n^{d-1},k}]}{n^{d-1}} = \nu ,$$

thus we can choose k_0 such that $\nu_{k_0} \leq \nu + \varepsilon/4$. Then we choose n_0 such that

$$\frac{\mathbb{E}[\tau_{n_0^{d-1},k_0}]}{n_0^{d-1}} < \nu_{k_0} + \frac{\varepsilon}{2} ,$$

and we fix $N = n_0 m$, with $m \in \mathbb{N}$. We have

$$\phi_{\mathcal{B}(k_0)} \leq \tau_{N^{d-1},k_0} \leq \sum_{i_1,\dots,i_{d-1}=0}^{m-1} \tau\left(\prod_{j=1}^{d-1}]i_j n_0, (i_j+1)n_0], k_0\right) .$$

The variables $(\tau(\prod_{j=1}^{d-1}]i_j n_0,(i_j+1)n_0], k_0), 0 \leq i_1, ..., i_{d-1} \leq m-1)$ are independent and identically distributed, with the same law as $\tau_{n_0^{d-1},k_0}$. Their common expectation is

$$\mathbb{E}[\tau_{n_0^{d-1},k_0}] \leq \left(\nu_{k_0} + \frac{\varepsilon}{2}\right) n_0^{d-1} .$$

Moreover for some positive θ we know that $\mathbb{E}[e^{\theta t}]$ is finite so

$$\mathbb{E}\left[e^{\theta \tau_{n_0^{d-1},k_0}}\right] \leq \mathbb{E}\left[e^{\theta \sum_{i=1}^{n_0^{d-1}} \widetilde{t}_i}\right] \leq \mathbb{E}\left[e^{\theta t}\right]^{n_0^{d-1}} < \infty ,$$

where $(\widetilde{t}_i, 1 \leq i \leq n_0^{d-1})$ are still the capacities of the vertical edges in $]0,n_0]^{d-1}\times]0,1]$. We can thus apply the Cramér theorem in $\mathbb{R}$ (see [**26**]), which states the existence of a negative constant $c(n_0,k_0,\varepsilon)$ such that

$$\lim_{m\to\infty} \frac{1}{m^{d-1}} \ln \mathbb{P}\left[\frac{1}{m^{d-1}} \sum_{i_1,\dots,i_{d-1}=0}^{m-1} \frac{\tau\Big(\prod_{j=1}^{d-1}]i_j n_0,(i_j+1)n_0],k_0\Big)}{n_0^{d-1}} \geq \nu_{k_0} \frac{3\varepsilon}{4}\right]$$
$$= c(n_0,k_0,\varepsilon) .$$

It follows that for $u = \lfloor \frac{h(n)}{2k_0} \rfloor$

$$\begin{aligned}
&\frac{1}{N^{d-1}h(N)} \ln \mathbb{P}\left[\phi_{N^{d-1},h(N)} \geq (\nu+\varepsilon)N^{d-1}\right] \\
&\leq \frac{u}{N^{d-1}h(N)} \ln \mathbb{P}\left[\phi_{\mathcal{B}(k_0)} \geq (\nu_{k_0} + \frac{3\varepsilon}{4})N^{d-1}\right] \\
&\leq \frac{u m^{d-1}}{N^{d-1}h(N)} \frac{1}{m^{d-1}} \ln \mathbb{P}\left[\frac{1}{m^{d-1}} \sum_{i_1,\dots,i_{d-1}=0}^{m-1} \frac{\tau\left(\prod_{j=1}^{d-1}]i_j n_0,(i_j+1)n_0],k_0\right)}{n_0^{d-1}} \geq \nu_{k_0} + \frac{3\varepsilon}{4}\right] \\
&\xrightarrow[m\to\infty]{} \frac{c(n_0,k_0,\varepsilon)}{2k_0 n_0^{d-1}} < 0 ,
\end{aligned}$$

so $\psi(\lambda) > 0$. This ends the proof of theorem 2.

REMARK 7. The existence of a positive θ satisfying $\mathbb{E}[e^{\theta t}] < \infty$ is probably not a necessary condition to have the positivity of the function ψ on $]\nu,+\infty[$. However, a condition on the moments of t is necessary. Indeed, if the tail of the distribution of t is too big, the probability to have a vertical path of edges with big capacities (bigger than λn^{d-1}) is large, thus the probability to have $\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}$ cannot decay exponentially fast in $n^{d-1}h(n)$.

4. Proof of Theorem 3

This is an adaptation of the proof of a large deviation principle in [**19**]. We take h such that $h(n)/\ln n \to \infty$ (we can do this again without loss of generality because ψ is independent of h) and we suppose that there exists a positive θ such that $\mathbb{E}[e^{\theta t}]$ is finite. We define

$$\beta \,=\, \inf\{\, v \,|\, \mathbb{P}[t(e) \leq v] > 0\,\}\,.$$

We remark that $\phi_{(n,\dots,n),h(n)}/n^{d-1}$ takes its values in $[\beta, +\infty[$. We have to prove that

- for any closed subset $\mathcal{F} \subset [\beta, +\infty[$, we have

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{(n,\dots,n),h(n)}}{n^{d-1}} \in \mathcal{F}\right] \,\leq\, -\inf_{\mathcal{F}} \psi\,,$$

- for any open subset $\mathcal{O} \subset [\beta, +\infty[$, we have

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{(n,\dots,n),h(n)}}{n^{d-1}} \in \mathcal{O}\right] \,\geq\, -\inf_{\mathcal{O}} \psi\,.$$

By definition of β, for all positive η, we have

$$s_\beta(\eta) \,=\, \mathbb{P}[t(e) \leq \beta + \eta] \,>\, 0\,.$$

4.1. Upper bound. Let $\mathcal{F}$ be a closed subset of $[\beta, +\infty[$, and $a = \inf \mathcal{F}$. Clearly

$$\mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in \mathcal{F}\right] \,\leq\, \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq a\right]\,,$$

so

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in \mathcal{F}\right] \,\leq\, -\psi(a) \,=\, -\inf_{\mathcal{F}} \psi$$

because ψ is non-decreasing on $\mathbb{R}^+$.

4.2. Lower bound. We shall prove the following local lower bound:
(2.12)

$$\forall \alpha \in [\beta, +\infty[\,,\ \forall \varepsilon > 0 \qquad \liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha-\varepsilon, \alpha+\varepsilon[\right] \,\geq\, -\psi(\alpha)\,.$$

If (2.12) holds, we have the desired lower bound. Indeed, if $\mathcal{O}$ is an open subset of $[\beta, +\infty[$, for every α in $\mathcal{O}$ there exists a positive ε such that $]\alpha-\varepsilon, \alpha+\varepsilon[\subset \mathcal{O}$, whence

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln\mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in \mathcal{O}\right] & \\ \geq\ & \liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha-\varepsilon, \alpha+\varepsilon[\right] \\ \geq\ & -\psi(\alpha)\,.\end{aligned}$$

By taking the supremum over α in $\mathcal{O}$, we obtain

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in O\right] \,\geq\, -\inf_{\mathcal{O}} \psi\,.$$

To prove (2.12), we have to consider again different cases.

- $\alpha \geq \nu$: When $\psi(\alpha) = +\infty$, the result is obvious. For a finite $\psi(\alpha)$ we have $\psi(\alpha+\varepsilon) > \psi(\alpha)$ because the function ψ is convex on $[\nu, +\infty[$, $\psi(\nu) = 0$ and ψ is positive on $]\nu, +\infty[$ so ψ

is increasing on $[\nu, +\infty[$ (or infinite). Now

$$\mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha-\varepsilon, \alpha+\varepsilon[\right] \;\geq\; \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq \alpha\right] - \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq \alpha+\varepsilon\right],$$

so

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha-\varepsilon, \alpha+\varepsilon[\right] \;\geq\; -\psi(\alpha)\,.$$

- $\beta \leq \alpha < \nu$: In our cylinder $B = B((n,...,n), h(n))$ we will isolate a smaller cylinder of

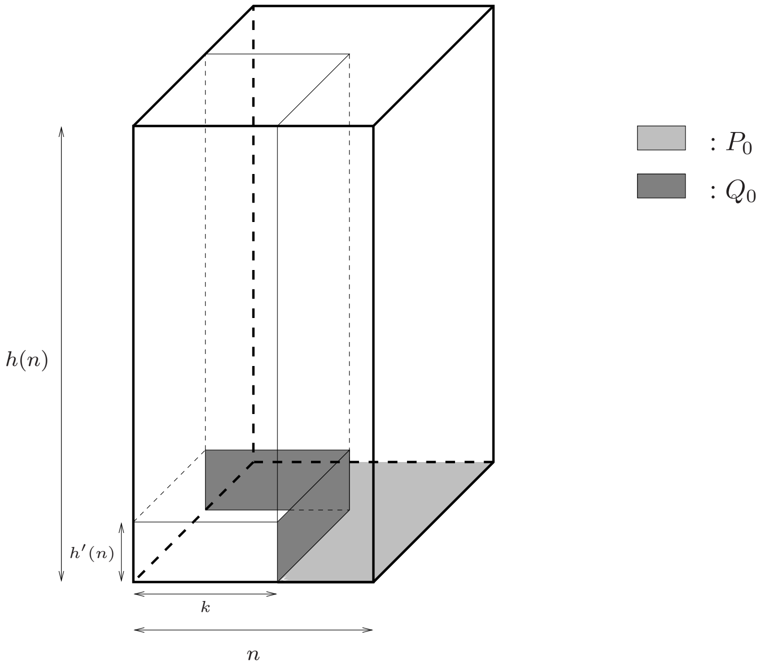

FIGURE 4. Control of the flow

adequate proportions in which we will impose that the rescaled flow is around its typical value ν, and we will control the amount of fluid that can circulate outside it (see figure 4). For that purpose, we consider a function h' such that

$$h' : \mathbb{N} \to \mathbb{N},\; h' \leq h\,,\; \lim_{n\to\infty} \frac{h'(n)}{n} = 0 \;\textit{and}\; \lim_{n\to\infty} \frac{h'(n)}{\ln n} = +\infty \;(\textit{then}\; \lim_{n\to\infty} h'(n) = +\infty)\,.$$

We define the constants

$$v = \left(\frac{\alpha-\beta}{\nu-\beta}\right)^{\frac{1}{d-1}},\; k = \lfloor vn \rfloor,\; 0 < \eta \leq \frac{\varepsilon}{4},$$

the set B' and the corresponding event A

$$B' = B((k,...,k), h(n))\,,\; A = \{\,\phi_{B'} \geq (\nu-\eta)k^{d-1}\,\}\,.$$

For $i \in \mathbb{N}$, $0 \leq i \leq (\lfloor h(n)/h'(n) \rfloor - 1)$, we finally define the sets B_i, P_i and Q_i and the corresponding events A_i, E_i and F_i, and the global events E and F as follows

$$B_i = B' \cap \left(\mathbb{R}^{d-1} \times]ih'(n), (i+1)h'(n)]\right),$$

$$P_i = (B \smallsetminus B') \cap (\mathbb{Z}^{d-1} \times \{\frac{1}{2} + ih'(n)\})\,,$$

$$Q_i = \bigcup_{j=1}^{d-1} \left([0,k]^{j-1} \times \{k+\frac{1}{2}\} \times [0,k]^{d-1-j} \times]ih'(n), (i+1)h'(n)]\right),$$

$$
\begin{aligned}
A_i \,&=\, \{\, \phi_{B_i} \leq (\nu+\eta) k^{d-1} \,\}\,,\\
E_i \,&=\, \left\{ \begin{array}{c} \textit{all the } (n^{d-1}-k^{d-1}) \textit{ vertical edges } e \textit{ of } B \smallsetminus B' \\ \textit{that intersect } P_i \textit{ satisfy } t(e) \leq \beta+\eta \end{array} \right\}\,,\\
E \,&=\, \bigcap_i E_i\,,\\
F_i \,&=\, \left\{ \begin{array}{c} \textit{all the } (d-1)h'(n)k^{d-2} \textit{ horizontal edges } e \\ \textit{that intersect } Q_i \textit{ satisfy } t(e) \leq \beta+\eta \end{array} \right\}\,,\\
F \,&=\, \bigcap_i F_i\,.
\end{aligned}
$$

Fix $n_0 \in \mathbb{N}$ such that

$$
\forall n \geq n_0 \qquad (d-1)(\beta+\eta)\frac{h'(n)}{n} \,\leq\, \frac{\varepsilon}{8} \qquad \textit{and} \qquad \left| \frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} - v^{d-1} \right| \leq \frac{\varepsilon}{8}\,.
$$

On one hand, on the event A, we have for $n \geq n_0$

$$
\begin{aligned}
\phi_B \,&\geq\, n^{d-1}(\nu-\eta)\frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} + \beta n^{d-1}\left(1 - \frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}}\right)\\
&\geq\, n^{d-1}\left(\nu v^{d-1} + \beta(1-v^{d-1}) - 2\frac{\varepsilon}{8} - \frac{\varepsilon}{4}\right)\\
&>\, n^{d-1}(\alpha-\varepsilon)\,.
\end{aligned}
$$

Here the term $\beta n^{d-1}(1-\lfloor vn \rfloor^{d-1}/n^{d-1})$ is the minimal amount of fluid that crosses $B \smallsetminus B'$ from its bottom to its top because the capacity of an edge cannot be smaller than β, by definition of β. On the other hand, if for some i in $\{0,...,(\lfloor \frac{h(n)}{h'(n)} \rfloor - 1)\}$ the event $A_i \cap E_i \cap F_i$ occurs then we have

$$
\begin{aligned}
\forall n \geq n_0 \qquad \phi_B \,&\leq\, n^{d-1}\left((\nu+\eta)\frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} + (\beta+\eta)\left(1 - \frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} + (d-1)\frac{\lfloor vn \rfloor h'(n)}{n^{d-1}}\right)\right)\\
&\leq\, n^{d-1}\left(\alpha + 2\frac{\varepsilon}{8} + 2\frac{\varepsilon}{4} + \frac{\varepsilon}{8}\right)\\
&<\, n^{d-1}(\alpha+\varepsilon)\,.
\end{aligned}
$$

We obtain then that

$$
\begin{aligned}
\forall n \geq n_0 \qquad \mathbb{P}\left[\frac{1}{n^{d-1}}\phi_B \in]\alpha-\varepsilon, \alpha+\varepsilon[\right] \,&\geq\, \mathbb{P}\left[A \cap \left(\bigcup_i A_i \cap E_i \cap F_i\right)\right]\\
&\geq\, \mathbb{P}[E] \times \mathbb{P}[F] \times \mathbb{P}\left[A \cap \left(\bigcup_i A_i\right)\right]\,.
\end{aligned}
$$

Now we know that

$$
\mathbb{P}[E] \,=\, s_\beta(\eta)^{(n^{d-1}-k^{d-1})\lfloor \frac{h(n)}{h'(n)} \rfloor}
$$

and

$$
\mathbb{P}[F] \,=\, s_\beta(\eta)^{(d-1)k^{d-2}\lfloor \frac{h(n)}{h'(n)} \rfloor h'(n)}\,,
$$

so

$$
\lim_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}[E] \,=\, \lim_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}[F] \,=\, 0\,.
$$

Moreover we have

$$
\begin{aligned}
\mathbb{P}\left[A \cap (\cup_i A_i)\right] \;&\geq\; \mathbb{P}[A] - \mathbb{P}\left[\cap_i A_i^c\right] \\
&\geq\; \mathbb{P}[A] - \mathbb{P}[A_0^c]^{\lfloor \frac{h(n)}{h'(n)} \rfloor} \\
&\geq\; \mathbb{P}\left[\frac{\phi_{k^{d-1},h(n)}}{k^{d-1}} \geq \nu - \eta\right] - \mathbb{P}\left[\frac{\phi_{k^{d-1},h'(n)}}{k^{d-1}} \geq \nu + \eta\right]^{\lfloor \frac{h(n)}{h'(n)} \rfloor} ,
\end{aligned}
$$

which leads, thanks to our previous study about ψ, to

$$\lim_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[A \cap (\cup_i A_i)\right] \;=\; 0\,.$$

We conclude that

$$\lim_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha-\varepsilon, \alpha+\varepsilon[\right] \;\geq\; 0 \;=\; -\psi(\alpha)\,.$$

This ends the proof of the lower bound.

CHAPTER 3

Upper large deviations for maximal flows through a cylinder: other cases

We consider the standard first passage percolation in $\mathbb{Z}^d$ for $d \geq 2$ and we study the maximal flow from the upper half part to the lower half part (respectively from the top to the bottom) of a cylinder whose basis is a hyperrectangle of sidelength proportional to n and whose height is $h(n)$ for a certain height function h. We denote this maximal flow by τ_n (respectively ϕ_n). We emphasize the fact that the cylinder may be tilted. We look at the probability that these flows, rescaled by the surface of the basis of the cylinder, are greater than $\nu(\vec{v}) + \varepsilon$ for some positive ε, where $\nu(\vec{v})$ is the limit of the expectation of the rescaled variable τ_n when n goes to infinity. On one hand, we prove that the speed of decay of this probability in the case of the variable τ_n depends on the tail of the distribution of the capacities of the edges: it can decays exponentially fast with n^{d-1}, or with $n^{d-1}\min(n, h(n))$, or at an intermediate regime. On the other hand, we prove that this probability in the case of the variable ϕ_n decays exponentially fast with the volume of the cylinder as soon as the law of the capacity of the edges admits one exponential moment; the importance of this result is however limited by the fact that $\nu(\vec{v})$ is not in general the almost sure limit of the rescaled maximal flow ϕ_n, but it is the case at least when the height $h(n)$ of the cylinder is negligible compared to n under some conditions on the law of the capacities.

Contents

1. Definitions and main results

Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$, the set of pairs of nearest neighbours for the standard L^1 norm. With each edge e in $\mathbb{E}^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}^d)$ is independent and identically distributed, with a common distribution function F: this is the standard model of first passage percolation on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$. We interpret $t(e)$ as the capacity of the edge e; it means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time.

The maximal flow $\phi(F_1 \rightarrow F_2 \textit{ in } C)$ from F_1 to F_2 in C, for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d$) can be defined properly this way. We will say that an edge $e = \langle x, y \rangle$ belongs to a subset A of $\mathbb{R}^d$, which we denote by $e \in A$, if the segment joining x to y (eventually excluding these points) is included in A. We define $\widetilde{\mathbb{E}}^d$ as the set of all the oriented edges, i.e., an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}^d$ is an ordered pair of vertices which are nearest neighbours. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}^d$ by $\langle\langle x, y\rangle\rangle$, where $x, y \in \mathbb{Z}^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from x towards y. We consider the set $\mathcal{S}$ of all pairs of functions (g, o), with $g : \mathbb{E}^d \to \mathbb{R}^+$ and $o : \mathbb{E}^d \to \widetilde{\mathbb{E}}^d$ such that $o(\langle x, y\rangle) \in \{\langle\langle x, y\rangle\rangle, \langle\langle y, x\rangle\rangle\}$, satisfying:

- for each edge e in C we have
$$0 \,\leq\, g(e) \,\leq\, t(e)\,,$$
- for each vertex v in $C \smallsetminus (F_1 \cup F_2)$ we have
$$\sum_{e\in C\,:\,o(e)=\langle\langle v,\cdot\rangle\rangle} g(e) \;=\; \sum_{e\in C\,:\,o(e)=\langle\langle \cdot,v\rangle\rangle} g(e)\,,$$

where the notation $o(e) = \langle\langle v, .\rangle\rangle$ (respectively $o(e) = \langle\langle ., v\rangle\rangle$) means that there exists $y \in \mathbb{Z}^d$ such that $e = \langle v, y\rangle$ and $o(e) = \langle\langle v, y\rangle\rangle$ (respectively $o(e) = \langle\langle y, v\rangle\rangle$). A couple $(g, o) \in \mathcal{S}$ is a possible stream in C from F_1 to F_2: $g(e)$ is the amount of fluid that goes through the edge e, and $o(e)$ gives the direction in which the fluid goes through e. The two conditions on (g, o) express only the fact that the amount of fluid that can go through an edge is bounded by its capacity, and that there is no loss of fluid in the graph. With each possible stream we associate the corresponding flow
$$flow(g,o) \;=\; \sum_{u\in F_2\,,\,v\notin C\,:\,\langle u,v\rangle\in\mathbb{E}_n^d} g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle u,v\rangle\rangle} - g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle v,u\rangle\rangle}\,.$$
This is the amount of fluid that crosses C from F_1 to F_2 if the fluid respects the stream (g, o). The maximal flow through C from F_1 to F_2 is the supremum of this quantity over all possible choices of streams
$$\phi(F_1 \to F_2 \text{ in } C) \;=\; \sup\{flow(g,o)\,|\,(g,o)\in\mathcal{S}\}\,.$$

The maximal flow $\phi(F_1 \to F_2 \text{ in } C)$ can be expressed differently thanks to the max-flow min-cut theorem (see [**12**]). We need some definitions to state this result. A path on the graph $\mathbb{Z}^d$ from v_0 to v_m is a sequence $(v_0, e_1, v_1, ..., e_m, v_m)$ of vertices $v_0, ..., v_m$ alternating with edges $e_1, ..., e_m$ such that v_{i-1} and v_i are neighbours in the graph, joined by the edge e_i, for i in $\{1, ..., m\}$. A set E of edges in C is said to cut F_1 from F_2 in C if there is no path from F_1 to F_2 in $C \smallsetminus E$. We call E an (F_1, F_2)-cut if E cuts F_1 from F_2 in C and if no proper subset of E does. With each set E of edges we associate its capacity which is the variable
$$V(E) \;=\; \sum_{e\in E} t(e)\,.$$
The max-flow min-cut theorem states that
$$\phi(F_1 \to F_2 \textit{ in } C) \;=\; \min\{\,V(E)\,|\,E \text{ is a } (F_1,F_2)\text{-cut}\,\}\,.$$

We need now some geometric definitions. For a subset X of $\mathbb{R}^d$, we denote by $\mathcal{H}^s(X)$ the s-dimensional Hausdorff measure of X (we will use $s = d-1$ and $s = d-2$). The r-neighbourhood $\mathcal{V}_i(X, r)$ of X for the distance d_i, that can be the Euclidean distance if $i = 2$ or the L^∞-distance if $i = \infty$, is defined by
$$\mathcal{V}_i(X,r) \;=\; \{y\in\mathbb{R}^d\,|\,d_i(y,X)<r\}\,.$$
If X is a subset of $\mathbb{R}^d$ included in an hyperplane of $\mathbb{R}^d$ and of co-dimension 1 (for example a non degenerate hyperrectangle), we denote by $\mathrm{hyp}(X)$ the hyperplane spanned by X, and we denote by $\mathrm{cyl}(X, h)$ the cylinder of basis X and of height $2h$ defined by
$$\mathrm{cyl}(X,h) \;=\; \{x+t\vec{v}\,|\,x\in X\,,\,t\in[-h,h]\}\,,$$
where $\vec{v}$ is one of the two unit vectors orthogonal to $\mathrm{hyp}(X)$.

Let A be a non degenerate hyperrectangle, i.e., a box of dimension $d-1$ in $\mathbb{R}^d$. All hyperrectangles will be supposed to be closed in $\mathbb{R}^d$. We denote by $\vec{v}$ one of the two unit vectors orthogonal to $\mathrm{hyp}(A)$. For h a positive real number, we consider the cylinder $\mathrm{cyl}(A,h)$. The set $\mathrm{cyl}(A,h) \smallsetminus \mathrm{hyp}(A)$ has two connected components, which we denote by $\mathcal{C}_1(A,h)$ and $\mathcal{C}_2(A,h)$. For $i=1,2$, let A_i^h be the set of the points in $\mathcal{C}_i(A,h) \cap \mathbb{Z}_n^d$ which have a nearest neighbour in $\mathbb{Z}^d \smallsetminus \mathrm{cyl}(A,h)$:

$$A_i^h \,=\, \{x \in \mathcal{C}_i(A,h) \cap \mathbb{Z}^d \,|\, \exists y \in \mathbb{Z}^d \smallsetminus \mathrm{cyl}(A,h)\,,\ \langle x,y\rangle \in \mathbb{E}^d\}\,.$$

Let $T(A,h)$ (respectively $B(A,h)$) be the top (respectively the bottom) of $\mathrm{cyl}(A,h)$, i.e.,

$$T(A,h) \,=\, \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h)\,,\ \langle x,y\rangle \in \mathbb{E}^d \text{ and } \langle x,y\rangle \text{ intersects } A+h\vec{v}\}$$

and

$$B(A,h) \,=\, \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h)\,,\ \langle x,y\rangle \in \mathbb{E}^d \text{ and } \langle x,y\rangle \text{ intersects } A-h\vec{v}\}\,.$$

For a given realization $(t(e), e \in \mathbb{E}^d)$ we define the variable $\tau(A,h) = \tau(\mathrm{cyl}(A,h),\vec{v})$ by

$$\tau(A,h) \,=\, \tau(\mathrm{cyl}(A,h),\vec{v}) \,=\, \phi(A_1^h \to A_2^h \text{ in } \mathrm{cyl}(A,h))\,,$$

and the variable $\phi(A,h) = \phi(\mathrm{cyl}(A,h),\vec{v})$ by

$$\phi(A,h) \,=\, \phi(\mathrm{cyl}(A,h),\vec{v}) \,=\, \phi(B(A,h) \to T(A,h) \text{ in } \mathrm{cyl}(A,h))\,,$$

where $\phi(F_1 \to F_2 \text{ } in \text{ } C)$ is defined previously.

We recall that as soon as

$$\int_{[0,+\infty[} x\, dF(x) < \infty\,,$$

for all function $h:\mathbb{N}\to\mathbb{R}^+$ such that $\lim_{n\to\infty} h(n) = +\infty$, the limit

$$\nu(\vec{v}) \,=\, \lim_{n\to\infty} \frac{\mathbb{E}[\tau(nA,h(n))]}{\mathcal{H}^{d-1}(nA)}$$

exists and depends only on F, d and $\vec{v}$, one of the two unit vectors normal to A, and not on A and h. Moreover, under assumptions on F, or on $\vec{v}$ and A, we also know that

$$\lim_{n\to\infty} \frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\vec{v}) \quad \text{a.s. and in } L^1\,,$$

(see the introduction of the thesis).

We will prove the following theorem:

THEOREM 5. *Let A be a non degenerate hyperrectangle, and $\vec{v}$ one of the two unit vectors normal to A. Let $h:\mathbb{N}\to\mathbb{R}^+$ such that $\lim_{n\to\infty} h(n) = +\infty$. The upper large deviations of $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$ depend on the tail of the distribution of the capacities. Indeed, we obtain that:*
i) if the law of the capacity of the edges has bounded support, then for every $\lambda > \nu(\vec{v})$ we have

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)\min(h(n),n)} \log \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda\right] \,>\, 0\,; \tag{3.1}$$

the upper large deviations are then of volume order for height functions h such that $h(n)/n$ is bounded, and of order n^d if $\lim_{n\to\infty} h(n)/n = +\infty$.
ii) if the capacity of the edges follows the exponential law of parameter 1, then there exists $n_0(d,A,h)$, and for every $\lambda > \nu(\vec{v})$ there exists a positive constant D depending only on d and λ such that for all $n \geq n_0$ we have

$$\mathbb{P}\left[\tau(nA,h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)\right] \,\geq\, \exp(-D\mathcal{H}^{d-1}(nA))\,. \tag{3.2}$$

iii) if the law of the capacity of the edges satisfies

$$\forall \gamma > 0 \qquad \int e^{\gamma x} dF(x) \,<\, \infty \,,$$

then for all $\lambda > \nu(\vec{v})$ we have

$$\lim_{n\rightarrow\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda \right] \,=\, -\infty \,. \tag{3.3}$$

We also prove the following partial result concerning the variable ϕ:

THEOREM 6. *Let A be a non degenerate hyperrectangle in $\mathbb{R}^d$, of normal unit vector $\vec{v}$, and $h : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function satisfying $\lim_{n\rightarrow\infty} h(n) = +\infty$. We suppose that the law of the capacities of the edges admits an exponential moment. Then for every $\lambda > \nu(\vec{v})$, we have*

$$\limsup_{n\rightarrow\infty} \frac{1}{\mathcal{H}^{d-1}(nA)h(n)} \log \mathbb{P}[\phi(nA,h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)] \,<\, 0 \,.$$

REMARK 8. We recall the reader that the asymptotic behaviour of $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ for large n is not known in general. For straight cylinders, i.e., cylinders of basis A of the form $\prod_{i=1}^{d-1}[a_i,b_i]\times\{c\}$ with real numbers a_i, b_i and c, we know thanks to the works of Kesten [**41**] and Zhang [**59**] that $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ converges a.s. towards $\nu((0,...,0,1))$ when n goes to infinity, and in this case the upper large deviations of $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ have been studied in the Chapter 2: they are of volume order, and we could even prove the corresponding large deviation principle. For tilted cylinders, we don't know the asymptotic behaviour of this variable (see Chapter 4 and Chapter 5 for partial answers), but looking at the trivial case where $t(e) = 1$ for every edge e, we can easily see that $\tau(nA,h(n))$ and $\phi(nA,h(n))$ do not have the same behaviour for large n. However, in the case where $\lim_{n\rightarrow\infty} h(n)/n = 0$, we also know that $\lim_{n\rightarrow\infty}\phi(nA,h(n))/\mathcal{H}^{d-1}(nA) = \nu(\vec{v})$ almost surely, at least under hypotheses on F, or on $\vec{v}$ and A, so in this case we really study here the upper large deviations of the variable $\phi(nA,h(n))$.

REMARK 9. Even for the variable $\tau(nA,h(n))$, when we know that $\nu(\vec{v})$ is the almost sure limit of the rescaled flow under some hypotheses on F or on $\vec{v}$ and A, we only investigated the upper large deviations of the variable, and we did not prove the corresponding large deviation principle. The idea used in Chapter 2 to prove a large deviation principle for the variable $\phi(nA,h(n))$ in straight cylinders is the following: we pile cylinders, and we let a large amount of flow cross the cylinders one after each other, using the fact that the top of a cylinder, i.e. the area through which the water goes out of this cylinder, is exactly the bottom of the cylinder above, i.e. the area through which the water can go into that cylinder. We cannot use the same method to prove a large deviation principle for $\tau(nA,h(n))$, even in straight cylinders, because in this case we cannot glue together the entire area through which the water goes out of a cylinder with the entire area through which the water goes into the cylinder above. In the case of tilted cylinders we even loose the symmetry of the graph with regard to the hyperplanes spanned by the faces of the cylinder. These symmetries were of huge importance in the proof of the large deviation principle from above for $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ in Chapter 2.

2. Upper large deviations for the rescaled variable $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$

2.1. Geometric construction. To study these upper large deviations, we will use exactly the same idea as in the proof of the strict positivity of the rate function of the large deviation principle we proved in Chapter 2. Thus the main tool is the Cramér Theorem in $\mathbb{R}$. We will consider two different scales on the graph, i.e., cylinders of two different sizes indexed by n and N, with N very large compared to n. We want to divide the cylinder $\mathrm{cyl}(NA,h(N))$ into images of $\mathrm{cyl}(nA, h(n))$ by integer translations, and to compare the maximal flows through these cylinders. In fact, we will first fill $\mathrm{cyl}(NA, h(N))$ with translates of $\mathrm{cyl}(nA, h(n))$ and then move slightly these translates to obtain integer translates. The problem is that we want to obtain disjoint small cylinders so that the associated flows are independent, therefore we need some extra space between the different images of $\mathrm{cyl}(nA, h(n))$ in order to move them separately and to obtain disjoint cylinders. The sets of edges we will have to add in order to glue together the different cutsets in the small cylinders to obtain a cutset in the big one are needed to correct the lack of subadditivity of the maximal flow in the case of tilted cylinders, as explained in the introduction of this thesis. The last remark we have to do before the beginning of the complete proof is that we may not divide the entire cylinder $\mathrm{cyl}(NA, h(N))$ into slabs, but a possibly smaller one, namely $\mathrm{cyl}(NA, Mh(n))$ for $Mh(n) \leq h(N)$. Indeed, we will see that the upper large deviations of $\tau(NA, h(N))$ are related to the behaviour of the edges of the cylinder that are "not too far" from NA, because the cutset is pinned at the boundary of NA so he cannot explore regions too far away from NA in $\mathrm{cyl}(NA, h(N))$.

Let $\lambda > \nu(\vec{v})$ and $\varepsilon > 0$ such that $\lambda > \nu(\vec{v}) + 3\varepsilon$. We take an h as in theorem 5, a large N (we will precise how large it is), and a smaller n. We define $\mathrm{cyl}'(nA, h(n))$ as

$$\mathrm{cyl}'(nA, h(n)) \,=\, \{x + t\vec{v} \,|\, x \in \mathrm{hyp}(A)\,,\; d(x, nA) \leq \zeta/2 \;and\; t \in [-h(n) - \zeta/2, h(n) + \zeta/2]\}\,.$$

We fix an $M = M(n, N)$ such that $M(2h(n) + \zeta) \leq 2h(N)$. We divide $\mathrm{cyl}(NA, M(h(n) + \zeta/2))$ into slabs S_i, $i = 1, ..., M(n, N)$, of the form

$$S_i \,=\, \{x + t\vec{v} \,|\, x \in NA\,,\; t \in \mathcal{T}_i\}$$

where

$$\mathcal{T}_i \,=\, [-M(h(n) + \zeta/2) + (i-1)(2h(n) + \zeta), -M(h(n) + \zeta/2) + i(2h(n) + \zeta)]$$

(see Figure 1). By a euclidean division of the dimensions of S_i, we divide then each S_i into m translates of $\mathrm{cyl}'(nA, h(n))$, which we denote by $S'_{i,j}$, $j = 1, ..., m$, plus a remaining part $S'_{i,m+1}$. Here m is smaller than $\mathcal{M}(n, N) = \lfloor \mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA) \rfloor$. Each $S'_{i,j}$ is a translate of $\mathrm{cyl}'(nA, h(n))$, which contains $\mathrm{cyl}(nA, h(n))$, and so we denote by $D_{i,j}$ the corresponding translate of $\mathrm{cyl}(nA, h(n))$ by the same translation ($D_{i,j} \subset S'_{i,j}$). See Figure 2 which illustrates these definitions.

For all (i, j) there exists a vector $\vec{u}_{i,j}$ in $\mathbb{R}^d$ such that $\|\vec{u}_{i,j}\|_\infty < 1$ and $B_{i,j} = D_{i,j} + \vec{u}_{i,j}$ is the image of $\mathrm{cyl}(nA, h(n))$ by an integer translation, i.e., a translation whose vector has integer coordinates; moreover we have $B_{i,j} \subset S'_{i,j}$, so the $B_{i,j}$ are disjoint. We define $\tau_i = \tau(S_i, \vec{v})$ and $\tau_{i,j} = \tau(B_{i,j}, \vec{v})$. Now we denote by E_1 the set of the edges which belong to $\mathcal{E}_1 \subset \mathbb{R}^d$ defined by

$$\mathcal{E}_1 \,=\, \{x + t\vec{v} \,|\, x \in NA\,,\; d(x, \partial(NA)) \leq 2\zeta \;and\; t \in [-M(h(n) + \zeta/2), M(h(n) + \zeta/2)]\}\,.$$

We denote also by $E_{0,i}$ the set of the edges which belong to $\mathcal{E}_{0,i} \subset \mathbb{R}^d$ defined by

$$\mathcal{E}_{0,i} \,=\, \{x + t\vec{v} \,|\, x \in NA\,,\; t \in \mathcal{T}'_i\} \cap \left(\bigcup_{j=1}^{m} \mathcal{V}(\partial S'_{i,j}, 3\zeta) \cup S'_{i,m+1} \right)\,,$$

where

$$\mathcal{T}'_i \,=\, [-h(N) + (i - 1/2)(2h(n) + \zeta) - 3\zeta, -h(N) + (i - 1/2)(2h(n) + \zeta) + 3\zeta]\,.$$

For all $i \in \{1, ..., M(n, N)\}$, if we denote by $\mathcal{F}_{i,j}$ a set of edges that cuts the lower half part from the upper half part of the cylinder $B_{i,j}$, then $\cup_{j=1}^{m} \mathcal{F}_{i,j} \cup E_{0,i} \cup E_1$ separates the lower half part from the upper half part of $\mathrm{cyl}(NA, h(N))$. Thus we obtain that

$$\forall i \in \{1, ..., M(n, N)\}, \qquad \tau(NA, h(N)) \,\leq\, \sum_{j=1}^{m} \tau_{i,j} + V(E_1 \cup E_{0,i})\,,$$

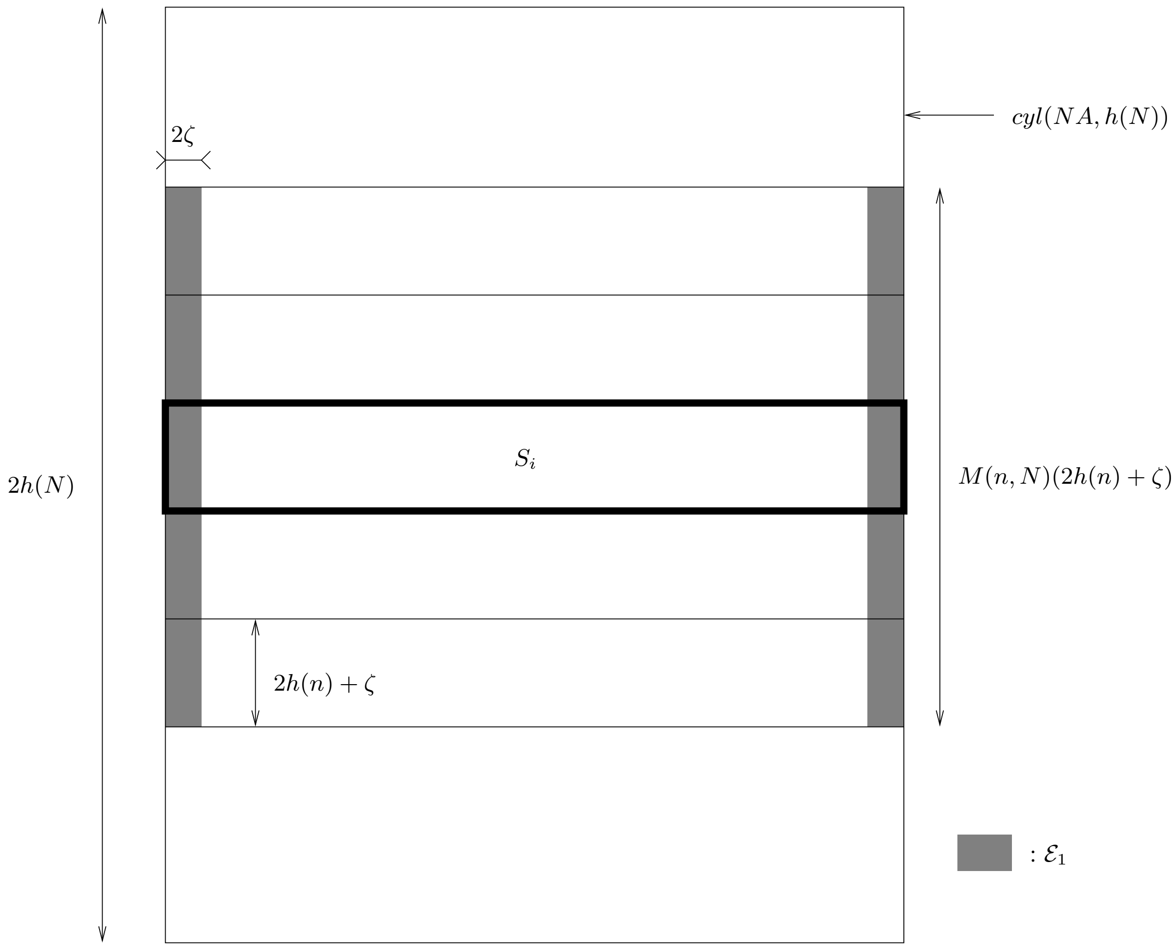

FIGURE 1. cyl$(NA, h(N))$ and S_i.

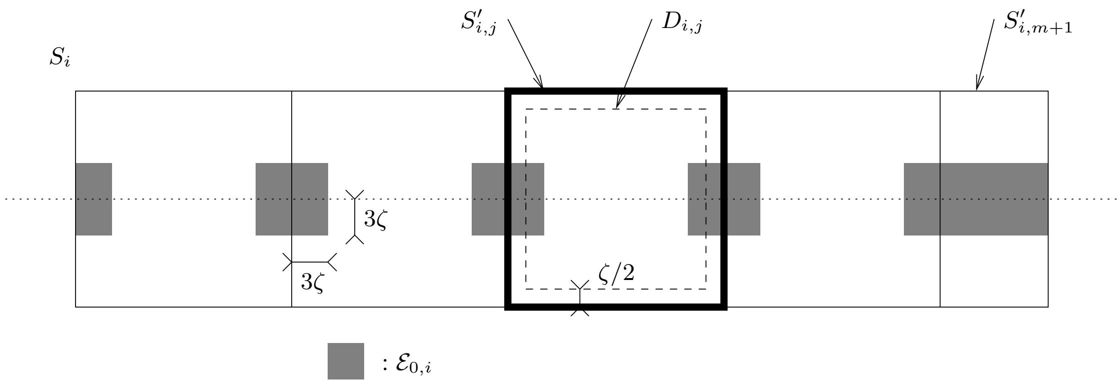

FIGURE 2. The slab S_i.

so

$$\mathbb{P}\Big[\tau(NA, h(N)) \geq \lambda\mathcal{H}^{d-1}(NA)\Big]$$
$$\leq \mathbb{P}\left[\forall i \in \{1, ..., M(n,N)\}\,,\ \sum_{j=1}^{m} \tau_{i,j} + V(E_1 \cup E_{0,i}) \geq \lambda\mathcal{H}^{d-1}(NA)\right]$$

$$\begin{aligned}&\leq \mathbb{P}\left[\forall i \in \{1, ..., M(n,N)\}\,,\ \sum_{j=1}^{m} \tau_{i,j} \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right]\\&\quad + \mathbb{P}\left[V(E_1) \geq \varepsilon \mathcal{H}^{d-1}(NA)/2\right]\\&\quad + \mathbb{P}\left[\exists i \in \{1, ..., M(n,N)\}\,,\ V(E_{0,i}) \geq \varepsilon \mathcal{H}^{d-1}(NA)/2\right]\,.\end{aligned}$$

We study the different probabilities appearing here separately.

• Let

$$\alpha(N,n) \,=\, \mathbb{P}\left[\forall i \in \{1, ..., M(n,N)\}\,,\ \sum_{j=1}^{m} \tau_{i,j} \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right]\,.$$

By independence of the families $(\tau_{i,j}, j = 1, ..., m)$ for different i we have

$$\begin{aligned}\alpha(N,n) \,&=\, \mathbb{P}\left[\sum_{j=1}^{m} \tau_{1,j} \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right]^{M(n,N)}\\&\leq\, \mathbb{P}\left[\sum_{j=1}^{\mathcal{M}(n,N,A)} \tau_{1,j} \geq (\lambda-\varepsilon)\mathcal{H}^{d-1}(NA)\right]^{M(n,N)}\\&\leq\, \mathbb{P}\left[\frac{1}{\mathcal{M}(n,N,A)} \sum_{j=1}^{\mathcal{M}(n,N,A)} \frac{\tau_n^{(j)}}{\mathcal{H}^{d-1}(nA)} \geq \lambda-\varepsilon\right]^{M(n,N)}\,,\end{aligned}$$

where we remember that

$$\mathcal{M}(n,N,A) \,=\, \lfloor \mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA) \rfloor\,,$$

and $(\tau_n^{(j)}, j \in \mathbb{N})$ is a family of independent and identically distributed variables with $\tau_n^{(j)} = \tau(nA, h(n))$ in law. We know that $\mathbb{E}(\tau(nA, h(n)))/\mathcal{H}^{d-1}(nA)$ converges to $\nu(\vec{v})$ when n goes to infinity so there exists n_0 large enough to have for all $n \geq n_0$

$$\frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(nA)} \,\leq\, \nu(\vec{v}) + \varepsilon \,<\, \lambda - \varepsilon\,.$$

In the three cases presented in Theorem 5, the law of the capacity of the edges admits at least one exponential moment, and by an easy comparison between $\tau(nA, h(n))$ and the capacity of a fixed flat cutset in $\mathrm{cyl}(nA, h(n))$, we obtain that $\tau(nA, h(n))$ admits an exponential moment. We can then apply the Cramér theorem to obtain that for fixed $n \geq n_0$ and λ there exists a constant c (depending on the law of $\tau(nA, h(n))$, λ and ε) such that

$$\limsup_{N\to\infty} \frac{1}{\mathcal{M}(n,N,A)} \log \mathbb{P}\left[\frac{1}{\mathcal{M}(n,N,A)} \sum_{j=1}^{\mathcal{M}(n,N,A)} \frac{\tau_n^{(j)}}{\mathcal{H}^{d-1}(nA)} \geq \lambda-\varepsilon\right] \,\leq\, c \,<\, 0\,,$$

and so for all $n \geq n_0$ and λ there exists a constant c' (depending on the law of $\tau(nA, h(n))$, λ and ε) such that

$$\limsup_{N\to\infty} \frac{1}{M(n,N)\mathcal{H}^{d-1}(NA)} \log \alpha(N,n) \,<\, c' < 0\,. \tag{3.4}$$

• To study the two other terms, we can study more generally the behaviour of

$$\gamma(n,N) \,=\, \mathbb{P}\left[\sum_{i=1}^{l(n,N)} t(e_i) \geq \varepsilon \mathcal{H}^{d-1}(NA)/2\right]\,.$$

We know that there exists a positive constant C depending on d, A and ζ such that

$$\operatorname{card}(E_{0,i}) \,\leq\, C\left(\frac{N^{d-1}}{n} + N^{d-2}n\right) \tag{3.5}$$

and

$$\operatorname{card}(E_1) \,\leq\, CN^{d-2}M(n,N)h(n)\,. \tag{3.6}$$

The values of $l(n,N)$ we will consider are

$$l_0(n,N) \,=\, C(N^{d-1}n^{-1} + N^{d-2}n) \quad \text{and} \quad l_1(n,N) \,=\, CN^{d-2}M(n,N)h(n)\,.$$

The behaviour of the quantity $\gamma(n,N)$ will depend on the law of the capacity we will consider.

2.2. Bounded capacities. We suppose that the capacity of the edges is bounded by a constant K. Then as soon as

$$2Kl(n,N) \,<\, \varepsilon\mathcal{H}^{d-1}(NA)\,, \tag{3.7}$$

we know that $\gamma(n,N) = 0$. It is obvious that there exists a n_0 such that for all fixed $n \geq n_0$, for all large N (how large depending on n), equation (3.7) is satisfied by $l_0(n,N)$. Moreover, there exists a constant $\kappa(n,A,d,\zeta,F)$ such that if $M(n,N) \leq \kappa N$, then equation (3.7) is also satisfied by $l_1(n,N)$ for all n. We choose $M(n,N)$ to be as large as possible according to the condition we just mentioned, and the fact that $M(n,N) \leq h(N)(h(n)+\zeta/2)^{-1}$; we define $\kappa'(n) = (h(n)+\zeta/2)^{-1}$ and we choose

$$M(n,N) \,=\, \min(\kappa(n)N, \kappa'(n)h(N))\,.$$

Thus, for a fixed $n \geq n_0$, for all N large enough, we obtain that

$$\mathbb{P}\left[V(E_1) \geq \varepsilon\mathcal{H}^{d-1}(NA)/2\right] + \mathbb{P}\left[\exists i \in \{1,...,M(n,N)\}\,,\, V(E_{0,i}) \geq \varepsilon\mathcal{H}^{d-1}(NA)/2\right] \,=\, 0$$

and then thanks to equation (3.4) we obtain that

$$\limsup_{N\to\infty} \frac{1}{\mathcal{H}^{d-1}(NA)\min(N,h(N))} \log \mathbb{P}\left[\frac{\tau(NA,h(N))}{\mathcal{H}^{d-1}(NA)} \geq \lambda\right] \,<\, 0\,,$$

so equation (3.1) is proved.

REMARK 10. The term $\min(n,h(n))$ can seem strange in (3.1). We try here to explain where it comes from. From the point of view of a minimal cutset, the heuristic is that a cutset in $\operatorname{cyl}(nA,h(n))$ separating the two half cylinders is pinned along the boundary of nA, so he cannot explore domains of $\operatorname{cyl}(nA,h(n))$ that are too far away from nA, i.e., at distance of order larger than n. We think it is this point of view that gives the best intuitive idea of how things work, but actually it is very difficult to study the position of a minimal cutset in the cylinder, so we cannot use this idea to prove the dependence of upper large deviations in $\min(n,h(n))$. From the point of view of the maximal flow, we can also understand where this term $\min(h(n),n)$ comes from. In fact, we can find of the order of n^{d-1} disjoint paths (i.e., with no common edge) that cross $\operatorname{cyl}(nA,h(n))$ from its upper half part to its lower half part using only the edges located at distance smaller than Kn of nA for some constant K (thus all the edges of the box if $h(n)/n$ is bounded). If $h(n)/n$ is bounded, we can consider paths that cross the cylinder from its top to its bottom, and if $h(n) \geq n$, we can consider paths that form a part of a loop around a point of $\partial(nA)$ - so they join two points of $\operatorname{cyl}(\partial(nA),Kn)$ that are on the same side of $\operatorname{cyl}(nA,h(n))$ and that are symmetric one to each other by the reflexion of axis the intersection of $\partial(nA)$ with this side (see figure 3 that shows these paths in dimension 2). Thus, if all the edges at distance smaller that Kn of nA in the cylinder have a big capacity, then the variable $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$ is abnormally big. The number of such edges is of order $n^{d-1}\min(n,h(n))$. We emphasize here

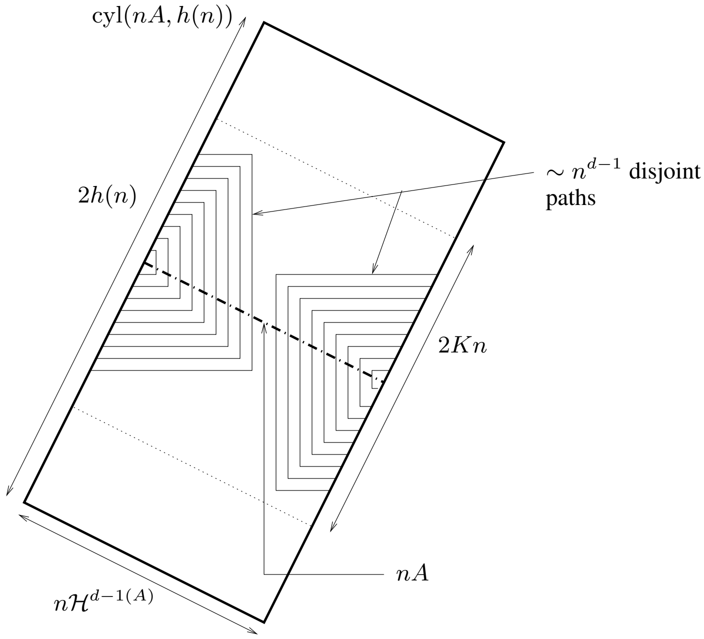

FIGURE 3. Disjoint paths near nA in dimension two.

the fact that $\phi(nA, h(n))$ does not have these properties, this is the reason why we expect for this variable upper large deviations of volume order for all functions h.

2.3. Capacities of exponential law. The goal of this short study is to emphasize the fact that the condition of having one exponential moment for the law of the capacity of the edges is not sufficient to obtain the speed of decay that we have with bounded capacities. We will consider a particular law, namely the exponential law of parameter 1, and show that we do not have upper deviations of volume order in this case.

We suppose that the law of the capacity of the edges is the exponential law of parameter 1. We know that $\mathbb{E}(\exp(\gamma t)) < \infty$ for all $\gamma < 1$. Let x_0 be a fixed point of the boundary $\partial(nA)$. We know that there exists a path from the lower half cylinder $(nA)_2^{h(n)}$ to the upper half cylinder $(nA)_1^{h(n)}$ in $\mathrm{cyl}(nA, h(n))$ that is included in the neighbourhood of x_0 of diameter $\zeta \geq 2d$ for the euclidean distance, as soon as $n \geq n_0(d, A, h)$, where $n_0(d, A, h)$ is the infimum of the n such that all the sidelengths of the cylinder $\mathrm{cyl}(nA, h(n))$ are larger than ζ (see figure 4). Thus for all $n \geq n_0$, every set of edges that cuts the upper half cylinder $(nA)_1^{h(n)}$ from the lower half cylinder $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA, h(n))$ must contain one of the edges of this neighbourhood of x_0. The number of such edges is at most $K(d, \zeta)$, where K is a constant depending only on d and ζ. Thus the probability that all of them have a capacity bigger than $\lambda\mathcal{H}^{d-1}(nA)$ for a $\lambda > \nu(\vec{v})$ is greater than $\exp(-K\lambda\mathcal{H}^{d-1}(nA))$. We obtain that for all $n \geq n_0(d, A, h)$,

$$\mathbb{P}\left[\tau(nA, h(n)) \geq \lambda\mathcal{H}^{d-1}(nA)\right] \geq \exp(-K\lambda\mathcal{H}^{d-1}(nA)) .$$

2.4. Capacities with exponential moments of all orders. We suppose that the capacity of the edges admits exponential moments of all order, i.e., for all $\theta > 0$ we have $\mathbb{E}(\exp(\theta t(e))) < \infty$. Then by a simple application of the Chebyshev inequality, we obtain that

$$\gamma(n, N) \leq \exp\left[-\mathcal{H}^{d-1}(NA)\left(\frac{\theta\varepsilon}{2} - \frac{l(n, N)\log\mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)}\right)\right] . \tag{3.8}$$

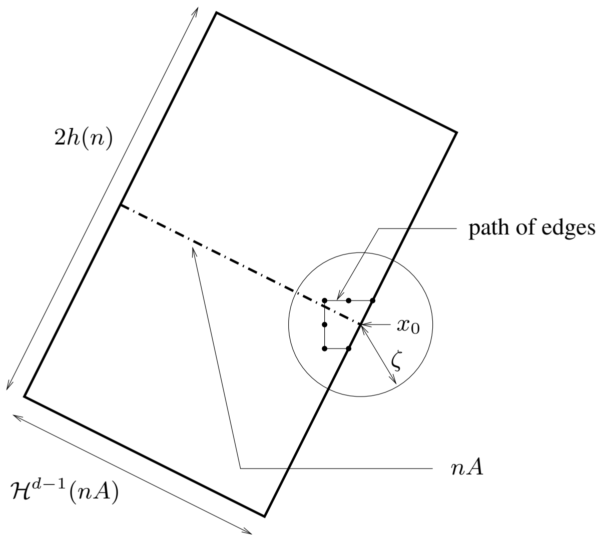

FIGURE 4. Path of edges included in a neighbourhood of x_0.

We want to be able to choose the term

$$\frac{\theta\varepsilon}{2} - \frac{l(n,N)\log\mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)}$$

as big as we want. For a fixed $R > 0$, we can take $\theta > 0$ large enough to have $\theta\varepsilon \geq 4R$. If there exists n_1 such that for a fixed $n \geq n_1$, for all N sufficiently large (how large depends on n), we have

$$\frac{l(n,N)}{\mathcal{H}^{d-1}(NA)}\log\mathbb{E}(e^{\theta t(e)}) \,\leq\, R \tag{3.9}$$

then for a fixed $n \geq n_1$, for all large N, we would obtain

$$\gamma(n,N) \,\leq\, \exp\left(-R\mathcal{H}^{d-1}(NA)\right)\,.$$

We consider now the values of $l_0(n,N)$ and $l_1(n,N)$. Looking at $l_1(n,N)$, we realize that we have to impose a condition on $M(n,N)$. Considering the result we want to prove, we can choose $M(n,N)$ satisfying, for each fixed n,

$$\lim_{N\rightarrow\infty}\frac{M(n,N)}{N} \,=\, 0 \qquad \text{and} \qquad \lim_{N\rightarrow\infty} M(n,N) \,=\, +\infty\,.$$

If $h(N)/N$ does not converge towards 0, we thus consider a small cylinder inside the cylinder $\mathrm{cyl}(NA,h(N))$, but we impose that its height goes to infinity with N. Under this hypothesis, we obtain that for all R, for every fixed n, for all large N, condition (3.9) is satisfied by $l_1(n,N)$. Thus, for all fixed n, thanks to (3.6), we obtain that

$$\limsup_{N\rightarrow\infty}\frac{1}{\mathcal{H}^{d-1}(NA)}\log\mathbb{P}\left[V(E_1)\geq\varepsilon\mathcal{H}^{d-1}(NA)/2\right] \,=\, -\infty\,. \tag{3.10}$$

For all R, we can find a n_1 such that for all $n \geq n_1$, for all large N, the condition (3.9) is satisfied by $l_0(n,N)$. Since our choice of $M(n,N)$ implies that

$$\lim_{N\rightarrow\infty}\frac{\log M(n,N)}{\mathcal{H}^{d-1}(NA)} \,=\, 0\,,$$

thanks to (3.5), we obtain that for all fixed $n \geq n_1$,

$$\limsup_{N\rightarrow\infty}\frac{1}{\mathcal{H}^{d-1}(NA)}\log\mathbb{P}\left[\exists i\in\{1,...,M(n,N)\}\,,\,V(E_{0,i})\geq\varepsilon\mathcal{H}^{d-1}(NA)/2\right] \,=\, -\infty\,. \tag{3.11}$$

Combining (3.10), (3.11) and (3.4), since $\lim_{N\to\infty} M(n,N) = +\infty$, we have proved (3.3). This ends the proof of Theorem 5.

3. Partial result concerning the upper large deviations for $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ through a tilted cylinder

We have already written the main part of the proof of Theorem 6 in the previous section. We keep all the notations introduced previously. The proof of Theorem 5 was based on the following inequality:

$$\forall i \in \{1, ..., M(n,N)\}, \qquad \tau(NA, h(N)) \,\leq\, \sum_{j=1}^{m} \tau_{i,j} + V(E_1 \cup E_{0,i})\,.$$

We recall that this inequality was obtained by noticing that if $\mathcal{F}_{i,j}$ is a cutset that separates the upper half part from the lower half part of $B_{i,j}$, then $\cup_{j=1}^{m} \mathcal{F}_{i,j} \cup E_{0,i} \cup E_1$ separates the upper half part from the lower half part of $\mathrm{cyl}(NA, h(N))$. Here we want to construct a cutset that separates the bottom from the top of $\mathrm{cyl}(NA, h(N))$. We have no need to add the set of edges E_1 in this context because we do not need to obtain a cutset that is pinned at $\partial(NA)$. Thus for all i, $\cup_{j=1}^{m} \mathcal{F}_{i,j} \cup E_{0,i}$ cuts the top from the bottom of $\mathrm{cyl}(NA, h(N))$, and then we have

$$\forall i \in \{1, ..., M(n,N)\}\,, \qquad \phi(NA, h(N)) \,\leq\, \sum_{j=1}^{m} \tau_{i,j} + V(E_{0,i})\,.$$

We obtain that for a fixed $\lambda > \nu(\vec{v})$, and ε such that $\lambda \geq \nu(\vec{v}) + 3\varepsilon$, we have by independence

$$\begin{aligned}
&\mathbb{P}[\phi(NA, h(N)) \geq \lambda \mathcal{H}^{d-1}(NA)] \\
&\leq \mathbb{P}\left[\bigcap_{i=1}^{M(n,N)} \left\{ \sum_{j=1}^{m} \tau_{i,j} + V(E_{0,i}) \geq \lambda \mathcal{H}^{d-1}(NA) \right\}\right] \\
&\leq \prod_{i=1}^{M(n,N)} \left(\mathbb{P}\left[\sum_{j=1}^{m} \tau_{i,j} \geq (\lambda - \varepsilon)\mathcal{H}^{d-1}(NA)\right] + \mathbb{P}\left[V(E_i) \geq \varepsilon \mathcal{H}^{d-1}(NA)\right] \right).
\end{aligned}$$

We consider here the maximal $M(n,N)$, i.e.,

$$M(n,N) \,=\, \left\lfloor \frac{h(N)}{h(n) + \zeta/2} \right\rfloor\,.$$

Indeed, we do not need to make any restriction on $M(n,N)$ because we do not have to consider the set of edges E_1 whose cardinality depends on $M(n,N)$.

From now on we suppose that the capacity of the edges admits an exponential moment. Thanks to the application of the Cramér theorem we have already done to obtain (3.4), we know that for all $n \geq n_0$ there exists a positive c' (depending on the law of $\tau(nA, h(n))$, λ and ε) such that

$$\limsup_{N\to\infty} \frac{1}{\mathcal{H}^{d-1}(NA)} \log \mathbb{P}\left[\sum_{j=1}^{m} \tau_{i,j} \geq (\lambda - \varepsilon)\mathcal{H}^{d-1}(NA)\right] \,\leq\, c' \,<\, 0\,. \tag{3.12}$$

On the other hand, let $\theta > 0$ be such that $\mathbb{E}(\exp(\theta t(e))) < \infty$. Thanks to equation (3.8), obtained by the Chebyshev inequality, and (3.5), we have for this fixed θ:

$$\begin{aligned}
\mathbb{P}[V(E_{0,i}) \geq \varepsilon \mathcal{H}^{d-1}(NA)] \,&\leq\, \mathbb{P}\left[\sum_{i=1}^{l_0(n,N)} t(e_i) \geq \varepsilon \mathcal{H}^{d-1}(NA)\right] \\
&\leq\, \exp\left[-\mathcal{H}^{d-1}(NA)\left(\frac{\theta\varepsilon}{2} - \frac{l_0(n,N)\log \mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)}\right)\right].
\end{aligned}$$

Since $l_0(n,N) \leq C(N^{d-1}n^{-1} + N^{d-2}n)$, we know that there exists n_1 such that for all $n \geq n_1$, for all N large enough (how large depending on n), we have

$$\frac{l_0(n,N)\log \mathbb{E}(\exp(\theta t(e)))}{\mathcal{H}^{d-1}(NA)} \leq \frac{\theta\varepsilon}{4},$$

and then

$$\mathbb{P}[V(E_{0,i}) \geq \varepsilon\mathcal{H}^{d-1}(NA)] \leq \exp\left(-\mathcal{H}^{d-1}(NA)\frac{\theta\varepsilon}{4}\right). \tag{3.13}$$

Combining equations (3.12) and (3.13), since $M(n,N)$ is proportional to $h(N)$ for a fixed n, Theorem 6 is proved.

Part 2

Lower large deviations for maximal flows in cylinders

CHAPTER 4

On the small maximal flows from the top to the bottom of a straight cylinder

We consider the standard first passage percolation on $\mathbb{Z}^d$: with each edge of the lattice we associate a random capacity. We are interested in the maximal flow through a cylinder in this graph. Under some assumptions Kesten proved in 1987 a law of large numbers for the rescaled flow. Chayes and Chayes established that the large deviations far away below its typical value are of surface order, at least for the Bernoulli percolation and cylinders of certain height. Thanks to another approach we extend here their result to higher cylinders, and we transport this result to the model of first passage percolation.

Contents

1. Definitions and main result

We use the notations introduced in [**40**] and [**41**]. Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$ the set of the pairs of nearest neighbours for the standard L^1 norm. With each edge e in $\mathbb{E}^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}^d)$ is independent and identically distributed, with a common distribution function F. More formally, we take the product measure $\mathbb{P}$ on $\Omega = \prod_{e\in\mathbb{E}^d}[0,\infty[$, and we write its expectation $\mathbb{E}$. We interpret $t(e)$ as the capacity of the edge e; it means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time. For a given realization $(t(e), e \in \mathbb{E}^d)$ we denote by $\phi_{\vec{k},m} = \phi_B$ the maximal flow through the box

$$B(\vec{k}, m) \,=\, \prod_{i=1}^{d-1}[0, k_i] \times [0, m]\,,$$

where $\vec{k} = (k_1, ..., k_{d-1}) \in \mathbb{Z}^{d-1}$, from its bottom

$$F_0 \,=\, \prod_{i=1}^{d-1}[0, k_i] \times \{0\}$$

to its top

$$F_m \,=\, \prod_{i=1}^{d-1} [0, k_i] \times \{m\}\,.$$

Let us define this quantity properly. We remember that $\mathbb{E}^d$ is the set of the edges of the graph. An edge $e \in \mathbb{E}^d$ can be written $e = \langle x, y \rangle$, where $x,\, y \in \mathbb{Z}^d$ are the endpoints of e. We will say that $e = \langle x, y \rangle$ belongs to a subset A of $\mathbb{R}^d$ ($e \in A$) if the segment joining x to y (eventually excluding these points) is included in A. Now we define $\widetilde{\mathbb{E}}^d$ as the set of all the oriented edges, i.e. an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}^d$ is an ordered pair of vertices. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}^d$ by $\langle\langle x, y \rangle\rangle$, where $x,\, y \in \mathbb{Z}^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from x towards y. We consider now the set $\mathcal{S}$ of all pairs of functions (g, o), with $g : \mathbb{E}^d \rightarrow \mathbb{R}^+$ and $o : \mathbb{E}^d \rightarrow \widetilde{\mathbb{E}}^d$ such that $o(\langle x, y \rangle) \in \{\langle\langle x, y \rangle\rangle, \langle\langle y, x \rangle\rangle\}$, satisfying

- for each edge e in B we have

$$0 \,\leq\, g(e) \,\leq\, t(e)\,,$$

- for each vertex v in $B \smallsetminus F_m$ we have

$$\sum_{e \in B \,:\, o(e) = \langle\langle v, \cdot \rangle\rangle} g(e) \,=\, \sum_{e \in B \,:\, o(e) = \langle\langle \cdot, v \rangle\rangle} g(e)\,.$$

A couple $(g, o) \in \mathcal{S}$ is a possible stream in B: $g(e)$ is the amount of fluid that goes through the edge e, and $o(e)$ gives the direction in which the fluid goes through e. The two conditions on (g, o) express only the fact that the amount of fluid that can go through an edge is bounded by its capacity, and that there is no loss of fluid in the cylinder. With each possible stream we associate the corresponding flow

$$\begin{aligned} flow(g, o) \,=\, \sum_{u \notin F_m\,,\, v \in F_m \,:\, \langle u, v \rangle \in \mathbb{E}^d \cap B} & g(\langle u, v \rangle) \mathbb{1}_{o(\langle u, v \rangle) = \langle\langle u, v \rangle\rangle} \\ & - g(\langle u, v \rangle) \mathbb{1}_{o(\langle u, v \rangle) = \langle\langle v, u \rangle\rangle}\,. \end{aligned}$$

This is the amount of fluid that crosses the cylinder B if the fluid respects the stream (g, o). The maximal flow through the cylinder B is the supremum of this quantity over all possible choices of stream

$$\phi_B \,=\, \phi_{\vec{k}, m} \,=\, \sup_{(g, o) \in \mathcal{S}} flow(g, o)\,.$$

We denote by $p_c(d)$ the critical value of the parameter of the Bernoulli percolation in dimension d. We will prove the following result:

THEOREM 7. *We suppose that*

$$F(0) \,<\, 1 - p_c(d)\,.$$

There exist a positive constant ε_0, depending only on d and F, and a positive constant C, depending only on d, such that for any function $h : \mathbb{N} \rightarrow \mathbb{N}$ satisfying

$$\lim_{n \rightarrow \infty} \frac{\ln h(n)}{n^{d-1}} \,=\, 0$$

we have

$$\forall \varepsilon < \varepsilon_0 \qquad \liminf_{n \rightarrow \infty} -\frac{1}{n^{d-1}} \ln \mathbb{P}\left[\phi_{(n, \dots, n), h(n)} \leq \varepsilon n^{d-1}\right] \geq C \,>\, 0\,.$$

The condition $F(0) < 1 - p_c$ is necessary for this result to hold. Indeed, Yu Zhang (see [**58**]) proved in dimension 3 that for a function F satisfying

$$F(0) \,=\, 1 - p_c \qquad and \qquad \int_{[0, +\infty[} x dF(x) \,<\, \infty$$

we have

$$\lim_{k,l,m\to\infty} \frac{\phi_{(k,l),m}}{kl} \,=\, 0\,.$$

The spirit of this result is not new, Chayes and Chayes proved in [**20**] (see Lemma 3.3) the following theorem:

THEOREM 8. *We suppose that the capacity t of each edge follows a Bernoulli law of parameter p satisfying $p > p_c$. Then there exist positive constants $\widetilde{\varepsilon}$, $\widetilde{C}$ such that*

$$\mathbb{P}\left[\phi_{(n,\dots,n),n} \geq \widetilde{\varepsilon} n^{d-1}\right] \,\geq\, 1 - e^{-\widetilde{C}n^{d-1}}$$

for n sufficiently large.

To prove it they divide the cylinder into thin layers, compare each one of them to objects of dimension 2 and use the results of [**2**]. Because of the passage in dimension 2, it seems to us that this proof can only be extended to cylinders $B((n,\dots,n),h(n))$ with a height satisfying $\lim_{n\to\infty} \ln h(n)/n \,=\, 0$. This is the constraint we have in dimension 2, but not in higher dimensions. Actually the condition $\lim_{n\to\infty} \ln h(n)/n^{d-1} \,=\, 0$ is the good one, in the sense that in the model of Bernoulli percolation if $h(n) \,=\, \exp(kn^{d-1})$ for a constant k sufficiently large, the maximal flow $\phi_{(n,\dots,n),h(n)}$ tends to 0 almost surely. Indeed if the n^{d-1} vertical edges of the cylinder that intersect one fixed horizontal plane have all 0 for capacity then $\phi_{(n,\dots,n),h(n)} = 0$. By independence and translation invariance of the model, we obtain, for k large enough

$$\mathbb{P}\left[\phi_{(n,\dots,n),h(n)} \neq 0\right] \,\leq\, \left[1-(1-p)^{n^{d-1}}\right]^{h(n)} \,\to_{n\to\infty}\, 0\,.$$

The proof of Theorem 7 is based on the coarse graining techniques of Pisztora (see [**49**]). Actually we don't need estimates as strong as those of Pisztora for the renormalization scheme. We will use a weaker version of these results as in [**19**]. Moreover we won't use the general stochastic domination inequality (see [**49**], [**46**]), it is sufficient here to use a partition of the space into equivalence classes to get rid of problems of dependence between random variables, as we will see in section 4.2.

We will first study two particular cases of this result in the model of Bernoulli percolation, that will allow us to deal very simply with the proof of the main theorem in general first passage percolation.

2. Max-flow min-cut theorem

The definition of the flow is not easy to deal with. The maximal flow ϕ_B can be expressed differently thanks to the max-flow min-cut theorem (see [**12**]). We need some definitions.

A path on a graph ($\mathbb{Z}^d$ for example) from v_0 to v_n is a sequence $(v_0, e_1, \dots, e_n, v_n)$ of vertices $v_0, \dots, v_n$ alternating with edges $e_1, \dots, e_n$ such that v_{i-1} and v_i are neighbours in the graph, joined by the edge e_i, for i in $\{1, \dots, n\}$. Two paths are said disjoint if they have no common edge.

A set E of edges of $B(\vec{k},m)$ is said to separate F_0 from F_m in $B(\vec{k},m)$ if there is no path from F_0 to F_m in $B(\vec{k},m) \smallsetminus E$. We call E an (F_0,F_m)-cut if E separates F_0 from F_m in $B(\vec{k},m)$ and if no proper subset of E does. With each set E of edges we associate the variable

$$V(E) \,=\, \sum_{e\in E} t(e)\,.$$

The max-flow min-cut theorem states that

$$\phi_B \,=\, \min\{\, V(E) \,|\, E \text{ } is \text{ } an \text{ } (F_0,F_m) - cut\,\}\,.$$

In the special case where $t(e)$ belongs to $\{0,1\}$, i.e. the law of t is a Bernoulli law, the flow has an other simple expression. In this case, let us consider the graph obtained from the initial graph $\mathbb{Z}^d$ by removing all the edges e with $t(e) = 0$. Menger's theorem (see [**12**]) states that the minimal number of edges in $B(\vec{k}, m)$ that have to be removed from this graph to disconnect F_0 from F_m is exactly the maximal number of disjoint paths that connect F_0 to F_m. By the max-flow min-cut theorem, it follows immediately that the maximal flow in the initial graph through B from F_0 to F_m is exactly the maximal number of disjoint open paths from F_0 to F_m, where a path is open if and only if the capacity of all its edges is one.

3. Bernoulli percolation for a parameter p near 1

We consider that the capacity t of each edge follows the Bernoulli law of parameter p, with $p = \mathbb{P}[t = 1]$ as close to 1 as we will need. Remember that here the maximal flow through a cylinder B is the maximal number of disjoint open paths from the bottom to the top of B. We will first prove the following theorem:

THEOREM 9. *For all ε in $[0,1[$, there exist $p_0(\varepsilon, d) < 1$ and a constant C' depending only on the dimension d such that for any function $h : \mathbb{N} \rightarrow \mathbb{N}$ satisfying*

$$\lim_{n\rightarrow\infty} \frac{\ln h(n)}{n^{d-1}} \,=\, 0$$

and for all $p \geq p_0$ we have

$$\liminf_{n\rightarrow\infty} -\frac{1}{n^{d-1}} \ln \mathbb{P}\left[\phi_{(n,\dots,n),h(n)} \leq \varepsilon n^{d-1}\right] \geq C' \,>\, 0\,.$$

To simplify the notations during the proof of this theorem, we define

$$\alpha(\varepsilon) \,=\, \mathbb{P}\left[\phi_{(n,\dots,n),h(n)} \leq \varepsilon n^{d-1}\right]\,.$$

Thanks to the max-flow min-cut theorem, we know that

$$\alpha(\varepsilon) \,=\, \mathbb{P}\left[there\ exists\ a\ (F_0, F_{h(n)}) - cut\ E\ satisfying\ V(E) \leq \varepsilon n^{d-1}\right]\,.$$

We need to define a notion of $\diamond$-connection. We associate with each edge e a plaquette $\mathcal{P}(e)$ which is the only unit square of $\mathbb{R}^d$ of the form $\mathcal{P}_i + (n_1, \dots, n_d)$ that intersects e in its middle, where $(n_1, \dots, n_d) \in \mathbb{Z}^d$ and $\mathcal{P}_i = [-1/2, 1/2]^{i-1} \times \{1/2\} \times [-1/2, 1/2]^{d-i}$ for $1 \leq i \leq d$. We say that two edges e_1 and e_2 are $\diamond$-connected if and only if $\mathcal{P}(e_1) \cap \mathcal{P}(e_2) \neq \emptyset$. According to Kesten (see [**41**]) a $(F_0, F_{h(n)})$-cut is $\diamond$-connected. Moreover, it is obvious that a cut contains at least n^{d-1} edges (to cut the n^{d-1} possible vertical paths). In particular, if we consider a fixed vertical path and if we denote by $(e_i, i = 1, \dots, h(n))$ the edges of this path, a $(F_0, F_{h(n)})$-cut E must contain at least one of these e_i. We can then find a subset E' of E which contains exactly n^{d-1} edges, including one of these e_i, and which is $\diamond$-connected. Obviously if $V(E) \leq \varepsilon n^{d-1}$ then $V(E') \leq \varepsilon n^{d-1}$. Finally we can relax the constraint for E' to be in B, and by translation invariance of the model we can suppose that E' contains a determined edge e_0. We deduce from these remarks that

$$\begin{aligned}
\alpha(\varepsilon) &\leq \sum_{i=1}^{h(n)} \mathbb{P}\left[there\ exists\ a\ (F_0, F_{h(n)}) - cut\ E\ s.t.\ V(E) \leq \varepsilon n^{d-1}\ and\ e_i \in E\right] \\
&\leq h(n)\, \mathbb{P}\left[\begin{array}{c} there\ exists\ a\ \diamond-connected\ set\ E'\ of\ n^{d-1}\ edges \\ such\ that\ V(E') \leq \varepsilon n^{d-1}\ and\ e_0 \in E' \end{array}\right] \\
&\leq h(n) \sum_{A} \mathbb{P}\left[\sum_{e \in A} t(e) \leq \varepsilon n^{d-1}\right],
\end{aligned}$$

where the sum is over the $\diamond$-connected sets A of n^{d-1} edges including e_0. We know (see [**41**]) that there exists a constant $c > 1$ depending only on the dimension d such that the number of such possible sets A is bounded by $c^{n^{d-1}}$. We deduce then, thanks to the exponential Chebyshev inequality, that, for all $\lambda > 0$,

$$\begin{aligned}\alpha(\varepsilon) &\le h(n)c^{n^{d-1}} e^{\lambda\varepsilon n^{d-1}} \mathbb{E}\left[e^{-\lambda t}\right]^{n^{d-1}} \\ &\le \exp\left(-n^{d-1}\left[-\frac{\ln h(n)}{n^{d-1}} - \ln c + \lambda(1-\varepsilon) - \ln(p+(1-p)e^{\lambda})\right]\right).\end{aligned}$$

We choose λ such that

$$\lambda(1-\varepsilon) \ge 3\ln c,$$

and then $p_0 < 1$ (depending on d and ε) such that for all $p \ge p_0$ we have

$$\ln\left(p+(1-p)e^{\lambda}\right) \le \ln c.$$

We conclude that

$$\forall p \ge p_0 \qquad \alpha(\epsilon) \le \exp\left(-n^{d-1}\left[\ln c - \frac{\ln h(n)}{n^{d-1}}\right]\right).$$

This ends the proof of theorem 9.

4. Bernoulli percolation

We consider now that the law of t is a Bernoulli law with a fixed parameter $p > p_c$. We will prove the following result:

THEOREM 10. *For any $p > p_c$, there exist a positive ε_0 (depending on d and p) and a positive constant C'' (depending only on the dimension d) such that for any function $h : \mathbb{N} \to \mathbb{N}$ satisfying*

$$\lim_{n\to\infty} \frac{\ln h(n)}{n^{d-1}} = 0$$

we have

$$\forall \varepsilon \le \varepsilon_0 \qquad \liminf_{n\to\infty} -\frac{1}{n^{d-1}} \ln \mathbb{P}\left[\phi_{(n,\dots,n),h(n)} \le \varepsilon n^{d-1}\right] \ge C'' > 0.$$

The proof of this theorem is based on the coarse graining techniques of Pisztora (see [**49**], [**19**]). The idea is to use a renormalization scheme: instead of looking at what happens for each edge, we try to understand what are the typical properties of the edges in a box, and to deduce some properties for the entire graph.

4.1. Coarse graining. Let Λ be a box. We define its inner vertex boundary as

$$\partial^{in}\Lambda = \{x \in \Lambda \,|\, \exists\, y \notin \Lambda,\ |x-y| = 1\}.$$

An open cluster within Λ is said crossing for Λ if it intersects each of the $2d$ faces of $\partial^{in}\Lambda$. The diameter of a set A is given by $diam(A) = \max_{i=1\dots d}\sup_{x,y\in A}|x_i - y_i|$. We now consider the event

$$U(\Lambda) = \{\, there\ exists\ an\ open\ crossing\ cluster\ in\ \Lambda \,\}$$

and, for m less than or equal to the diameter of Λ,

$$W(\Lambda, m) = \{\, there\ exists\ a\ unique\ open\ cluster\ in\ \Lambda\ with\ diameter\ \ge m \,\}.$$

Let $\Lambda(n)$ be the square box $]-n/2, n/2]^d$. We know that

LEMMA 2. *For all dimension $d \ge 2$ and for all $p > p_c$, we have*

$$\lim_{n\to\infty} \mathbb{P}[U(\Lambda(n))] = 1.$$

Moreover, there exists a finite constant γ (depending on d and p) such that

$$\lim_{n\to\infty} \mathbb{P}[W(\Lambda(n), \gamma\ln n)] = 1.$$

For a proof, see [**19**]. In particular, this lemma implies that

$$\lim_{n\to\infty} \mathbb{P}(W(\Lambda(n), n/3)) \,=\, 1\,.$$

To use this estimate, we will rescale the lattice. Let K be a positive integer. We divide $\mathbb{Z}^d$ into small boxes called blocks of size K in the following way. For $\underline{x} = (\underline{x}_1, ..., \underline{x}_d) \in \mathbb{Z}^d$, we define the block indexed by $\underline{x}$ as

$$B_K(\underline{x}) \,=\, K\underline{x} + \Lambda(K)\,,$$

where $K\underline{x}$ is the vertex $(K\underline{x}_1, ..., K\underline{x}_d)$. We remark that the blocks partition $\mathbb{R}^d$. Let A be a region of $\mathbb{R}^d$, we define the rescaled region $\underline{A}_K$ as

$$\underline{A}_K \,=\, \left\{\underline{x} \in \mathbb{Z}^d \,|\, B_K(\underline{x}) \cap A \neq \emptyset\right\}\,.$$

For $\underline{x} \in \mathbb{Z}^d$, we define next a neighbourhood of the block $B_K(\underline{x})$, called the event-block, as

$$B'_K(\underline{x}) \,=\, \bigcup_{\underline{u}} B_K(\underline{u})\,,$$

where the union is over the vertices $\underline{u} = (\underline{u}_1, ..., \underline{u}_d) \in \mathbb{Z}^d$ satisfying $\max_{1\leq i\leq d} |\underline{x}_i - \underline{u}_i| \leq 1$. Finally we define the block process $(X_K(\underline{x}), \underline{x} \in \mathbb{Z}^d)$ as

$$\forall \underline{x} \in \mathbb{Z}^d \qquad X_K(\underline{x}) \,=\, \mathbb{1}_{U(B_K(\underline{x}))} \times \mathbb{1}_{W(B_K(\underline{x}), \frac{K}{3})} \times \prod_{\underline{y}\in Y} \mathbb{1}_{W(B_K(\underline{x})+\underline{y}, \frac{K}{3})}\,,$$

where $Y = \{(\pm K/2, 0, ..., 0), (0, \pm K/2, 0, ..., 0), ..., (0, ..., 0, \pm K/2)\}$. We say that the event-block $B'_K(\underline{x})$ is good if $X_K(\underline{x}) = 1$; it is bad otherwise. According to lemma 2, we know that for a fixed $\underline{x}$ the variable $X_K(\underline{x})$ is a Bernoulli random variable with parameter $1 - \delta_K$, where $\lim_{K\to\infty} \delta_K = 0$. This way we obtain a dependent percolation by edges on the rescaled lattice.

Now if we have a L^1-connected path of good event-blocks $(B'_K(\underline{x}_i), i \in I)$ in the rescaled lattice from the bottom to the top of a rescaled cylinder $\underline{B}$, we can find an open path from the bottom to the top of the corresponding cylinder B in the initial graph which is completely included in $\cup_{i\in I} B_K(\underline{x}_i)$. Indeed, take $\underline{x}$, $\underline{y}$ and $\underline{z}$ three successive elements of the L^1-connected sequence $(\underline{x}_i, i \in I)$. Suppose (for the recurrence) that we have already constructed an open path γ in $B_K(\underline{x}) \cup B_K(\underline{y})$ which joins the two opposite faces of $\partial^{in}(B_K(\underline{x}) \cup B_K(\underline{y}))$ at distance $2K$ (the ones perpendicular to the direction of the vector $\underline{y} - \underline{x}$). We know that $B'_K(\underline{y})$ and $B'_K(\underline{z})$ are good event-blocks, so the events $U(B_K(\underline{y}))$, $U(B_K(\underline{z}))$ and $W(B_K(\underline{y}) + (\underline{z} - \underline{y})K/2, K/3)$ occur. On $U(B_K(\underline{y})) \cap U(B_K(\underline{z}))$, we know that there exists an open path γ'_1 (respectively γ'_2) in $B_K(\underline{y})$ (respectively $B_K(\underline{z})$) that join the two opposite faces of $\partial^{in} B_K(\underline{y})$ (respectively $\partial^{in} B_K(\underline{z})$) perpendicular to the direction of the vector $\underline{z} - \underline{y}$. Moreover, since $W(B_K(\underline{y}) + (\underline{z} - \underline{y})K/2, K/3)$ occurs, γ'_1 and γ'_2 are connected by an open path γ'_3 in $B_K(\underline{y}) + (\underline{z} - \underline{y})K/2$ because of the uniqueness of the open cluster of diameter greater than $K/3$ in $B_K(\underline{y}) + (\underline{z} - \underline{y})K/2$ (see figure 1). So $\gamma' = \gamma'_1 \cup \gamma'_2 \cup \gamma'_3$ contains an open path in $B_K(\underline{y}) \cup B_K(\underline{z})$ which joins the two opposite faces of $\partial^{in}(B_K(\underline{y}) \cup B_K(\underline{z}))$ at distance $2K$ (the ones perpendicular to the direction of the vector $\underline{z} - \underline{y}$). Finally the event $W(B_K(\underline{y}), K/3)$ occurs so we know that γ and γ' are connected by an open path γ'' in $B_K(\underline{y})$ (see figure 2). From an event-block to another, we can build the desired open path from the bottom to the top of the cylinder B, and it lies indeed in $\cup_{i\in I} B_K(\underline{x}_i)$. Moreover, if we have N disjoint L^1-connected paths of good event-blocks, that we denote by $(B'_K(\underline{x}_i), i \in I_j)$, $j = 1, ..., N$, in the rescaled lattice from the bottom to the top of a rescaled cylinder $\underline{B}$, we can find N disjoint open paths from the bottom to the top of the corresponding cylinder B in the initial graph, because the sets I_j are pairwise disjoint and so are the sets $\cup_{i\in I_j} B_K(\underline{x}_i)$, which contains the different open paths constructed as previously.

4.2. Proof of theorem 10. As previously we use the notation

$$\alpha(\varepsilon) \,=\, \mathbb{P}\left[\phi_{(n,...,n),h(n)} \leq \varepsilon n^{d-1}\right]\,.$$

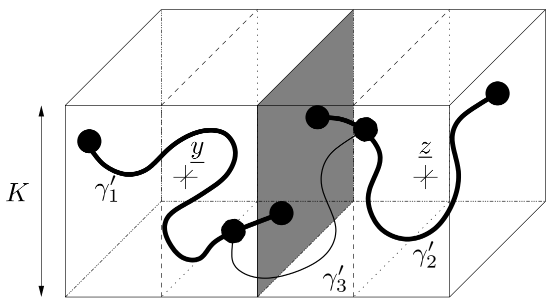

FIGURE 1. Construction of the open path - 1.

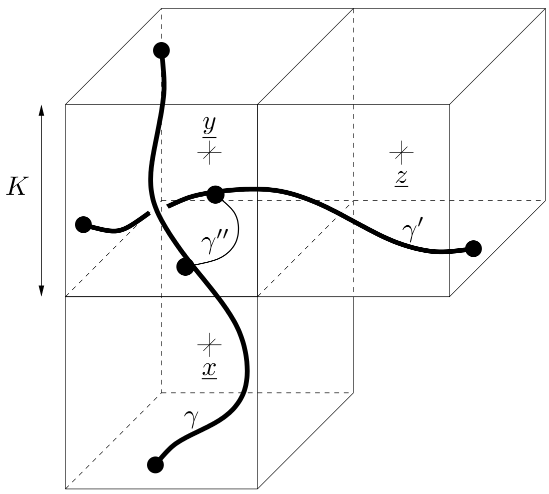

FIGURE 2. Construction of the open path - 2.

We define the cylinder

$$A(n) \,=\, [0,n]^{d-1} \times [0,h(n)]\,,$$

and $\underline{A}_K$ is the rescaled cylinder for an integer K which will be chosen soon.

According to the remark at the end of the previous subsection, we know that if there exist εn^{d-1} disjoint paths of L^1-connected good event-blocks from the bottom to the top of the rescaled cylinder $\underline{A}_K(n)$, then there exist at least εn^{d-1} disjoint open paths from the bottom to the top of $A(n)$. Therefore

$$\alpha(\varepsilon) \,\leq\, \mathbb{P}\left[\begin{array}{c} there\ exist\ less\ than\ \varepsilon n^{d-1}\ disjoint\ paths\ of\ good \\ event-blocks\ from\ the\ bottom\ to\ the\ top\ of\ \underline{A}_K(n) \end{array} \right].$$

Now the arguments which will be used are very similar to those used in the proof of theorem 9. The main difference is that the random variables $(X_K(\underline{x}), \underline{x} \in \mathbb{Z}^d)$ are not independent.

We work for the rest of the proof in the rescaled lattice. The notion of cut can be adapted easily in the model of site percolation and the max-flow min-cut theorem remains valid in this model. Thanks to the max-flow min-cut theorem applied in the rescaled lattice we obtain that

$$\alpha(\varepsilon) \,\leq\, \mathbb{P}\left[there\ exists\ a\ (\underline{F_0}_K, \underline{F_{h(n)}}_K) - cut\ \underline{E}\ satisfying\ \underline{V}(\underline{E}) \leq \varepsilon n^{d-1}\right],$$

where here

$$\underline{V}(\underline{E}) \,=\, \sum_{\underline{x}\in\underline{E}} X_K(\underline{x})\,.$$

Note that such a $(\underline{F_0}_K, \underline{F_{h(n)}}_K)$-cut contains at least $u = \lfloor (n/K)^{d-1} \rfloor$ vertices. As previously, we obtain

$$\alpha(\varepsilon) \leq \frac{h(n)}{K} \sum_{\underline{A}} \mathbb{P}\left[\sum_{\underline{x} \in \underline{A}} X_K(\underline{x}) \leq \varepsilon n^{d-1} \right] ,$$

where the sum is over the L^1-connected sets $\underline{A}$ of u vertices containing a fixed vertex $\underline{x}_0$ of $\mathbb{Z}^d$.

To deal with the variables $(X_K(\underline{x}), \underline{x} \in \mathbb{Z}^d)$ we introduce an equivalence relation on $\mathbb{Z}^d$: $x \sim y$ if and only if 3 divides all the coordinates of $x - y$. There exist 3^d equivalence classes $V_1, ..., V_{3^d}$ in $\mathbb{Z}^d$. For a set of vertices $\underline{E}$, we define

$$\underline{E}^l \;=\; \underline{E} \cap V_l \,.$$

Now the variables $(X_K(\underline{x}), \underline{x} \in V_l)$ are independent for a fixed $l \in \{1, ..., 3^d\}$, so we want to consider only sums of variables indexed by vertices in the same equivalence class. For that purpose, we remark that if $\sum_{\underline{x} \in \underline{A}} X_K(\underline{x}) \leq \varepsilon n^{d-1}$ for some set $\underline{A}$ of u vertices and for some $\varepsilon \leq 1/K^{d-1}$, then $\underline{A}$ contains at least $u - \lfloor \varepsilon n^{d-1} \rfloor$ bad event-blocks which are included in the subsets $\underline{A}^1, ..., \underline{A}^{3^d}$. Thus there exists $l \in \{1, ..., 3^d\}$ such that $\underline{A}^l$ contains at least $(u - \lfloor \varepsilon n^{d-1} \rfloor)/3^d$ bad event-blocks, so $\sum_{\underline{x} \in \underline{A}^l} X_K(\underline{x}) \leq |\underline{A}^l| - (u - \lfloor \varepsilon n^{d-1} \rfloor)/3^d$. This remark leads to

$$\mathbb{P}\left[\sum_{\underline{x} \in \underline{A}} X_K(\underline{x}) \leq \varepsilon n^{d-1} \right] \;\leq\; \sum_{l=1}^{3^d} \mathbb{P}\left[\sum_{\underline{x} \in \underline{A}^l} X_K(\underline{x}) \leq |\underline{A}^l| - \frac{1}{3^d}(u - \varepsilon n^{d-1}) \right] ,$$

where $|\underline{A}^l|$ is the cardinal of $\underline{A}^l$. Now, thanks again to the bound on the number of possible sets $\underline{A}$ and to the exponential Chebyshev inequality, we obtain as in the proof of theorem 9 that, for all $\lambda > 0$,

$$\begin{aligned}
\alpha(\varepsilon) \;&\leq\; \frac{h(n)}{K} \sum_{\underline{A}} \sum_{l=1}^{3^d} \exp\left(\lambda \left[|\underline{A}^l| - \frac{1}{3^d}(u - \varepsilon n^{d-1}) \right] \right) \mathbb{E}\left[e^{-\lambda X_K(\underline{x}_0)} \right]^{|\underline{A}^l|} \\
&\leq\; \frac{h(n)}{K} \sum_{\underline{A}} \sum_{l=1}^{3^d} \exp\left(\lambda |\underline{A}^l| - \lambda \frac{u - \varepsilon n^{d-1}}{3^d} + |\underline{A}^l| \ln\left[e^{-\lambda}(1 + \delta_K(e^\lambda - 1)) \right] \right) \\
&\leq\; \frac{3^d h(n)}{K} \exp\left(-u \left[-\ln c' + \frac{\lambda}{3^d}(1 - \frac{\varepsilon n^{d-1}}{u}) - \ln(1 + \delta_K(e^\lambda - 1)) \right] \right) ,
\end{aligned}$$

because $|\underline{A}^l| \leq u$. Now we choose first λ such that

$$\frac{\lambda}{2 \times 3^d} \;\geq\; 3 \ln c' \,,$$

and then K large enough (depending on d and p), so δ_K small enough, to have

$$\ln\left(1 + \delta_K(e^\lambda - 1) \right) \;\leq\; \ln c' \,.$$

We obtain

$$\forall \varepsilon \leq \frac{1}{2K^{d-1}} \qquad \alpha(\varepsilon) \;\leq\; \frac{3^d h(n)}{K} e^{-u \ln c'} \,,$$

and this ends the proof.

5. Proof of theorem 7

We consider finally the general case of the first passage percolation model, with the condition

$$F(0) \;<\; 1 - p_c \,.$$

The distribution function F is right continuous, so there exists a positive η such that

$$p' = \mathbb{P}[t > \eta] = 1 - F(\eta) > p_c \, .$$

Now we consider a new family of random variables on $\mathbb{E}^d$ defined as

$$t'(e) = \begin{cases} 1 & if\ t(e) > \eta \\ 0 & otherwise \end{cases}$$

The family $(t'(e), e \in \mathbb{E}^d)$ defines an independent Bernoulli percolation of parameter p' on the lattice. We consider the rescaled lattice, and we say that an event-block is good if it is good for this Bernoulli percolation according to the definition given in the previous section. We remark that the existence of a path of good event-blocks in the rescaled lattice implies the existence of a path of edges with a capacity greater than η in the initial graph. Therefore

$$\begin{aligned} \alpha(\varepsilon) &= \mathbb{P}\left[\phi_{(n,\dots,n),h(n)} \leq \varepsilon n^{d-1}\right] \\ &\leq \mathbb{P}\left[\begin{array}{c} there\ exist\ less\ than\ \frac{\varepsilon}{\eta} n^{d-1}\ disjoint\ paths\ of\ good \\ event-blocks\ from\ the\ bottom\ to\ the\ top\ of\ \underline{A}_K(n) \end{array}\right] . \end{aligned}$$

We proceed as in the proof of theorem 10 to obtain the desired estimate.

CHAPTER 5

Lower large deviations for maximal flows through a cylinder

This chapter is a joint work with Raphaël Rossignol.

We consider the standard first passage percolation model in $\mathbb{Z}^d$ for $d \geq 2$. We are interested in two quantities, the maximal flow τ between the lower half and the upper half of the box, and the maximal flow ϕ between the top and the bottom of the box. A standard subadditive argument yields the law of large numbers for τ. Kesten and Zhang have proved the law of large numbers for ϕ. The two variables grow linearly with the surface s of the basis of the box, with the same deterministic speed. We study the probabilities that the rescaled variables τ/s and ϕ/s are abnormally small. Using a concentration inequality, we show that these probabilities decay exponentially fast with s, when s grows to infinity. Moreover, we prove an associated large deviation principle of speed s for τ/s, and for ϕ/s. For ϕ, we require either that the box is sufficiently flat, or that its sides are parallel to the coordinate hyperplanes.

Contents

1. Introduction

The model of maximal flow in a randomly porous medium with independent and identically distributed capacities has been initially studied by Kesten (see [**41**]), who introduced it as a “higher dimensional version of FPP". The purpose of this model is to understand the behaviour of the maximum amount of flow that can cross the medium from one part to another.

All the precise definitions will be given in section 2, but let us be a little more accurate. The random medium is represented by the lattice $\mathbb{Z}^d$. We see each edge as a microscopic pipe which the fluid can flow through. To each edge e, we attach a nonnegative capacity $t(e)$ which represents the amount of fluid (or the amount of fluid per unit of time) that can effectively go through the edge e. Capacities are then supposed to be random, identically and independently distributed with common distribution function F. Let A be some hyperrectangle in $\mathbb{R}^d$ (i.e., a box of dimension $d-1$) and n be an integer. The portion of media that we will look at is a box B_n of basis nA and of height $2h(n)$, which nA splits into two equal parts. The boundary of B_n is thus split into two parts, A_n^1 and A_n^2. We define two flows through B_n: the flow τ_n for which the fluid can enter the box through A_n^1 and leave it through A_n^2, and the flow ϕ_n for which the fluid enters B_n only through its bottom side and leaves it through its top side. Existing results for ϕ_n and τ_n are essentially of two types: laws of large numbers and large deviation results. It is important to note that all the results presented here were obtained for "straight" hyperrectangles A, i.e., hyperrectangles of the form $\prod_{i=1}^{d-1}[0,a_i]\times\{0\}$. In some ways, especially concerning the study of ϕ_n, this simplifies the task. Subadditivity implies a law of large numbers for τ_n. Kesten proved a law of large numbers for ϕ_n (see [**41**]), under various conditions on the height $h(n)$ and the value of $F(0)$. Recently, in a remarkable paper Zhang (see [**59**]) improved Kesten's conditions (see Theorem 13 below). Théret proved a large deviation principle for ϕ_n at volume order for upper deviations (see [**55**]). Lower large deviations for ϕ_n far from its typical behaviour were investigated for Bernoulli capacities in [**20**], and for general functions in [**56**], and are shown to be of surface order, although a full large deviation principle was not proved.

The main results of this paper are the lower large deviation principles for τ_n and ϕ_n under various conditions. More precisely, we shall show lower large deviation principles at the surface order for τ_n for general A and height $h(n)$, and for ϕ_n when $h(n)$ is small compared to n (see Theorems 12 and Corollary 2.2). We also show a lower large deviation principle at the surface order for ϕ_n when $\log h(n)$ is small compared to n^{d-1} and when A is straight (see Theorem 15). Unfortunately, when $d\geq 3$, we are not able to prove the lower large deviation principle for ϕ_n through general hyperrectangles and heights (see Remark 23). Incidentally, we prove concentration results which are interesting on their own for ϕ_n and τ_n for general hyperrectangles A (see Theorem 11, Corollary 2.1 and Theorem 14 below).

The paper is organized as follows. In section 2, we give the precise definitions and state the main results of the paper. Section 3 is devoted to the concentration results for τ_n and ϕ_n. Since we need, for the large deviation principles, to have concentration of τ_n/n^{d-1} around its almost sure limit, we prove the law of large numbers for τ_n in this section. We prove the large deviation principle for τ in section 4, and the ones for ϕ in section 5.

2. Definitions and main results

2.1. Precise definitions. We use many notations introduced in [**40**] and [**41**]. Let $d\geq 2$. We consider the graph $(\mathbb{Z}^d,\mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$, the set of pairs of nearest neighbours for the standard L^1 norm. With each edge e in $\mathbb{E}^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e\in\mathbb{E}^d)$ is independent and identically distributed, with a common distribution function F: this is the standard model of first passage percolation on the graph $(\mathbb{Z}^d,\mathbb{E}^d)$. More formally, we take the product measure $\mathbb{P}$ on $\Omega=\prod_{e\in\mathbb{E}^d}[0,\infty[$, and we write its expectation $\mathbb{E}$. We interpret $t(e)$ as the capacity of the edge e; it means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time.

For a given realization $(t(e), e\in\mathbb{E}^d)$, we define the maximal flow $\phi(F_0\rightarrow F_1 \; in \; C)$ from F_0 to F_1 in C, where $C\subset\mathbb{R}^d$ (or by commodity in the corresponding graph $C\cap\mathbb{Z}^d$) and F_0 and

F_1 are two disjoint subsets of C. We remember that $\mathbb{E}^d$ is the set of edges of the lattice $\mathbb{Z}^d$. We will say that an edge $e = \langle x, y \rangle$ belongs to a subset A of $\mathbb{R}^d$, which we denote by $e \in A$, if the segment joining x to y (eventually excluding these points) is included in A. Now we define $\widetilde{\mathbb{E}}^d$ as the set of all the oriented edges, i.e., an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}^d$ is an ordered pair of vertices. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}^d$ by $\langle\langle x, y \rangle\rangle$, where x, $y \in \mathbb{Z}^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from x towards y. We consider now the set $\mathcal{S}$ of all pairs of functions (g, o), with $g : \mathbb{E}^d \rightarrow \mathbb{R}^+$ and $o : \mathbb{E}^d \rightarrow \widetilde{\mathbb{E}}^d$ such that $o(\langle x, y \rangle) \in \{\langle\langle x, y \rangle\rangle, \langle\langle y, x \rangle\rangle\}$, satisfying:

- for each edge e in C we have
$$0 \,\leq\, g(e) \,\leq\, t(e)\,,$$
- for each vertex v in $C \smallsetminus (F_0 \cap F_1)$ we have
$$\sum_{e \in C \,:\, o(e) = \langle\langle v, \cdot \rangle\rangle} g(e) \,=\, \sum_{e \in C \,:\, o(e) = \langle\langle \cdot, v \rangle\rangle} g(e)\,,$$

where the notation $o(e) = \langle\langle v, . \rangle\rangle$ (respectively $o(e) = \langle\langle ., v \rangle\rangle$) means that there exists $y \in \mathbb{Z}^d$ such that $e = \langle v, y \rangle$ and $o(e) = \langle\langle v, y \rangle\rangle$ (respectively $o(e) = \langle\langle y, v \rangle\rangle$). A couple $(g, o) \in \mathcal{S}$ is a possible stream in C from F_0 to F_1: $g(e)$ is the amount of fluid that goes through the edge e, and $o(e)$ gives the direction in which the fluid goes through e. The two conditions on (g, o) express only the fact that the amount of fluid that can go through an edge is bounded by its capacity, and that there is no loss of fluid in the cylinder. With each possible stream we associate the corresponding flow

$$flow(g, o) \,=\, \sum_{u \in F_1\,,\, v \notin C \,:\, \langle u, v \rangle \in \mathbb{E}^d} g(\langle u, v \rangle) \mathbb{1}_{o(\langle u, v \rangle) = \langle\langle u, v \rangle\rangle} - g(\langle u, v \rangle) \mathbb{1}_{o(\langle u, v \rangle) = \langle\langle v, u \rangle\rangle}\,.$$

This is the amount of fluid that crosses C from F_0 to F_1 if the fluid respects the stream (g, o). The maximal flow through C from F_0 to F_1 is the supremum of this quantity over all possible choices of stream

$$\phi(F_0 \rightarrow F_1 \;in\; C) \,=\, \sup\{flow(g, o) \,|\, (g, o) \in \mathcal{S}\}\,.$$

We stress here that our capacities $(t(e), e \in \mathbb{E}^d)$ are non-negative, so for every triplet (C, F_0, F_1), the maximal flow $\phi(F_0 \rightarrow F_1 \;in\; C)$ is non-negative.

For a subset X of $\mathbb{R}^d$, we denote by $\mathcal{H}^s(X)$ the s-dimensional Hausdorff measure of X (we will use $s = d-1$ and $s = d-2$). Let $A \subset \mathbb{R}^d$ be a non degenerate hyperrectangle (for the usual scalar product), i.e., a box of dimension $d-1$ in $\mathbb{R}^d$. We stress here the fact that all the hyperrectangle we will consider are non-degenerate, and they all will be supposed to be closed in $\mathbb{R}^d$. We denote by $\vec{v}$ one of the two vectors of unit euclidean norm, orthogonal to $\mathrm{hyp}(A)$, the hyperplane spanned by A. For h a positive real number, we denote by $\mathrm{cyl}(A, h)$ the cylinder of basis A and height $2h$, i.e., the set

$$\mathrm{cyl}(A, h) \,=\, \{x + t\vec{v} \,|\, x \in A\,,\, t \in [-h, h]\}\,.$$

The set $\mathrm{cyl}(A, h) \smallsetminus \mathrm{hyp}(A)$ has two connected components, which we denote by $\mathcal{C}_1(A, h)$ and $\mathcal{C}_2(A, h)$. For $i = 1, 2$, let A_i^h be the set of the points in $\mathcal{C}_i(A, h) \cap \mathbb{Z}^d$ which have a nearest neighbour in $\mathbb{Z}^d \smallsetminus \mathrm{cyl}(A, h)$:

$$A_i^h \,=\, \{x \in \mathcal{C}_i(A, h) \cap \mathbb{Z}^d \,|\, \exists y \in \mathbb{Z}^d \smallsetminus \mathrm{cyl}(A, h)\,,\, \|x - y\|_1 = 1\}\,,$$

where $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i|$ if $x = (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$. Let $T(A, h)$ (respectively $B(A, h)$) be the top (respectively the bottom) of $\mathrm{cyl}(A, h)$, i.e.,

$$T(A, h) \,=\, \{x \in \mathrm{cyl}(A, h) \,|\, \exists y \notin \mathrm{cyl}(A, h)\,,\; \langle x, y \rangle \in \mathbb{E}^d \;and\; \langle x, y \rangle \;intersects\; A + h\vec{v}\}$$

and

$$B(A, h) \,=\, \{x \in \mathrm{cyl}(A, h) \,|\, \exists y \notin \mathrm{cyl}(A, h)\,,\; \langle x, y \rangle \in \mathbb{E}^d \;and\; \langle x, y \rangle \;intersects\; A - h\vec{v}\}\,.$$

The notation $\langle x,y \rangle$ corresponds to the edge of endpoints x and y. We define two particular maximal flows through the cylinder $\mathrm{cyl}(A,h)$, the variable $\tau(A,h) = \tau(\mathrm{cyl}(A,h),\vec{v})$ by
$$\tau(A,h) \,=\, \tau(\mathrm{cyl}(A,h),\vec{v}) \,=\, \phi(A_1^h \rightarrow A_2^h \ in \ \mathrm{cyl}(A,h))\,,$$
and the variable $\phi(A,h)$ by
$$\phi(A,h) \,=\, \phi(B(A,h) \rightarrow T(A,h) \ in \ \mathrm{cyl}(A,h))\,.$$

The maximal flow $\phi(F_0 \rightarrow F_1 \ in \ C)$ can be expressed differently thanks to the max-flow min-cut theorem (see [**12**]). We need some definitions. A path on the graph $\mathbb{Z}^d$ from v_0 to v_n is a sequence $(v_0, e_1, v_1, ..., e_n, v_n)$ of vertices $v_0, ..., v_n$ alternating with edges $e_1, ..., e_n$ such that v_{i-1} and v_i are neighbours in the graph, joined by the edge e_i, for i in $\{1, ..., n\}$. Two paths are said disjoint if they have no common edge. A set E of edges in C is said to cut F_0 from F_1 in C if there is no path from F_0 to F_1 in $C \smallsetminus E$. We call E an (F_0, F_1)-cut if E cuts F_0 from F_1 in C and if no proper subset of E does. With each set E of edges we associate its capacity which is the variable
$$V(E) \,=\, \sum_{e\in E} t(e)\,.$$
The max-flow min-cut theorem states that
$$\phi(F_0 \rightarrow F_1 \ in \ C) \,=\, \min\{\, V(E) \,|\, E \ is \ an \ (F_0,F_1) - cut\,\}\,.$$

We finally define the r-neighbourhood $\mathcal{V}(H,r)$ of a subset H of $\mathbb{R}^d$ as
$$\mathcal{V}(H,r) \,=\, \{x \in \mathbb{R}^d \,|\, d(x,H) < r\}\,,$$
where the distance is taken for the euclidean norm:
$$d(x,H) \,=\, \inf\{\|x-y\|_2 \,|\, y \in H\}\,,$$
where $\|x-y\|_2 = \sqrt{\sum_{i=1}^d (x_i-y_i)^2}$ for $x = (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$.

2.2. Results concerning τ. We have a law of large numbers for $\tau(A,h)$ in big cylinders $\mathrm{cyl}(A,h)$. The following result, that we will prove in section 3.2, is a consequence of a subadditive argument:

PROPOSITION 2.1. We suppose that the capacity of an edge is in L^1. For every function $h : \mathbb{N} \rightarrow \mathbb{R}^+$ satisfying $\lim_{n\rightarrow\infty} h(n) = +\infty$, for every non degenerate hyperrectangle A, the limit
$$\lim_{n\rightarrow\infty} \frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(nA)}$$
exists and depends on the direction of $\vec{v}$, one of the two unit vectors orthogonal to $\mathrm{hyp}(A)$, and not on A itself. We denote it by $\nu(\vec{v})$ (the dependence in F and d is implicit). Moreover, if $F(0) < 1 - p_c(d)$ and F admits an exponential moment:
$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) \,<\, \infty\,,$$
we also know that for every function $h : \mathbb{N} \rightarrow \mathbb{R}^+$ satisfying $\lim_{n\rightarrow\infty} h(n) = +\infty$ we have
$$\lim_{n\rightarrow\infty} \frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\vec{v}) \qquad a.s.$$
Here $p_c(d)$ is the critical parameter for the Bernoulli percolation in dimension d.

REMARK 11. In fact, as we explained in the introduction of the thesis, we know that if $F(0) \geq 1 - p_c(d)$ and $\mathbb{E}(t(e)) < \infty$, then $\nu(\vec{v}) = 0$ for all $\vec{v}$ so there is no interest to study $\mathbb{P}(X(nA,h(n)) \leq \lambda \mathcal{H}^{d-1}(nA))$ for $X = \tau$ or ϕ and $\lambda < \nu(\vec{v}) = 0$.

We will show two results, the first gives the speed of decay of the probability that the rescaled flow τ is abnormally small, and the second one states a large deviation principle for the rescaled variable τ.

The estimate of lower large deviations is the following:

THEOREM 11. *Suppose that $F(0) < 1 - p_c(d)$ and that F admits an exponential moment:*

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) \, < \, \infty \, .$$

Then for every $\varepsilon > 0$ there exists a positive constant $C(\varepsilon, F, d)$ such that for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, for every unit vector $\vec{v}$ and every non degenerate hyperrectangle A orthogonal to $\vec{v}$, there exists a constant $\widetilde{C}(d, F, A, h, \varepsilon)$ (possibly depending on all the parameters d, F, A, h, ε) such that

$$\mathbb{P}\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v}) - \varepsilon \right) \leq \widetilde{C}(d, F, A, h, \varepsilon) \exp\left(-C(\varepsilon, F, d) \mathcal{H}^{d-1}(A) n^{d-1} \right) .$$

Theorem 2 in [**59**] is the key to obtain the relevant condition $F(0) < 1 - p_c(d)$. Now we can state a large deviation principle:

THEOREM 12. *Suppose that $F(0) < 1 - p_c(d)$ and that F admits exponential moments of all orders:*

$$\forall \gamma > 0 \qquad \int e^{\gamma x} dF(x) \, < \, \infty \, ,$$

Then for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, for every unit vector $\vec{v}$ and every non degenerate hyperrectangle A orthogonal to $\vec{v}$, the sequence

$$\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)}, n \in \mathbb{N} \right)$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the good rate function $\mathcal{J}_{\vec{v}}$. Moreover we know that $\mathcal{J}_{\vec{v}}$ is convex on $\mathbb{R}^+$, infinite on $[0, \delta\|\vec{v}\|_1[\cup]\nu(\vec{v}), +\infty[$, where $\delta = \inf\{\lambda \,|\, \mathbb{P}(t(e) \leq \lambda) > 0\}$, equal to 0 at $\nu(\vec{v})$, and if $\delta\|\vec{v}\|_1 < \nu(\vec{v})$ this function is finite on $]\delta\|\vec{v}\|_1, \nu(\vec{v})]$, continuous and strictly decreasing on $[\delta\|\vec{v}\|_1, \nu(\vec{v})]$ and positive on $[\delta\|\vec{v}\|_1, \nu(\vec{v})[$.

Obviously, this result is interesting only if $\nu(\vec{v}) > \delta\|\vec{v}\|_1$. We shall prove in section 4.5 the following property, that implies that lower large deviations of surface order occur at least if $F(\delta) < 1 - p_c(d)$:

PROPOSITION 2.2. We suppose that the capacities are in L^1: $\int x dF(x) < \infty$. If $F(\delta) < 1 - p_c(d)$, then $\nu(\vec{v}) > \delta\|\vec{v}\|_1$ for all unit vector $\vec{v}$. In the case $\delta = 0$, the previous implication is in fact an equivalence.

REMARK 12. In his PhD-thesis [**57**], section 2, Wouts shows a similar lower large deviations result in the context of the dilute Ising model. More precisely, for every temperature T, a Gibbs measure $\Phi_{n,T}$ with i.i.d nonnegative, bounded random interactions $(J_e)_{e\in\mathbb{E}^d}$ is constructed on the set of configurations $\{0,1\}^{E_n}$, where E_n is the set of edges of a cube B_n of length n, and 0 (resp. 1) means the edge is closed (resp. open). Wouts defines the quenched surface tension in this box as the normalized logarithm of the $\Phi_{n,T}$-probability of the event that there is a disconnection between the upper and lower parts of the boundary of B_n. Then, Wouts shows that for Lebesgue-almost every temperature T, the quenched surface tension satisfies a large deviation principle at surface order. A remarkable feature of this work is that the proof, quite simple, relies on a concentration property that avoids the use of any estimate like that of Theorem 16. A similar treatment could be done in our setting, with the value of $F(0)$ playing the role of the inverse temperature. Of course, this is quite artificial and unsatisfactory for our purpose, since one would not obtain any information for a precise distribution function F, but rather for almost all distributions of the form $p\delta_0 + (1-p)dF$, $p \in [0,1]$. Still, it seems to us that Wouts' method deserves further investigation.

2.3. Results concerning ϕ in flat cylinders. Under the additional assumption that the cylinder we study is sufficiently flat, in the sense that we suppose $\lim_{n\to\infty} h(n)/n = 0$, we can transport results from τ to ϕ even in non-straight boxes, because the behaviour of these two variables are very similar in that case. We obtain the two following corollaries:

COROLLARY 2.1 (of Theorem 11). *Suppose that $F(0) < 1 - p_c(d)$ and that F admits an exponential moment:*

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) \,<\, \infty\,.$$

Then for every $\varepsilon > 0$ there exists a positive constant $C'(\varepsilon, F, d)$ such that for every unit vector $\vec{v}$, every non degenerate hyperrectangle A orthogonal to $\vec{v}$ and for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} h(n)/n = 0$, there exists a constant $\widetilde{C}'(d, F, A, h, \varepsilon)$ (possibly depending on all the parameters d, F, A, h, ε) such that

$$\mathbb{P}\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v}) - \varepsilon\right) \,\leq\, \widetilde{C}'(d, F, A, h, \varepsilon) \exp\left(-C'(\varepsilon, F, d)\mathcal{H}^{d-1}(A) n^{d-1}\right)\,.$$

COROLLARY 2.2 (of Theorem 12). *Suppose that $F(0) < 1 - p_c(d)$ and that F admits exponential moments of all orders:*

$$\forall \gamma > 0 \qquad \int e^{\gamma x} dF(x) \,<\, \infty\,,$$

Then for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} h(n)/n = 0$, for every unit vector $\vec{v}$ and every non degenerate hyperrectangle A orthogonal to $\vec{v}$, the sequence

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)}, n \in \mathbb{N}\right)$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the good rate function $\mathcal{J}_{\vec{v}}$ (the same as in Theorem 12).

2.4. Results concerning ϕ in straight but high cylinders. Kesten proved in 1987 the law of large numbers for ϕ in vertical boxes in dimension 3 under the additional assumption that $F(0)$ is sufficiently small and $h(n)$ not too large (see [**41**]). In a remarkable paper, Zhang recently improved Kesten's result by relaxing the assumption on $F(0)$ to the relevant one $F(0) < 1 - p_c(d)$, and extended it to any dimension $d \geq 3$ (see [**59**]). If we denote by $A(\vec{k})$ the hyperrectangle $\prod_{i=1}^{d-1}[0, k_i] \times \{0\}$, Zhang proved the following result:

THEOREM 13 (Zhang 2007). *Suppose $F(0) < 1 - p_c(d)$ and F admits an exponential moment:*

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) \,<\, \infty\,.$$

If k_1, ..., k_{d-1}, m go to infinity in such a way that for some $0 < \eta \leq 1$, we have

$$\log m \,\leq\, \max_{1 \leq i \leq d-1} k_i^{1-\eta}\,,$$

then

$$\lim_{k_1, \dots, k_{d-1}, m \to \infty} \frac{\phi(A(\vec{k}), m)}{k_1 \cdots k_{d-1}} \,=\, \nu((0, \dots, 0, 1)) \qquad a.s. \text{ and in } L^1\,.$$

We shall say that a hyperrectangle A is *straight* if it is of the form $\prod_{i=1}^{d-1}[0, a_i] \times \{0\}$ ($a_i \in \mathbb{R}_*^+$ for all i). In particular, for a straight A, for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) =$

$+\infty$ and $\log h(n) \leq n^{1-\eta}$ for some $0 < \eta \leq 1$, we have

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu((0, ..., 0, 1)) \qquad a.s. \text{ and in } L^1 .$$

Then we obtain a result similar to Theorem 11 for ϕ under the assumptions of Theorem 13:

THEOREM 14. *Suppose that $F(0) < 1 - p_c(d)$ and that F admits an exponential moment:*

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) < \infty .$$

Then, for every $\varepsilon > 0$ there exists a positive constant $C''(\varepsilon, F, d)$ such that for every non degenerate hyperrectangle A of the form $\prod_{i=1}^{d-1}[0, a_i] \times \{0\}$ ($a_i \in \mathbb{R}_*^+$ for all i) and for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\log h(n) \leq n^{1-\eta}$ for some $0 < \eta \leq 1$, there exists a constant $\widetilde{C}''(d, F, A, h, \varepsilon)$ (possibly depending on all the parameters d, F, A, h, ε) such that

$$\mathbb{P}\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu((0, ..., 0, 1)) - \varepsilon \right) \leq \widetilde{C}''(d, F, A, h, \varepsilon) \exp\left(-C''(\varepsilon, F, d)\mathcal{H}^{d-1}(A) n^{d-1}\right) .$$

This result answers a question asked by Kesten in [**41**]. All the tools used to prove it are contained in Zhang's paper [**59**].

REMARK 13. Actually, we have this lower large deviations result under the assumptions that $F(0) < 1 - p_c(d)$, F admits an exponential moment, $h(n) \leq \exp(\mathcal{H}^{d-1}(nA))$, and that the quantity $\mathbb{E}(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA))$ converges towards some limit μ (replace ν by a general μ in the previous inequality). Zhang gives us some conditions that imply the convergence of $\mathbb{E}(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA))$ towards $\nu((0, ..., 0, 1))$ in straight boxes. In the case where we have $\lim_{n\to\infty} h(n)/n = 0$ as in the previous section, it is easy to see that $\mathbb{E}(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA))$ converges towards $\nu(\vec{v})$, and so Corollary 2.1 could also be proved directly like Theorem 14.

Using subadditivity and symmetry arguments we can prove a large deviation principle for ϕ in straight boxes.

THEOREM 15. *Suppose that $F(0) < 1 - p_c(d)$ and that F admits an exponential moment:*

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) < \infty ,$$

Then for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \frac{\log h(n)}{n^{d-1}} = 0$, for every non degenerate straight hyperrectangle A, the sequence

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)}, n \in \mathbb{N} \right)$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the good rate function $\mathcal{J}_{\vec{v}}$ with $\vec{v} = (0, ..., 0, 1)$ (the same as in Theorem 12).

REMARK 14. The strict positivity of the limit

$$\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}(\phi(nA, h(n)) \geq (\nu(\vec{v}) + \varepsilon)\mathcal{H}^{d-1}(nA))$$

(which is proved during the proof of Theorem 15), together with the upper large deviations result obtained in Chapter 2 and a simple Borel-Cantelli lemma, implies the law of large numbers for ϕ as the one that Zhang proved in [**59**]. The difference is that we send to infinity the lenghts of the sides of the basis of our cylinder to infinity as the same speed, whereas Zhang consider in [**59**] possible different speeds. Thanks to this restriction, we obtain a better condition on h than he can, because the problem is easier.

REMARK 15. Actually the condition $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$ is essentially the good one. For instance, if $A = [0,1]^{d-1} \times \{0\}$, $h(n) \geq \exp(kn^{d-1})$ for a constant k sufficiently large and $F(0) > 0$, then the maximal flow $\phi(nA, h(n))$ eventually equals 0, almost surely. Indeed if the n^{d-1} vertical edges of the cylinder that intersect one fixed horizontal plane have all 0 for capacity then $\phi(nA, h(n)) = 0$. By independence and translation invariance of the model, we obtain:

$$\mathbb{P}\left[\phi(nA, h(n)) \neq 0\right] \leq \left[1 - F(0)^{n^{d-1}}\right]^{2\exp(kn^{d-1})}$$

which is summable for k large enough, and so we conclude by the Borel-Cantelli lemma.

3. Lower large deviations

3.1. Concentration of the maximal flow around its mean. In this section, we obtain concentration inequalities for $\phi(A,h)$ and $\tau(A,h)$ around their means. These inequalities, stated below in Proposition 3.1, give the right speed for the lower large deviation probabilities. This will be used in section 3.2 to prove the law of large numbers for τ, but above all this will be essential to show the positivity of the rate function for lower large deviations in section 4.4.

Recall that through the max-flow min-cut theorem, we can express $\tau(A,h)$ and $\phi(A,h)$ as follows:

$$\tau(A,h) = \min\{V(E) | E \text{ is a } (A_1^h, A_2^h)\text{-cut in } \operatorname{cyl}(A,h)\}, \tag{5.1}$$

$$\phi(A,h) = \min\{V(E) | E \text{ is a } (B(A,h), T(A,h))\text{-cut in } \operatorname{cyl}(A,h)\}. \tag{5.2}$$

We define $E_\tau(A,h)$ (resp. $E_\phi(A,h)$) to be a cut achieving the minimum in (5.1) (resp. in (5.2)). If there are more than one cut achieving the minimum, we use a deterministic method to select one with the minimum number of edges among these. To control the deviations of the flow $\tau(A,h)$ (resp. $\phi(A,h)$), the main ingredient is a control on the number of edges of $E_\tau(A,h)$ (resp. $E_\phi(A,h)$). This has been done recently under the condition that $F(0) < 1 - p_c(d)$ by Zhang (see Theorem 2 in [**59**]). Since we are interested in hyperrectangles A whose size is big, in the rest of this section we shall always suppose that the side lengths of A are all larger than 1.

THEOREM 16 (Zhang 2007). *Suppose that $F(0) < 1 - p_c(d)$ and F admits an exponential moment:*

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) < \infty .$$

Then, there exist positive constants β, l_0, C_1 and C_2, depending only on F and d and such that, for every hyperrectangle A having all its sidelengths bigger than l_0, every $h > 0$ and every $n \geq \beta \mathcal{H}^{d-1}(A)$,

$$\mathbb{P}(\operatorname{card}(E_\tau(A,h)) \geq n) \leq C_1 e^{-C_2 n} .$$

Furthermore, for every $0 < h \leq \exp(\mathcal{H}^{d-1}(A))$ and every $n \geq \beta \mathcal{H}^{d-1}(A)$,

$$\mathbb{P}(\operatorname{card}(E_\phi(A,h)) \geq n) \leq C_1 e^{-C_2 n} .$$

REMARK 16. Actually, Theorem 2 in [**59**] is stated for $\phi(A,h)$ only, and only with A a hyperrectangle of the form $\prod_{i=1}^{d-1}[0,k_i] \times \{0\}$. But this result can be extended, as mentioned by Zhang (see remark 2 in [**59**]). Theorem 1 in [**59**] states that under the same hypothesis on F (existence of an exponential moment and $F(0) < 1 - p_c(d)$), the minimal number of edges $N(B)$ of a set of edges of minimal capacity among those that cut a straight (not too high) box B from the infinity satisfies the same kind of property: the probability that $N(B)$ is bigger than n decays exponentially fast in n for sufficiently large n. Zhang uses his Theorem 1 in the proof of his Theorem 2, but the proof of his Theorem 1 could also be directly adapted to prove his Theorem 2, or more generally an equivalent result about the number of edges of a set of edges separating two given subsets of the graph: for us $B(A,h)$ and $T(A,h)$, or A_1^h and A_2^h. The difference of condition on h in the two parts of Theorem 16 is linked to the fact that $E_\tau(A,h)$ is pinned at the boundary of A and $E_\phi(A,h)$ is not.

In the sequel, the following simple deviation inequality shall be useful:

LEMMA 3. *Suppose that F admits an exponential moment:*

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) \,<\, \infty\,.$$

Then, there are positive constants C_3, C_4 and C_5, depending only on F and d, such that, for any hyperrectangle A, any h, and any $u \geq C_5 \mathcal{H}^{d-1}(A)$,

$$\mathbb{P}(\tau(A,h) \geq u) \,\leq\, C_3 e^{-C_4 u}\,,$$

and

$$\mathbb{P}(\phi(A,h) \geq u) \,\leq\, C_3 e^{-C_4 u}\,.$$

Proof :
Obviously, there is an (A_1^h, A_2^h)-cut in $\operatorname{cyl}(A,h)$ containing at most $c(d)\mathcal{H}^{d-1}(A)$ edges, where $c(d)$ is a constant depending on d only: take a cut close to A itself (a more precise statement appears below in Lemma 5, we can choose $c(d) = d + d^2$). The flow through this specific cut is bigger than $\tau(A,h)$ and $\phi(A,h)$, and it is a sum of $c(d)\mathcal{H}^{d-1}(A)$ independent capacities. Thus a simple exponential Chebyshev inequality gives the result.

■

We can now state the main result of this section, the concentration property of ϕ and τ around their means.

PROPOSITION 3.1. Suppose that $F(0) < 1 - p_c(d)$ and F admits an exponential moment:

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) \,<\, \infty\,.$$

Then, there are positive constants l_0, D_1 and D_2, depending only on F and d and such that, for every hyperrectangle A having all its sidelengths bigger than l_0, every $h > 0$ and every $u > 0$,

$$\mathbb{P}(|\tau(A,h) - \mathbb{E}(\tau(A,h))| \geq u) \,\leq\, D_1 \exp\left(-\frac{u^2}{D_2 \mathcal{H}^{d-1}(A)}\right) + D_1 \exp\left(-\frac{1}{D_2}\mathcal{H}^{d-1}(A)\right)\,.$$

Furthermore, for every $h \leq \exp(\mathcal{H}^{d-1}(A))$ and every $u > 0$,

$$\mathbb{P}(|\phi(A,h) - \mathbb{E}(\phi(A,h))| \geq u) \,\leq\, D_1 \exp\left(-\frac{u^2}{D_2 \mathcal{H}^{d-1}(A)}\right) + D_1 \exp\left(-\frac{1}{D_2}\mathcal{H}^{d-1}(A)\right)\,.$$

Proof :
Here we reason as Talagrand in the proof of (8.3.2) in [**54**]. We shall prove the result for ϕ, the proof for τ being entirely analogous. Let us denote by M a median of $\phi(A,h)$. An immediate consequence of Lemma 3 is that:

$$M \,\leq\, \max\left\{\frac{1}{C_4}\log(2C_3), C_5\mathcal{H}^{d-1}(A)\right\}\,.$$

Suppose that $h \leq \exp(\mathcal{H}^{d-1}(A))$. Let β, C_1, C_2, C_3 and C_4 be the constants appearing in Theorem 16 and Lemma 3. Let us say that a $(B(A,h),T(A,h))$-cut is a *good* $(B(A,h),T(A,h))$-cut if it contains no more than

$$r \,=\, \max\left\{\beta\mathcal{H}^{d-1}(A), \frac{1}{C_2}\log(4C_1), \frac{1}{C_4}\log(2C_3), C_5\mathcal{H}^{d-1}(A)\right\}$$

edges. Define

$$\phi_g(A,h) \,=\, \min\{V(E) | E \text{ is a good } (B(A,h),T(A,h))\text{-cut in } \operatorname{cyl}(A,h)\}\,,$$

and let M_g be a median of $\phi_g(A,h)$. Let $\mathcal{S}$ denote a family of sets of edges, $(X_e)_{e\in\mathbb{E}^d}$ be a collection of positive i.i.d. random variables, and $Z_{\mathcal{S}} = \inf_{S\in\mathcal{S}}\sum_{e\in S} X_e$. By Proposition 8.3 in [**54**], there is a universal constant K, such that, if $\mathbb{E}(\exp(X_e/K)) \leq 2$, then, for all $u>0$,

$$\mathbb{P}(|Z_{\mathcal{S}} - M| \geq u) \leq 4\exp\left(-\frac{1}{K}\min\left(\frac{u^2}{R},u\right)\right) ,$$

where M is any median of $Z_{\mathcal{S}}$, and $R = \sup_{S\in\mathcal{S}} \operatorname{card} S$. Define:

$$\gamma_0 = \gamma\min\left\{\frac{\log(2)}{\log\mathbb{E}(e^{\gamma t(e)})},1\right\} .$$

We apply this result with $X_e = K\gamma_0 t(e)$ and $\mathcal{S}$ the set of good $(B(A,h),T(A,h))$-cuts. These cuts contain no more than r edges, and thus, $r\geq R$. Noticing that $\gamma_0\leq\gamma$, Jensen's inequality ensures that $\mathbb{E}(\exp(X_e/K))\leq 2$:

$$\mathbb{E}(\exp(X_e/K)) = \mathbb{E}(\exp(\gamma_0 t(e))) \leq (\mathbb{E}(\exp(\gamma t(e))))^{\frac{\gamma_0}{\gamma}} \leq 2 .$$

Thus, for every $u>0$,

$$\mathbb{P}(|\phi_g(A,h) - M_g| > u/(K\gamma_0)) \;\leq\; 4\exp\left(-\frac{1}{K}\min\left\{\frac{u^2}{r},u\right\}\right) .$$

Thus, there is a constant K', depending only on F, and such that, for every $u>0$:

$$\mathbb{P}(|\phi_g(A,h) - M_g| > u) \;\leq\; 4\exp\left(-\frac{1}{K'}\min\left\{\frac{u^2}{r},u\right\}\right) .$$

Also, Theorem 16 implies that for every hyperrectangle A having all its sidelengths bigger than l_0:

$$\mathbb{P}(\phi_g(A,h)\neq\phi(A,h)) \;\leq\; C_1 e^{-C_2 r} .$$

Thus, for every $u>0$,

$$\mathbb{P}(|\phi(A,h) - M_g| > u) \;\leq\; 4\exp\left(-\frac{1}{K'}\min\left\{\frac{u^2}{r},u\right\}\right) + C_1\exp\left(-C_2 r\right) .$$

Taking $u = \max\{\sqrt{rK'\log(16)}, K'\log(16)\}$ in the above inequality, the right hand side becomes lower than $\frac{1}{2}$. Thus,

$$|M - M_g| \;\leq\; \max\{\sqrt{rK'\log(16)}, K'\log(16)\} .$$

Therefore, we can find a positive constant K'', depending only on F, such that:

$$\mathbb{P}(|\phi(A,h) - M| > u) \;\leq\; K''\exp\left(-\frac{1}{K''}\min\left\{\frac{u^2}{r},u\right\}\right) + C_1\exp\left(-C_2 r\right) . \tag{5.3}$$

Finally, the mean and median of $\phi(A,h)$ are not far apart. Indeed, noticing that $r\geq M$, and using Lemma 3 and inequality (5.3),

$$\begin{aligned}
|\mathbb{E}(\phi(A,h)) - M| \;&\leq\; \int_0^\infty \mathbb{P}(|\phi(A,h) - M| > u)\,du \\
&\leq\; \int_0^r \mathbb{P}(|\phi(A,h) - M| > u)\,du + \int_r^\infty \mathbb{P}(\phi(A,h) > u)\,du \\
&\leq\; \int_0^r \left(K''\exp\left(-\frac{1}{K''}\min\left\{\frac{u^2}{r},u\right\}\right) + C_1\exp\left(-C_2 u\right)\right)\,du \\
&\qquad + \int_r^\infty C_3\exp\left(-C_4 u\right)\,du \\
&\leq\; K''\frac{\sqrt{\pi K'' r}}{2} + \frac{C_1}{C_2} + \frac{C_3}{C_4} .
\end{aligned}$$

Recalling that $r \geq \beta \mathcal{H}^{d-1}(A)$, we then deduce from inequality (5.3) that we can find positive constants D_1 and D_2, depending only on F and d and such that for every hyperrectangle A having all its sidelengths bigger than l_0, every $h \leq \exp(\mathcal{H}^{d-1}(A))$ and every $u > 0$,

$$\mathbb{P}(|\phi(A,h) - \mathbb{E}(\phi(A,h))| > u) \leq D_1 \exp\left(-\frac{u^2}{D_2 \mathcal{H}^{d-1}(A)}\right) + D_1 \exp\left(-\frac{1}{D_2}\mathcal{H}^{d-1}(A)\right).$$

Proposition 3.1 follows.

∎

REMARK 17. A technical remark is that one would obtain directly concentration of the maximal flow around its mean, rather than around its median, using the method of [**14**]. We chose to use Talagrand's result because of the "ready-to-use" aspect of Proposition 8.3 in [**54**], whereas in order to use [**14**] for the upper deviations, one needs a very slight modification of their proof.

REMARK 18. More importantly, it should be noted that Proposition 3.1 certainly does not give the right order of the "typical fluctuations", i.e., fluctuations that occur with a non negligible probability. Indeed, let S_n be the square:

$$S_n = \partial\left(\left[-\frac{1}{2}, n - \frac{1}{2}\right]^{d-1} \times \left\{\frac{1}{2}\right\}\right).$$

We say that a set of edges E "is a cut based on S_n" if it is finite, and if every path in $\mathbb{Z}^d$ which is not contractible to one point in $\mathbb{R}^d \smallsetminus S_n$ has to contain one edge of E. Let $\mathcal{E}_n$ be the set of all sets of edges which are a cut based on S_n and define:

$$\tilde{\tau}_n = \inf\{V(E) | E \in \mathcal{E}_n\}.$$

Then, mimicking the work of [**8**], one can prove that the variance of $\tilde{\tau}_n$ is at most of order $C(n^{d-1}/\log n)$ where C is a constant (and there is no reason for this bound to be optimal). It is then very reasonable to think that $\tau(A,h)$ and $\phi(A,h)$ will inherit this property to have "submean" variance, i.e. their typical fluctuations should be small with respect to $(\mathcal{H}^{d-1}(A))^{1/2}$ when the side lengths of A tend to infinity.

3.2. Convergence of $\mathbb{E}(\tau)$ and τ. We suppose that the capacity of the edges is in L^1. Let us consider two hyperrectangles A, A' which have a common orthogonal unit vector $\vec{v}$, and two functions $h, h' : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{n\to\infty} h(n) = \lim_{n\to\infty} h'(n) = +\infty$. We take $n, N \in \mathbb{N}$ such that $N \geq N_0(n)$ with $N_0(n)$ large enough to have $h(N) \geq h'(n) + 1$ and $N \operatorname{diam}(A) > n \operatorname{diam}(A')$ for all $N \geq N_0(n)$ (here $\operatorname{diam}(A) = \sup\{\|x-y\|_2 \,|\, x, y \in A\}$). We define

$$D(n,N) = \{x \in NA \,|\, d(x, \partial(NA)) > 2n \operatorname{diam} A'\}.$$

There exists a finite collection of sets $(T(i), i \in I)$ such that each $T(i)$ is a translate of nA' intersecting the set $D(n,N)$, the sets $(T(i), i \in I)$ have pairwise disjoint interiors, and their union $\cup_{i\in I} T(i)$ contains the set $D(n,N)$ (see Figure 1). For all i, there exists a vector $\vec{t_i}$ in $\mathbb{R}^d$ such that $\|\vec{t_i}\|_\infty < 1$ and $T'(i) = T(i) + \vec{t_i}$ is the image of nA' by an integer translation (that leaves $\mathbb{Z}^d$ globally invariant). The cylinders $(\operatorname{cyl}(T'(i), h'(n)), i \in I)$ are included in $\operatorname{cyl}(NA, h(N))$, and the family $(\tau(T'(i), h'(n)), i \in I)$ is identically distributed (but not indenpendant in general). For each i, by the max-flow min-cut theorem, we know that $\tau(T'(i), h'(n))$ is equal to the minimal capacity $V(E) = \sum_{e\in E} t(e)$ of a set of edges $E \subset \operatorname{cyl}(T'(i), h'(n))$ that cuts $T'(i)_1^{h'(n)}$ from $T'(i)_2^{h'(n)}$. For each $i \in I$, let E_i be such a set of edges of minimal capacity, i.e., $\tau(T'(i), h'(n)) = V(E_i)$.

We take a fixed $\zeta \geq 4d$. Let E_0 be the set of the edges included in $\mathcal{E}_0$, where we define

$$\mathcal{E}_0 = \operatorname{cyl}(NA \smallsetminus D(n,N), \zeta) \cup \bigcup_{i\in I} \left(\mathcal{V}(\operatorname{cyl}(\partial T'(i), +\infty), \zeta) \cap \mathcal{V}(\operatorname{hyp}(NA), \zeta)\right).$$

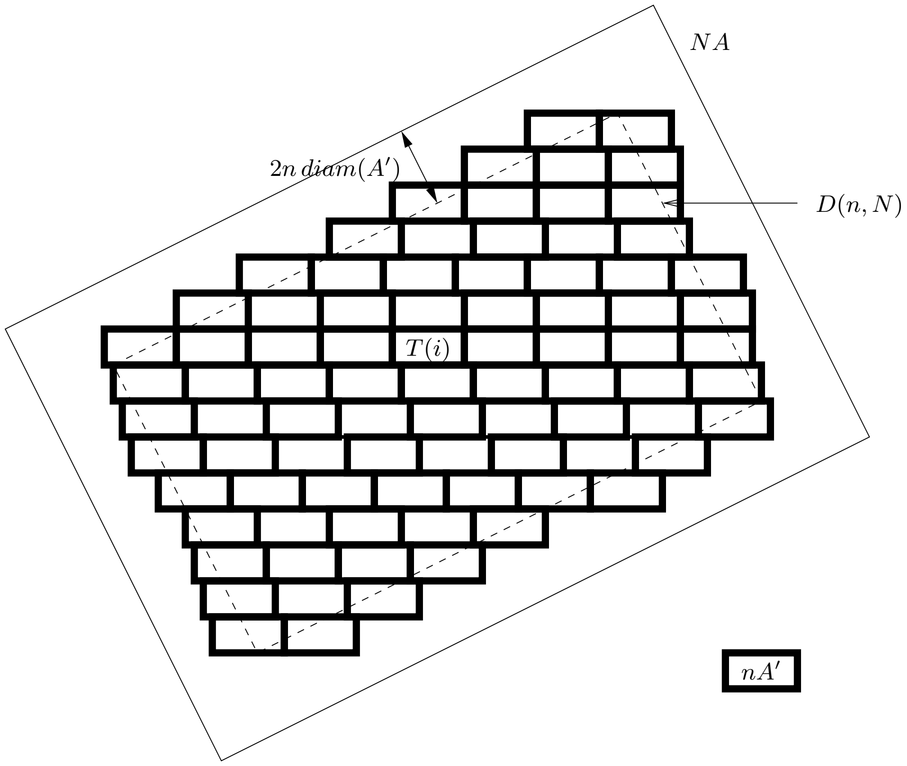

FIGURE 1. The hyperplane $\mathrm{hyp}(A)$.

The set of edges $E_0 \cup \bigcup_{i\in I} E_i$ cuts $(NA)_1^{h(N)}$ from $(NA)_2^{h(N)}$ in cyl$(NA, h(N))$, so

$$\begin{aligned} \tau(NA, h(N)) &\leq V(E_0) + \sum_{i\in I} V(E_i) \\ &\leq V(E_0) + \sum_{i\in I} \tau(T'(i), h'(n)) \,. \end{aligned} \tag{5.4}$$

Taking the expectation of (5.4), we obtain

$$\begin{aligned} \frac{\mathbb{E}\left(\tau(NA, h(N))\right)}{\mathcal{H}^{d-1}(NA)} &\leq \frac{\mathrm{card}(E_0)}{\mathcal{H}^{d-1}(NA)}\mathbb{E}(t) + \frac{\mathrm{card}(I)\mathbb{E}(\tau(nA', h'(n)))}{\mathcal{H}^{d-1}(NA)} \\ &\leq \frac{\mathrm{card}(E_0)}{\mathcal{H}^{d-1}(NA)}\mathbb{E}(t) + \frac{\mathbb{E}(\tau(nA', h'(n)))}{\mathcal{H}^{d-1}(nA')} \,. \end{aligned}$$

There exists a constant $c(d, \zeta, A, A')$ such that

$$\mathrm{card}(E_0) \leq c(d, \zeta, A, A') \left(N^{d-2}n + N^{d-1}/n + 1\right) ,$$

and so

$$\lim_{n\to\infty} \lim_{N\to\infty} \frac{\mathrm{card}(E_0)}{\mathcal{H}^{d-1}(NA)} = 0 \,.$$

By sending N to infinity, and then n to infinity, we obtain that

$$\limsup_{N\to\infty} \frac{\mathbb{E}\left(\tau(NA, h(N))\right)}{\mathcal{H}^{d-1}(NA)} \leq \liminf_{n\to\infty} \frac{\mathbb{E}\left(\tau(nA', h'(n))\right)}{\mathcal{H}^{d-1}(nA')} \,.$$

For $A = A'$ and $h = h'$, we deduce from this inequality that $\lim_{n\to\infty} \mathbb{E}\left(\tau(nA, h(n))\right)/\mathcal{H}^{d-1}(nA)$ exists. For different A, A', and h, h', we conclude that this limit does not depend on A and h, but only on the direction of $\vec{v}$ (and on F and d of course). We denote this limit by $\nu(\vec{v})$.

It remains to prove that the sequence $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$ converges almost surely towards $\nu(\vec{v})$. For that purpose, we will use Proposition 3.1. We suppose that $F(0) < 1 - p_c(d)$ and that there exists a positive γ such that $\int \exp(\gamma x)dF(x) < \infty$. Proposition 3.1 says there exist constants D_1 and D_2 such that for all real number $u > 0$,

$$\begin{aligned}\mathbb{P}\Big(|\tau(nA,h(n)) - \mathbb{E}(\tau(nA,h(n)))| \geq u\Big) \\ \leq D_1 \exp\left(\frac{-u^2}{D_2\mathcal{H}^{d-1}(nA)}\right) + D_1 \exp\left(\frac{-\mathcal{H}^{d-1}(nA)}{D_2}\right),\end{aligned}$$

so for all positive ε,

$$\mathbb{P}\left(\left|\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(NA)} - \frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(NA)}\right| > \varepsilon\right) \leq 2D_1 \exp\left(-\min\{1,\varepsilon^2\}\frac{\mathcal{H}^{d-1}(nA)}{D_2}\right).$$

By the Borel-Cantelli lemma we obtain

$$\forall \varepsilon > 0\,,\ a.s. \qquad \limsup_{n\to\infty}\left|\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} - \frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(nA)}\right| \leq \varepsilon\,.$$

We conclude that

$$\limsup_{n\to\infty}\left|\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} - \frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(nA)}\right| = 0 \qquad a.s.\,,$$

and then since $\lim_{n\to\infty}\mathbb{E}\left(\tau(nA,h(n))\right)/\mathcal{H}^{d-1}(nA) = \nu(\vec{v})$, we have

$$\lim_{n\to\infty}\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}) \qquad a.s.$$

This ends the proof of Proposition 2.1.

REMARK 19. If there exists a real M such that all the coordinates of $M\vec{v}$ are rational, and if the origin of the graph belongs to A, then the convergence of $\tau(nA,h(n)/\mathcal{H}^{d-1}(nA)$ a.s. and in L^1 can also be obtained as a consequence of multiparameter subadditive ergodic theorems proved in [**43**], [**53**], as explained in the introduction of the thesis, and the conditions on F can be released (we need only $\mathbb{E}(t(e)) < \infty$). The independance of the limit with regard to the particular form of A is a consequence of [**43**]. Unfortunately, we cannot apply these subadditive ergodic theorems for general A and $\vec{v}$. We choose here to use a concentration inequality to obtain simply the almost convergence of $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA,n\in\mathbb{N}))$ because we already stated it.

REMARK 20. The almost sure convergence of $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$ is not necessary to prove Theorem 11, but we need it to prove Theorem 12 (in fact we need only the convergence in probability), and it is natural to study it with the convergence of the sequence $(\mathbb{E}(\tau(nA,h(n)))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$.

3.3. Proof of Theorem 11, Corollary 2.1 and Theorem 14. Now, we can prove Theorem 11, and so we consider F, h, $\vec{v}$ and A as in the statement of this Theorem. If $\nu(\vec{v}) = 0$, there is nothing to prove. Suppose now that $\nu(\vec{v}) > 0$ and let $\varepsilon \leq \nu(\vec{v})$ be a positive real number. Let $u = \varepsilon/(2\nu_{max})$, where $\nu_{max} = \max\{\nu(\vec{v}) \,|\, \vec{v}\ unit\ vector\}$. Then $u > 0$ and we have

$$\frac{\nu(\vec{v}) - \varepsilon}{\nu(\vec{v}) - \varepsilon/2} \leq 1 - u\,.$$

Let n_0 be large enough to have

$$\forall n \geq n_0 \qquad \frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(nA)} \geq \nu(\vec{v}) - \frac{\varepsilon}{2}\,.$$

Then, for all $n \geq n_0$,

$$\mathbb{P}\Big[\tau(nA,h(n)) \leq (\nu(\vec{v}) - \varepsilon)\,\mathcal{H}^{d-1}(nA)\Big]$$

$$\begin{aligned}
&\leq \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathbb{E}(\tau(nA,h(n)))} \leq (\nu(\vec{v})-\varepsilon)\frac{\mathcal{H}^{d-1}(nA)}{\mathbb{E}(\tau(nA,h(n)))}\right]\\
&\leq \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathbb{E}(\tau(nA,h(n)))} \leq 1-u\right].
\end{aligned}$$

Now, Proposition 3.1 gives us for all $n \geq n_0$

$$\begin{aligned}
\mathbb{P}\left[\tau(nA,h(n)) \leq (\nu(\vec{v})-\varepsilon)\,\mathcal{H}^{d-1}(nA)\right] \;&\leq\; D_1 \exp\left(-\frac{u^2\mathbb{E}(\tau(nA,h(n)))^2}{D_2\mathcal{H}^{d-1}(nA)}\right)\\
&\quad + D_1 \exp\left(-\frac{\mathcal{H}^{d-1}(nA)}{D_2}\right)\\
&\leq\; D_1 \exp\left(-D_2^{-1}u^2(\nu(\vec{v})-\varepsilon/2)^2\mathcal{H}^{d-1}(nA)\right)\\
&\quad + D_1 \exp\left(-D_2^{-1}\mathcal{H}^{d-1}(nA)\right)\\
&\leq\; D_1 \exp\left(-\left(\frac{\varepsilon^4}{16D_2\nu_{max}^2}+\frac{1}{D_2}\right)\mathcal{H}^{d-1}(nA)\right).
\end{aligned}$$

This gives us the desired result for τ.

To prove Corollary 2.1, we have to compare ϕ and τ. We suppose that $\lim_{n\to\infty} h(n)/n = 0$, and fix $\zeta \geq 4d$. Let E_1^+ be the set of the edges that belong to $\mathcal{E}_1^+$, defined as

$$\mathcal{E}_1^+ \;=\; \mathcal{V}(\mathrm{cyl}(\partial(nA),h(n)),\zeta) \cap \mathrm{cyl}(nA,h(n)).$$

We have, for all n,

$$\tau(nA,h(n)) \;\geq\; \phi(nA,h(n)) \;\geq\; \tau(nA,h(n)) - V(E_1^+).$$

For all positive ε, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v})-\varepsilon\right)\\
&\qquad\leq\; \mathbb{P}\left(\frac{\tau(nA,h(n))-V(E_1^+)}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v})-\varepsilon\right)\\
&\qquad\leq\; \mathbb{P}\left(\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v})-\frac{\varepsilon}{2}\right) + \mathbb{P}\left(V(E_1^+) \geq \frac{\varepsilon}{2}\mathcal{H}^{d-1}(nA)\right).
\end{aligned}$$

On one hand, by Theorem 11 we know that for large n we have

$$\mathbb{P}\left(\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v})-\frac{\varepsilon}{2}\right) \;\leq\; \exp\left(-C(\varepsilon,F,d)\mathcal{H}^{d-1}(A)n^{d-1}\right).$$

On the other hand, there exists a constant C^+ such that

$$\mathrm{card}(E_1^+) \leq C^+ n^{d-2}h(n),$$

so for all $\varepsilon > 0$, for a family (t_k) of independent variables with the same law as the capacities of the edges, we have

$$\begin{aligned}
\mathbb{P}\left(V(E_1^+) \geq \frac{\varepsilon}{2}\mathcal{H}^{d-1}(nA)\right) \;&\leq\; \mathbb{P}\left(\sum_{k=1}^{C^+n^{d-2}h(n)} t_k \geq \frac{\varepsilon}{2}\mathcal{H}^{d-1}(nA)\right)\\
&\leq\; \mathbb{E}(e^{\gamma t})^{C^+n^{d-2}h(n)} \exp\left(-\gamma\frac{\varepsilon}{2}\mathcal{H}^{d-1}(nA)\right)
\end{aligned}$$

$$\leq \exp\left(-\mathcal{H}^{d-1}(nA)\left(\gamma\frac{\varepsilon}{2} - C^+\frac{n^{d-2}h(n)}{\mathcal{H}^{d-1}(nA)}\log\mathbb{E}(e^{\gamma t})\right)\right).$$

For n large enough, we have

$$C^+\frac{n^{d-2}h(n)}{\mathcal{H}^{d-1}(nA)}\log\mathbb{E}(e^{\gamma t}) \leq \gamma\frac{\varepsilon}{4},$$

so for large n we have

$$\mathbb{P}\left(V(E_1^+) \geq \frac{\varepsilon}{2}\mathcal{H}^{d-1}(nA)\right) \leq \exp\left(-\gamma\frac{\varepsilon}{4}\mathcal{H}^{d-1}(nA)\right).$$

So Corollary 2.1 is proved.

Exactly the same proof as the one of Theorem 11 for τ holds for ϕ under the assumption that the sequence $(\phi(nA,h(n))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$ converges almost surely towards $\nu(\vec{v})$, and so under the assumptions in Theorem 14. As said in Remark 13, we also could have proved Corollary 2.1 this way.

4. Large deviation principle for τ

In this section, we show the large deviation principle for τ. We construct a precursor of the rate function in section 4.1, and then study its properties. Precisely, we show it is convex in section 4.2, finite (and thus continuous) on $]\delta\|\vec{v}\|_1, +\infty[$ in section 4.3, and positive on $[0,\nu(\vec{v})[$ in section 4.4. In section 4.5 we show that $\delta\|\vec{v}\|_1 < \nu(\vec{v})$ if $F(\delta) < 1-p_c(d)$. Using the result of chapter 3, we know that upper large deviations occur at an order bigger than the surface order, so we can complete the proof of the full large deviation principle in section 4.6.

4.1. Construction of the rate function.

We will prove the following lemma:

LEMMA 4. *For every function $h:\mathbb{N}\to\mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, for every hyperrectangle A, for all λ in $\mathbb{R}^+$, the limit*

$$\lim_{n\to\infty}\frac{-1}{\mathcal{H}^{d-1}(nA)}\log\mathbb{P}\left[\tau(nA,h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right]$$

exists in $[0,+\infty]$ and depends only on the direction of $\vec{v}$, one of the two unit vectors orthogonal to $\mathrm{hyp}(A)$. *We denote it by $\mathcal{I}_{\vec{v}}(\lambda)$.*

We introduce a factor $1/\sqrt{n}$ in the definition of $\mathcal{I}_{\vec{v}}(\lambda)$ because we want to work with subadditive objects, but $\tau(A,h)$ is not subadditive in A, except for straight cylinders. Indeed, if A and B are two hyperrectangles with a common orthogonal vector and with a common side, to glue together a set of edges in $\mathrm{cyl}(A,h)$ that cuts A_1^h from A_2^h and a set of edges in $\mathrm{cyl}(B,h)$ that cuts B_1^h from B_2^h, we have to add edges at the common side of A and B (see the set of edges E_0 defined in section 3.2). These edges may not have a capacity 0, so they perturb the subadditivity of τ. We add the factor $1/\sqrt{n}$ to compensate.

Proof :
For the proof of Lemma 4, we consider the same construction as in section 3.2 (see Figure 1). From (5.4) we deduce that for all $\lambda\in\mathbb{R}^+_*$, we have

$$\mathbb{P}\left[\tau(NA,h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\right]$$
$$\geq \mathbb{P}\left[V(E_0) + \sum_{i\in I}\tau(T'(i),h'(n)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\right].$$

Let $\mathcal{D} = \{\lambda \,|\, \mathbb{P}(t(e) \leq \lambda) > 0\}$, and $\delta = \inf \mathcal{D}$. We take $u = \delta + \zeta$, so $p = \mathbb{P}(t(e) \leq u) > 0$. Using the FKG inequality and the fact that the family $(\tau(T'(i), h'(n)), i \in I)$ is identically distributed, we obtain that

$$\begin{aligned}\mathbb{P}\left[\tau(NA, h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\right] \\ &\geq \mathbb{P}\left[V(E_0) \leq u \,\mathrm{card}(E_0)\right] \\ &\quad \times \prod_{i\in I}\mathbb{P}\left[\tau(T'(i), h'(n)) \leq \frac{(\lambda - 1/\sqrt{N})\mathcal{H}^{d-1}(NA) - u\,\mathrm{card}(E_0)}{\mathrm{card}(I)}\right] \\ &\geq \mathbb{P}\left[t(e) \leq u\right]^{\mathrm{card}(E_0)} \\ &\quad \times \mathbb{P}\left[\tau(nA', h'(n)) \leq \frac{(\lambda - 1/\sqrt{N})\mathcal{H}^{d-1}(NA) - u\,\mathrm{card}(E_0)}{\mathrm{card}(I)}\right]^{\mathrm{card}(I)}.\end{aligned}$$

We have immediately that $\mathrm{card}(I) \leq \mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA')$, so

$$\begin{aligned}\frac{-1}{\mathcal{H}^{d-1}(NA)}\log\mathbb{P}\left[\tau(NA, h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\right] \\ \leq \frac{-1}{\mathcal{H}^{d-1}(nA')}\log\mathbb{P}\left[\tau(nA', h'(n)) \leq \beta\right] - \frac{\mathrm{card}(E_0)}{\mathcal{H}^{d-1}(NA)}\log p\,,\end{aligned}$$

where

$$\beta \;=\; \frac{\left(\lambda - 1/\sqrt{N}\right)\mathcal{H}^{d-1}(NA) - u\,\mathrm{card}(E_0)}{\mathrm{card}(I)}\,.$$

As we saw in section 3.2, there exists a constant $c(d, \zeta, A, A')$ such that

$$\mathrm{card}(E_0) \;\leq\; c(d, \zeta, A, A')\left(N^{d-2}n + N^{d-1}/n + 1\right)\,.$$

On one hand, we obtain that

$$\lim_{n\to\infty}\lim_{N\to\infty}\frac{\mathrm{card}(E_0)}{\mathcal{H}^{d-1}(NA)}\log p \;=\; 0\,.$$

On the other hand we want to compare β with $(\lambda - 1/\sqrt{n})\mathcal{H}^{d-1}(nA')$. Obviously we have

$$\frac{\lambda\mathcal{H}^{d-1}(NA)}{\mathrm{card}(I)} \;\geq\; \lambda\mathcal{H}^{d-1}(nA')\,.$$

We also know that

$$\mathrm{card}(I) \;\geq\; \frac{\mathcal{H}^{d-1}(D(n,N))}{\mathcal{H}^{d-1}(nA')}$$

so there exist a constant $c'(d, A, A')$ and an integer $N_1(n)$ large enough to have, for all $N \geq N_1(n)$,

$$\mathrm{card}(I) \;\geq\; c'(d, A, A')\left(\frac{N}{n}\right)^{d-1}\,.$$

Thus, there exist constants $c_1(d, \zeta, A, A')$ and $c_2(d, \zeta, F, A, A')$ such that for all $N \geq N_1(n)$, we have

$$\frac{\mathcal{H}^{d-1}(NA)}{\mathrm{card}(I)\sqrt{N}} \;\leq\; \frac{c_1(d, \zeta, A, A')}{\sqrt{N}}\mathcal{H}^{d-1}(nA')$$

and

$$\frac{u\,\mathrm{card}(E_0)}{\mathrm{card}(I)} \;\leq\; c_2(d, \zeta, F, A, A')\left(\frac{n}{N} + \frac{1}{n}\right)\mathcal{H}^{d-1}(nA')\,.$$

There exists n_0 such that for all $n \geq n_0$, $c_2/n \leq 1/(4\sqrt{n})$. Then there exists $N_2(n) \geq N_0(n) \vee N_1(n)$ such that for all $N \geq N_2(n)$, $c_2 n/N \leq 1/(4\sqrt{n})$ and $c_1/\sqrt{N} \leq 1/(2\sqrt{n})$. Thus for a fixed $n \geq n_0$, for all $N \geq N_2(n)$, we have

$$\beta \geq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA')\,.$$

Now in the following inequality, obtained for $n \geq n_0$ and $N \geq N_2(n)$,

$$\begin{aligned}&\frac{-1}{\mathcal{H}^{d-1}(NA)}\log\mathbb{P}\left[\tau(NA,h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\right]\\ &\leq \frac{-1}{\mathcal{H}^{d-1}(nA')}\log\mathbb{P}\left[\tau(nA',h'(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA')\right] - \frac{\operatorname{card}(E_0)}{\mathcal{H}^{d-1}(NA)}\log p\,,\end{aligned}$$

we send N to infinity for a fixed $n \geq n_0$, and then we send n to infinity. We thus obtain

$$\begin{aligned}&\limsup_{N\to\infty}\frac{-1}{\mathcal{H}^{d-1}(NA)}\log\mathbb{P}\left[\tau(NA,h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\right]\\ &\qquad\leq \liminf_{n\to\infty}\frac{-1}{\mathcal{H}^{d-1}(nA')}\log\mathbb{P}\left[\tau(nA',h'(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA')\right]\,.\end{aligned}$$

For $A = A'$ and $h = h'$, this gives us the existence of

$$\lim_{n\to\infty}\frac{-1}{\mathcal{H}^{d-1}(nA)}\log\mathbb{P}\left[\tau(nA,h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right]$$

for all $\lambda \in \mathbb{R}^+_*$, and for different A, A', h, h' this shows that the limit is independent of A and h. We denote this limit by $\mathcal{I}_{\vec{v}}(\lambda)$.

For $\lambda = 0$,

$$\mathbb{P}\left[\tau(nA,h(n)) \leq -\frac{\mathcal{H}^{d-1}(nA)}{\sqrt{n}}\right] = 0$$

for all $n \in \mathbb{N}$, so the previous limit equals $+\infty$, independently of A and $\vec{v}$. This ends the proof of Lemma 4.

∎

REMARK 21. The function $\mathcal{I}_{\vec{v}}$ is not exactly the rate function we will consider later: we will change its value from 0 to $+\infty$ on $]\nu(\vec{v}), +\infty]$ and we will regularize it at $\|\vec{v}\|_1\delta$.

4.2. Convexity of $\mathcal{I}_{\vec{v}}$. We will prove that $\mathcal{I}_{\vec{v}}$ is convex, i.e., for all $\lambda_1 \geq \lambda_2 \in \mathbb{R}^+$ and $\alpha \in]0,1[$, we have

$$\mathcal{I}_{\vec{v}}(\alpha\lambda_1 + (1-\alpha)\lambda_2) \leq \alpha\mathcal{I}_{\vec{v}}(\lambda_1) + (1-\alpha)\mathcal{I}_{\vec{v}}(\lambda_2)\,.$$

For $\lambda_2 = 0$, the result is obvious, so we suppose $\lambda_2 > 0$. We keep the same notations as in the previous section, for $D(n,N)$, $T(i)$, E_i, etc..., except that we take $A = A'$. We define

$$\gamma = \lfloor \alpha \operatorname{card}(I) \rfloor\,.$$

If we have

$$\tau(T'(i),h(n)) \leq (\lambda_1 - 1/\sqrt{n})\mathcal{H}^{d-1}(nA) \qquad \textit{for } i = 1,\dots,\gamma\,, \tag{5.5}$$

$$\tau(T'(i),h(n)) \leq (\lambda_2 - 1/\sqrt{n})\mathcal{H}^{d-1}(nA) \qquad \textit{for } i = \gamma+1,\dots,\operatorname{card}(I)\,, \tag{5.6}$$

and

$$V(E_0) \leq u \operatorname{card}(E_0)\,,$$

then we obtain that

$$\begin{aligned}
\tau(NA,h(N)) \,&\leq\, \left(\gamma(\lambda_1-\frac{1}{\sqrt{n}})+(\mathrm{card}(I)-\gamma)(\lambda_2-\frac{1}{\sqrt{n}})\right)\mathcal{H}^{d-1}(nA)+u\,\mathrm{card}(E_0)\,,\\
&\leq\, (\alpha\lambda_1+(1-\alpha)\lambda_2)\,\mathrm{card}(I)\mathcal{H}^{d-1}(nA)-\frac{\mathrm{card}(I)\mathcal{H}^{d-1}(nA)}{\sqrt{n}}+u\,\mathrm{card}(E_0)\,,\\
&\leq\, (\alpha\lambda_1+(1-\alpha)\lambda_2)\mathcal{H}^{d-1}(NA)-\rho\,,
\end{aligned}$$

where

$$\rho \,=\, \frac{\mathrm{card}(I)\mathcal{H}^{d-1}(nA)}{\sqrt{n}}-u\,\mathrm{card}(E_0)\,.$$

We want to prove that $\rho\geq\mathcal{H}^{d-1}(NA)/\sqrt{N}$ for N large enough. We have seen in the previous section that there exists a constant $c(d,\zeta,A)$ such that

$$\mathrm{card}(E_0)\,\leq\, c(d,\zeta,A)N^{d-1}\left(\frac{n}{N}+\frac{1}{n}\right)\,,$$

and that there exists a constant $c'(d,A)$ and a $N_1(n)$ large enough to have, for all $N\geq N_1(n)$,

$$\mathrm{card}(I)\geq c'(d,A)\left(\frac{N}{n}\right)^{d-1}\,.$$

There exists n_1 such that for all $n\geq n_1$, $2c/n\leq c'/(2\sqrt{n})$. For a fixed $n\geq n_1$, there exists $N_3(n)$ such that for all $N\geq N_3(n)$ we have

$$\frac{u\,\mathrm{card}(E_0)}{\mathcal{H}^{d-1}(NA)}\leq\frac{2c}{n}\leq\frac{c'}{2\sqrt{n}}\,,\qquad \frac{\mathrm{card}(I)\mathcal{H}^{d-1}(nA)}{\mathcal{H}^{d-1}(NA)\sqrt{n}}\geq\frac{c'}{\sqrt{n}}\qquad \textit{and}\qquad \frac{c'}{2\sqrt{n}}\geq\frac{1}{\sqrt{N}}\,.$$

We conclude that for $n\geq n_1$ and $N\geq N_3(n)$, $\rho\geq\mathcal{H}^{d-1}(NA)/\sqrt{N}$ and then

$$\tau(NA,h(N))\,\leq\,\left(\alpha\lambda_1+(1-\alpha)\lambda_2-\frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\,,$$

as long as (5.5) and (5.6) hold. Then, for all $n\geq n_1$ and $N\geq N_3(n)$, we have, by the FKG inequality:

$$\begin{aligned}
\mathbb{P}\Bigg(\tau(NA,h(N))\leq&\left(\alpha\lambda_1+(1-\alpha)\lambda_2-\frac{1}{\sqrt{N}}\right)\mathcal{H}^{d-1}(NA)\Bigg)\\
&\geq\,\mathbb{P}\left(\tau(nA,h(n))\leq(\lambda_1-\frac{1}{\sqrt{n}})\mathcal{H}^{d-1}(nA)\right)^{\gamma}\\
&\quad\times\mathbb{P}\left(\tau(nA,h(n))\leq(\lambda_2-\frac{1}{\sqrt{n}})\mathcal{H}^{d-1}(nA)\right)^{\mathrm{card}(I)-\gamma}p^{\mathrm{card}(E_0)}\,.
\end{aligned}$$

We take the logarithm of this expression, we divide it by $\mathcal{H}^{d-1}(NA)$, we send N to infinity and then n to infinity to obtain

$$\mathcal{I}_{\vec{v}}(\alpha\lambda_1+(1-\alpha)\lambda_2)\,\leq\,\alpha\mathcal{I}_{\vec{v}}(\lambda_1)+(1-\alpha)\mathcal{I}_{\vec{v}}(\lambda_2)\,.$$

The convexity of $\mathcal{I}_{\vec{v}}$ is so proved.

4.3. Continuity of $\mathcal{I}_{\vec{v}}$. For every hyperrectangle A, we denote by $\mathcal{N}(A,h)$ the minimal number of edges in A that can disconnect A_1^h from A_2^h in $\mathrm{cyl}(A,h)$. We first need the following lemma:

LEMMA 5. *Let $\vec{v}$ be a unitary vector. Then for all hyperrectangle A orthogonal to $\vec{v}$, and for all function $h : \mathbb{N} \to \mathbb{R}^+$, there exists a constant $D(d,A)$ depending only on d and A such that*

$$\left| \frac{\mathcal{N}(nA,h(n))}{\mathcal{H}^{d-1}(nA)} - \|\vec{v}\|_1 \right| \leq \frac{D(d,A)}{n}.$$

Proof :
To prove it, we introduce some definitions. For A a hyperrectangle orthogonal to $\vec{v}$, we denote by $P_i(A)$ the orthogonal projection of A on the i-th hyperplane of coordinates, i.e., the hyperplane $\{(x_1,...,x_d) \in \mathbb{R}^d \,|\, x_i = 0\}$. We have the property

$$\frac{\sum_{i=1}^d \mathcal{H}^{d-1}(P_i(A))}{\mathcal{H}^{d-1}(A)} = \|\vec{v}\|_1 .$$

Indeed, $\mathcal{H}^{d-1}(P_i(A)) = |v_i|\mathcal{H}^{d-1}(A)$, where $\vec{v} = (v_1,...,v_d)$. We define now $E_i(nA)$ the set of edges orthogonal to the i-th hyperplane of coordinates that 'intersect' the hyperrectangle nA in the following sense:

$$E_i(nA) = \{e = \langle x,y\rangle \in \mathbb{E}^d \,|\, y_i = x_i + 1 \text{ and } [x,y[\cap nA \neq \emptyset\}.$$

We exclude here the extremity y in the segment $[x,y[$ to avoid problems of non uniqueness of such an edge intersecting nA at a given point. On one hand, we have a straight path that goes from $(nA)_1^{h(n)}$ to $(nA)_2^{h(n)}$ through each edge of $E_i(nA)$, $i = 1,...,d$, maybe except the edges that intersect nA along $\partial(nA)$, and these paths are disjoint, so a set of edges that disconnect $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA,h(n))$ must cut each one of these paths, thus

$$\begin{aligned}\mathcal{N}(nA,h(n)) &\geq \sum_{i=1}^d \left(\mathcal{H}^{d-1}(P_i(nA)) - \mathcal{H}^{d-2}(\partial P_i(nA))\right)\\ &\geq \left(\|\vec{v}\|_1 - d\frac{\mathcal{H}^{d-2}(\partial(nA))}{\mathcal{H}^{d-1}(nA)}\right)\mathcal{H}^{d-1}(nA).\end{aligned}$$

On the other hand, each path from $(nA)_1^{h(n)}$ to $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA,h(n))$ must go through nA and so contains an edge of one of the $E_i(nA)$, $i = 1,...,d$. It suffices then to remove all the edges in the union of the sets $E_i(nA), i = 1,\dots,d$ to disconnect $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA,h(n))$, and so

$$\begin{aligned}\mathcal{N}(nA,h(n)) &\leq \sum_{i=1}^d \left(\mathcal{H}^{d-1}(P_i(nA)) + \mathcal{H}^{d-2}(\partial P_i(nA))\right)\\ &\leq \left(\|\vec{v}\|_1 + d\frac{\mathcal{H}^{d-2}(\partial(nA))}{\mathcal{H}^{d-1}(nA)}\right)\mathcal{H}^{d-1}(nA).\end{aligned}$$

We conclude that

$$\left| \frac{\mathcal{N}(nA,h(n))}{\mathcal{H}^{d-1}(nA)} - \|\vec{v}\|_1 \right| \leq d\,\frac{\mathcal{H}^{d-2}(\partial(nA))}{\mathcal{H}^{d-1}(nA)} = \frac{d\mathcal{H}^{d-2}(\partial A)}{n\mathcal{H}^{d-1}(A)}.$$

■

Now we come back to the problem of the continuity of $\mathcal{I}_{\vec{v}}$. Since $\mathcal{I}_{\vec{v}}$ is convex, we first try to determine its domain.

•$\lambda > \|\vec{v}\|_1\delta$: there exists $\varepsilon > 0$ such that $\lambda > (\|\vec{v}\|_1 + \varepsilon)(\delta + 2\varepsilon)$. Then there exists n_0 such that, for all $n \geq n_0$, there exists a set of edges $E_0(n)$ that disconnects $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA,h(n))$ and such that $\mathrm{card}(E_0(n)) \leq (\|\vec{v}\|_1 + \varepsilon)\mathcal{H}^{d-1}(nA)$. We obtain for $n \geq n_0$

$$\mathbb{P}\left(\tau(nA,h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right) \geq \mathbb{P}\left(V(E_0(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right)$$

$$\geq \mathbb{P}\left(t(e) \leq \frac{\lambda - 1/\sqrt{n}}{\|\vec{v}\|_1 + \varepsilon}\right)^{\lfloor (\|\vec{v}\|_1+\varepsilon)\mathcal{H}^{d-1}(nA)\rfloor} .$$

But there exists n_1 large enough to have for all $n \geq n_1$, $\lambda - 1/\sqrt{n} \geq (\|\vec{v}\|_1 + \varepsilon)(\delta + \varepsilon)$, so for all $n \geq n_0 \vee n_1$, we have

$$\mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right) \geq \mathbb{P}(t(e) \leq \delta + \varepsilon)^{\lfloor (\|\vec{v}\|_1+\varepsilon)\mathcal{H}^{d-1}(nA)\rfloor},$$

and finally

$$\mathcal{I}_{\vec{v}}(\lambda) \leq -(\|\vec{v}\|_1 + \varepsilon)\log \mathbb{P}(t(e) \leq \delta + \varepsilon) < \infty .$$

•$\lambda \leq \|\vec{v}\|_1\delta$: for $\lambda > 0$, there exists n_0 such that for all $n \geq n_0$,

$$\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \delta \frac{\mathcal{N}(nA, h(n))}{\mathcal{H}^{d-1}(nA, h(n))} \geq \delta\|\vec{v}\|_1 - \frac{1}{2\sqrt{n}} > \lambda - \frac{1}{\sqrt{n}},$$

and so for all $n \geq n_0$,

$$\mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right) = 0 .$$

The same result is true for $\lambda = 0$. We obtain that $\mathcal{I}_{\vec{v}}(\lambda) = +\infty$.

We now know that $\mathcal{I}_{\vec{v}}$ is convex and finite on $]\delta\|\vec{v}\|_1, +\infty[$ so it is continuous on $]\delta\|\vec{v}\|_1, +\infty[$, and it is infinite on $[0, \delta\|\vec{v}\|_1]$.

REMARK 22. The only point we didn't study is the behaviour of the function near $\delta\|\vec{v}\|_1$. In fact, we will eventually change the value of $\mathcal{I}_{\vec{v}}(\delta\|\vec{v}\|_1)$ to obtain a lower semicontinuous function. Moreover, the fact that $\mathcal{I}_{\vec{v}}(\delta\|\vec{v}\|_1) = +\infty$ even if there exists an atom of the law of $t(e)$ at δ is linked with the fact that we added a term $1/\sqrt{n}$ and not with the behaviour of $\mathbb{P}(\tau(nA, h(n)) \leq \delta\|\vec{v}\|_1\mathcal{H}^{d-1}(nA))$. This remark can be illustrated by an example in dimension 2: let $A = [-1/2, 1/2] \times \{1/2\}$. Here $\vec{v} = (0, 1)$ so $\|\vec{v}\|_1 = 1$. We consider a law of capacities with an atom at δ. We remark (see Figure 2) that $\mathcal{N}((2n+1)A, 2n+1) = 2n+1$. Moreover,

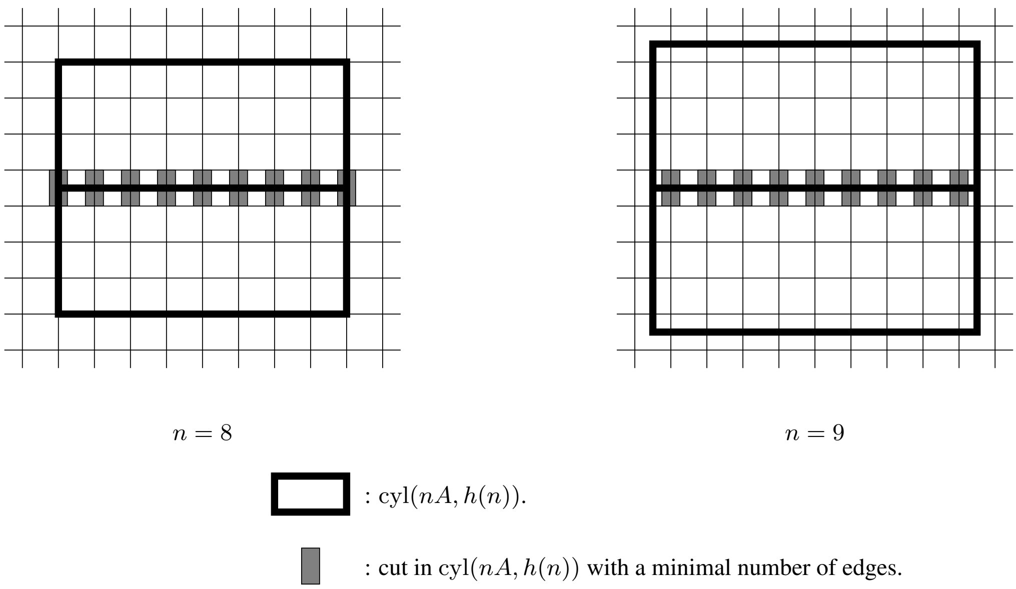

FIGURE 2. Examples of cuts.

there exists a unique cut $E_0(2n+1)$ in cyl$((2n+1)A, 2n+1)$ composed by $2n+1$ edges (see it on the Figure). So we have

$$\mathbb{P}(\tau((2n+1)A, 2n+1) \leq (2n+1)\delta) \;=\; \mathbb{P}(V(E_0(2n+1)) = (2n+1)\delta) \;=\; \mathbb{P}(t(e)=\delta)^{2n+1}$$

and

$$\lim_{n\to\infty} \frac{-1}{2n+1} \log \mathbb{P}(\tau((2n+1)A, 2n+1) \leq (2n+1)\delta) \;=\; -\log \mathbb{P}(t(e)=\delta) \;<\; \infty\,.$$

We also remark that $\mathcal{N}(2nA, 2n) \;=\; 2n+1$ because a cut in $\mathrm{cyl}(2nA, 2n)$ must contain a vertical edge of first coordinate i for $i = 0, ..., 2n$. Then we have

$$\mathbb{P}(\tau(2nA, 2n) \leq 2n\delta) \;=\; 0$$

and

$$\lim_{n\to\infty} \frac{-1}{2n} \log \mathbb{P}(\tau(2nA, 2n) \leq 2n\delta) \;=\; +\infty\,.$$

This example shows that the behaviour of $\mathbb{P}(\tau(nA, h(n)) \;\leq\; \delta\|\vec{v}\|_1 \mathcal{H}^{d-1}(nA))$ is not clear, and we will avoid the problem by taking later at $\|\vec{v}\|_1\delta$ the value of the limit

$$\lim_{\lambda > \|\vec{v}\|_1\delta, \lambda\to\|\vec{v}\|_1\delta} \mathcal{I}_{\vec{v}}(\lambda)$$

instead of $\mathcal{I}_{\vec{v}}(\|\vec{v}\|_1\delta)$. The value of this limit will be a little bit discussed in section 6.

4.4. Positivity of $\mathcal{I}_{\vec{v}}$. From now on we need the assumptions that $F(0) < 1 - p_c(d)$, and that there exists a positive γ such that

$$\int e^{\gamma x} dF(x) \;<\; \infty\,.$$

• $\mathcal{I}_{\vec{v}} = 0$ **on** $]\nu(\vec{v}), +\infty]$

Under the previous hypothesis, Proposition 2.1 states that for every unit vector $\vec{v}$, for every hypersquare A orthogonal to $\vec{v}$, for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, we have

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \;=\; \nu(\vec{v}) \qquad a.s.$$

Then for all $\lambda > \nu(\vec{v})$, so $\lambda - 1/\sqrt{n} > \nu(\vec{v})$ for $n \geq n_0 = 4/(\lambda - \nu(\vec{v}))^2$, we have

$$\mathcal{I}_{\vec{v}}(\lambda) \;=\; \lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left(\tau(nA, h(n)) \leq (\lambda - \frac{1}{\sqrt{n}})\mathcal{H}^{d-1}(nA)\right) \;=\; 0\,.$$

• $\mathcal{I}_{\vec{v}} > 0$ **on** $[0, \nu(\vec{v})[$

Thanks to Theorem 11, we have for all $\varepsilon > 0$

$$\begin{aligned}
\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}&\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v}) - \varepsilon - \frac{1}{\sqrt{n}}\right) \\
&\geq \lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v}) - \varepsilon\right) \\
&\geq C(\varepsilon, F, d) \;>\; 0\,.
\end{aligned}$$

4.5. Study of $\nu(\vec{v})$. We did not study what happens at $\nu(\vec{v})$, i.e., if $\mathcal{I}_{\vec{v}}(\nu(\vec{v})) = 0$ or not. It is obvious that $\nu(\vec{v}) \geq \delta\|\vec{v}\|_1$, because

$$\begin{aligned}
\tau(nA, h(n)) \;&=\; \min\{V(E)\,|\, E \textit{ is an } ((nA)_1^{h(n)}, (nA)_2^{h(n)}) - \textit{cut}\} \\
&\geq\; \delta\mathcal{N}(nA, h(n))\,.
\end{aligned}$$

If $\nu(\vec{v}) > \delta\|\vec{v}\|_1$, then $\mathcal{I}_{\vec{v}}$ is continuous at $\nu(\vec{v})$ and so $\mathcal{I}_{\vec{v}}(\nu(\vec{v})) = 0$. If $\nu(\vec{v}) = \delta\|\vec{v}\|_1$, the value of $\mathcal{I}_{\vec{v}}(\nu(\vec{v}))$ itself is not relevant for the understanding of the system as explained in Remark 22, and the large deviation principles from below that we will prove are of no interest. Now we can wonder under what condition we have $\nu(\vec{v}) > \delta\|\vec{v}\|_1$. We will prove the property 2.2, that we recall here:

PROPOSITION 4.1. We suppose that the capacities are in L^1: $\int x dF(x) < \infty$. If $F(\delta) < 1 - p_c(d)$, then $\nu(\vec{v}) > \delta\|\vec{v}\|_1$ for all unit vector $\vec{v}$. In the case $\delta = 0$, the previous implication is in fact an equivalence.

We first need to state the following result on $\nu(\vec{v})$:

PROPOSITION 4.2. We suppose that the capacities are in L^1: $\int x dF(x) < \infty$. Let (ABC) be a non degenerate triangle in $\mathbb{R}^d$ and let $\vec{v_A}$, $\vec{v_B}$ and $\vec{v_C}$ be the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane spanned by A, B, C. Then

$$\mathcal{H}^1([AB])\nu(\vec{v_C}) \,\leq\, \mathcal{H}^1([AC])\nu(\vec{v_B}) + \mathcal{H}^1([BC])\nu(\vec{v_A})\,.$$

Proof :
We will follow the proof of Proposition 11.2 in [**19**]. We consider first the case where the triangle (ABC) is such that $AC\cdot AB \geq 0$ and $BC\cdot BA \geq 0$. Let $\varepsilon > 0$. Let $(e_1, ..., e_d)$ be an orthonormal basis of $\mathbb{R}^d$ such that e_1 and e_2 belong to the plane $\mathcal{P}$ spanned by A, B, C. We define the sets

$$\begin{aligned}
\mathcal{R}^{\varepsilon}_{BC} &= \{x + \sum_{i=3}^{d} u_i e_i \,|\, x \in [BC]\,,\ d(x, \{B\}\cup\{C\}) \geq \varepsilon \ and\ (u_3, ..., u_d) \in [0,1]^{d-2}\}\,,\\
\mathcal{R}^{\varepsilon}_{AC} &= \{x + \sum_{i=3}^{d} u_i e_i \,|\, x \in [AC]\,,\ d(x, \{A\}\cup\{C\}) \geq \varepsilon \ and\ (u_3, ..., u_d) \in [0,1]^{d-2}\}\,,\\
\widetilde{\mathcal{R}}^{\varepsilon}_{AB} &= \{x + \sum_{i=3}^{d} u_i e_i \,|\, x \in (AB)\,,\ d(x, [AB]) \leq \varepsilon \ and\ (u_3, ..., u_d) \in [0,1]^{d-2}\}\,,
\end{aligned}$$

where (AB) denotes the line and $[AB]$ the segment. We define also the sets

$$\begin{aligned}
S^{\varepsilon}_{BC} &= \{x + \sum_{i=3}^{d} u_i e_i \,|\, x \in [BC]\,,\ d(x, \{B\}\cup\{C\}) \leq 2\varepsilon \ and\ (u_3, ..., u_d) \in [0,1]^{d-2}\}\,,\\
S^{\varepsilon}_{AC} &= \{x + \sum_{i=3}^{d} u_i e_i \,|\, x \in [AC]\,,\ d(x, \{B\}\cup\{C\}) \leq 2\varepsilon \ and\ (u_3, ..., u_d) \in [0,1]^{d-2}\}\,,\\
\widetilde{S}^{\varepsilon}_{AB} &= \{x + \sum_{i=3}^{d} u_i e_i \,|\, x \in (AB)\smallsetminus[AB]\,,\ d(x, \{A\}\cup\{B\}) \leq \varepsilon \ and\ (u_3, ..., u_d) \in [0,1]^{d-2}\}\,.
\end{aligned}$$

We now consider the triangle (nA, nB, nC). Let $\zeta \geq 4d$ be fixed. For a fixed n, let $E(n)$ be the set of the edges that belong to $\mathcal{E}(n)$, defined as

$$\begin{aligned}
\mathcal{E}(n) \,=\, &\operatorname{cyl}(nS^{\varepsilon}_{BC}, \zeta) \cup \operatorname{cyl}(nS^{\varepsilon}_{AC}, \zeta) \cup \operatorname{cyl}(n\widetilde{S}^{\varepsilon}_{AB}, \zeta)\\
&\cup \{x + \sum_{i=3}^{d} u_i e_i \,|\, x \in \mathcal{V}(\{A\}\cup\{B\}\cup\{C\}, \zeta) \cap \mathcal{P} \ and\ (u_3, ..., u_d) \in [0,n]^{d-2}\}\,.
\end{aligned}$$

See Figure 3. There exists h large enough and η small enough such that, for n sufficiently large,

$$\operatorname{cyl}(n\mathcal{R}^{\varepsilon}_{BC}, n\eta) \cup \operatorname{cyl}(n\mathcal{R}^{\varepsilon}_{AC}, n\eta) \cup \mathcal{E}(n) \,\subset\, \operatorname{cyl}(n\widetilde{\mathcal{R}}^{\varepsilon}_{AB}, nh)$$

and then

$$\tau(n\widetilde{\mathcal{R}}^{\varepsilon}_{AB}, nh) \,\leq\, \tau(n\mathcal{R}^{\varepsilon}_{BC}, n\eta) + \tau(n\mathcal{R}^{\varepsilon}_{AC}, n\eta) + V(E(n))\,.$$

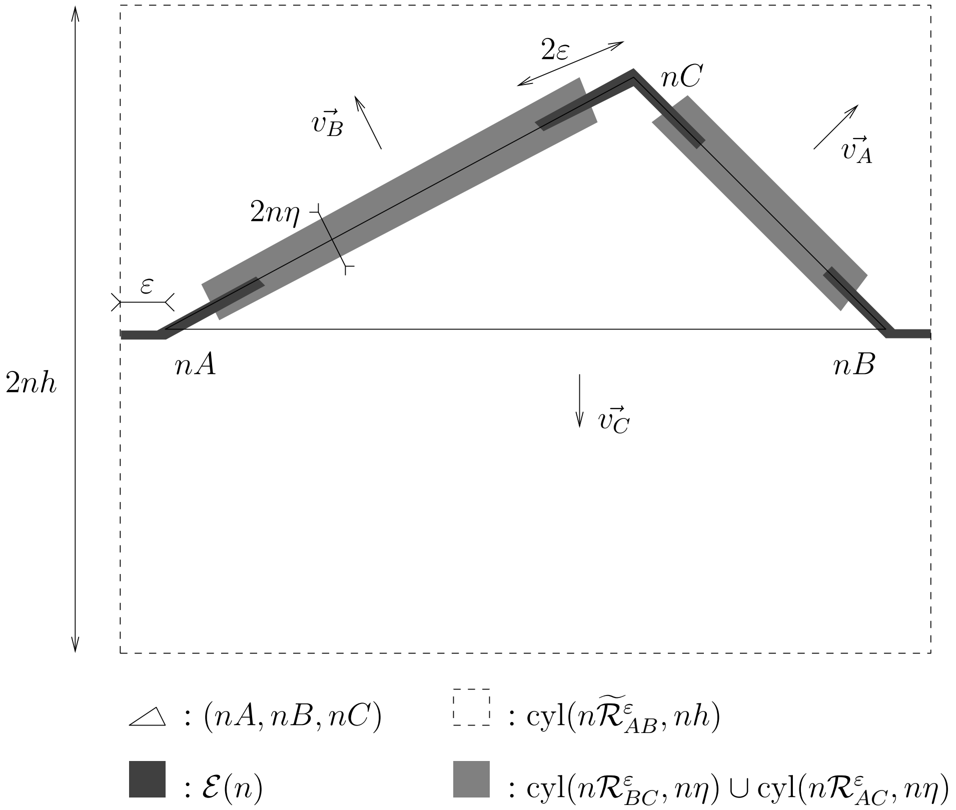

FIGURE 3. The plane $\mathcal{P}$.

Equivalently, we have

$$\frac{\mathbb{E}(\tau(n\widetilde{\mathcal{R}}^{\varepsilon}_{AB}, nh))}{n^{d-1}} \leq \frac{\mathbb{E}(\tau(n\mathcal{R}^{\varepsilon}_{BC}, n\eta))}{n^{d-1}} + \frac{\mathbb{E}(\tau(n\mathcal{R}^{\varepsilon}_{AC}, n\eta))}{n^{d-1}} + \frac{\mathbb{E}(V(E(n)))}{n^{d-1}} . \tag{5.7}$$

We know that there exists a constant $K(\zeta)$ such that

$$\mathrm{card}(E(n)) \leq K(\zeta)(\varepsilon n^{d-1} + n^{d-2}) ,$$

so

$$\lim_{n\to\infty} \frac{\mathbb{E}(V(E(n)))}{n^{d-1}} \leq K(\zeta)\varepsilon .$$

Thus by taking the expectation of (5.7) and sending n to the infinity, thanks to Proposition 2.1, we obtain, for all $\varepsilon > 0$,

$$\Big(\mathcal{H}^1([AB]) + 2\varepsilon\Big)\,\nu(\vec{v_C}) \leq \Big(\mathcal{H}^1([BC]) - 2\varepsilon\Big)\,\nu(\vec{v_A}) + \Big(\mathcal{H}^1([AC]) - 2\varepsilon\Big)\,\nu(\vec{v_B}) + K(\zeta)\varepsilon .$$

We send ε to zero, and the desired result follows.

In the case where $AC \cdot AB < 0$ (the same proof holds for the case $BC \cdot BA < 0$), we define the orthogonal projection D of A on $[BC]$ (see Figure 4). We then can apply the previous result to the triangles (ABD) and (ADC), thus, if $\vec{v_D}$ denotes one of the two unit vectors orthogonal to (AD) in $\mathcal{P}$, we have

$$\mathcal{H}^1([AB])\nu(\vec{v_C}) \leq \mathcal{H}^1([AD])\nu(\vec{v_D}) + \mathcal{H}^1([BD])\nu(\vec{v_A})$$

and

$$\mathcal{H}^1([AD])\nu(\vec{v_D}) \leq \mathcal{H}^1([AC])\nu(\vec{v_B}) + \mathcal{H}^1([DC])\nu(\vec{v_A}) ,$$

and so the result follows easily by combining these two inequalities.

■

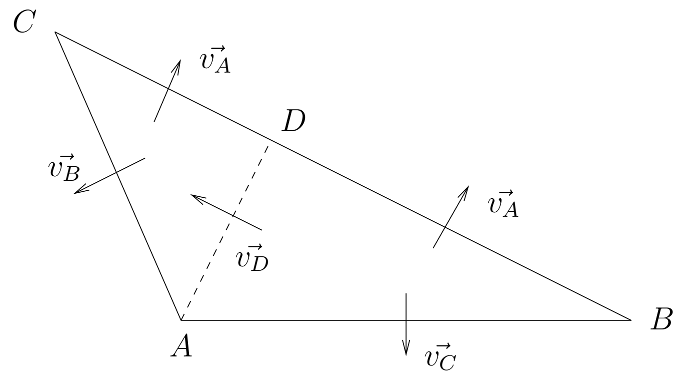

FIGURE 4. A triangle with $AC \cdot AB < 0$.

As in [**40**], one can extend ν as a function on $\mathbb{R}^d$ as follows:

$$\nu(\vec{0}) = 0, \quad \text{and} \quad \forall \vec{u} \neq \vec{0},\ \nu(\vec{u}) := \|\vec{u}\|.\nu\left(\frac{\vec{u}}{\|\vec{u}\|}\right) .$$

Then, Proposition 4.2 shows that ν is subadditive:

$$\forall \vec{u}_1, \vec{u}_2,\ \nu(\vec{u}_1 + \vec{u}_2) \leq \nu(\vec{u}_1) + \nu(\vec{u}_2) . \tag{5.8}$$

We will now prove Proposition 2.2. **Proof** :
First we consider the case $\delta = 0$. We define $\vec{v_0} = (0, ..., 0, 1)$. It is proved in [**56**] (see also [**20**] for capacities equal to zero or one) that $F(0) < 1 - p_c(d)$ implies $\nu(\vec{v_0}) > 0$. Conversely, Zhang proved in [**58**] that $\nu(\vec{v_0}) = 0$ if $F(0) = 1 - p_c(d)$, and so by a simple coupling of probability if $F(0) \geq 1 - p_c(d)$. Actually, he wrote the proof for $d = 3$ but said himself that the argument works for $d \geq 3$ (see Remark 1 of [**58**]). This property is also satisfied in dimension $d = 2$ where we can use duality arguments (see [**40**] Theorem (6.1) and Remark (6.2)). So we obtain

$$\nu(\vec{v_0}) > 0 \iff F(0) < 1 - p_c(d) .$$

Now let us prove that

$$\exists \vec{v} \neq 0\ s.t.\ \nu(\vec{v}) = 0 \iff \forall \vec{v},\ \nu(\vec{v}) = 0 . \tag{5.9}$$

Suppose there exists a vector $\vec{w} = (w_1, ..., w_d) \neq 0$ such that $\nu(\vec{w}) = 0$. It exists i such that $w_i \neq 0$, and by symmetry of the lattice $\mathbb{Z}^d$ and the edge-capacities distribution, we can consider that $w_d \neq 0$. We denote by $\vec{w}'$ the unit vector $\vec{w}' = (-w_1, ..., -w_{d-1}, w_d)$. By symmetry, $\nu(\vec{w}') = 0$ too. Property (5.8) gives us that

$$\nu(\vec{v_0}) = \frac{\nu(\vec{w} + \vec{w}')}{\|\vec{w} + \vec{w}'\|} \leq \frac{\nu(\vec{w}) + \nu(\vec{w}')}{\|\vec{w} + \vec{w}'\|} = 0 .$$

By symmetry, for all i, $\nu(\vec{v_i}) = \nu(-\vec{v_i}) = 0$ for $\vec{v_i} = (0, ..., 0, 1, ..., 0)$ (the i-th coordinate is equal to 1). Finally, property (5.8) shows that ν is identically zero. So for all unit vector $\vec{v}$,

$$\nu(\vec{v}) > 0 \iff F(0) < 1 - p_c(d) .$$

Now we study the case $\delta > 0$. For a given realization of $(t(e), e \in \mathbb{E}^d)$, we define the family of variables $(t'(e), e \in \mathbb{E}^d)$ by $t'(e) = t(e) - \delta$ for all e. Then the variables $(t'(e), e \in \mathbb{E}^d)$ are independent and identically distributed, and if we denote by F' their distribution function, we have $F'(\lambda) = F(\lambda + \delta)$ for all $\lambda \in \mathbb{R}$. We compare the variable $\tau(nA, h(n))$ and the corresponding variable $\tau'(nA, h(n))$ for the capacities $(t'(e))$, for a given hyperrectangle A of normal unit vector $\vec{v}$, and a given height function h such that $\lim_{n\to\infty} h(n) = +\infty$. We still denote by $\mathcal{N}(nA, h(n))$ the minimal number of edges that can disconnect $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA, h(n))$. By the max-flow min-cut theorem, we easily obtain that

$$\tau(nA,h(n)) \,\geq\, \tau'(nA,h(n)) + \delta\mathcal{N}(nA,h(n))\,,$$

and so

$$\frac{\mathbb{E}(\tau(nA,h(n)))}{\mathcal{H}^{d-1}(nA)} \,\geq\, \frac{\mathbb{E}(\tau'(nA,h(n)))}{\mathcal{H}^{d-1}(nA)} + \delta\frac{\mathcal{N}(nA,h(n))}{\mathcal{H}^{d-1}(nA)}\,.$$

Proposition 2.1 and Lemma 5 give us that

$$\nu_F(\vec{v}) \,\geq\, \nu_{F'}(\vec{v}) + \delta\|\vec{v}\|_1$$

with trivial notations. Now $F(\delta) = F'(0) < 1 - p_c(d)$ implies that $\nu_{F'}(\vec{v}) > 0$, so Proposition 2.2 is proved.

■

4.6. Proof of Theorem 12. We define the function $\mathcal{J}_{\vec{v}}$ on $\mathbb{R}^+$ by

$$\mathcal{J}_{\vec{v}}(\lambda) \,=\, \begin{cases} \mathcal{I}_{\vec{v}}(\lambda) & if\ \lambda \leq \nu(\vec{v})\ and\ \lambda \neq \|\vec{v}\|_1\delta\,, \\ \lim_{\mu>\|\vec{v}\|_1\delta, \mu\to\|\vec{v}\|_1\delta} \mathcal{I}_{\vec{v}}(\mu) & if\ \lambda = \|\vec{v}\|_1\delta\,, \\ +\infty & if\ \lambda > \nu(\vec{v})\,. \end{cases}$$

The study of the function $\mathcal{I}_{\vec{v}}$ made previously and the construction of $\mathcal{J}_{\vec{v}}$ gives us immediately that the function $\mathcal{J}_{\vec{v}}$ is a good rate function.

• **Lower bound**

We have to prove that for all open subset $\mathcal{O}$ of $\mathbb{R}^+$,

$$\liminf_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log\mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{O}\right] \,\geq\, -\inf_{\mathcal{O}} \mathcal{J}_{\vec{v}}\,.$$

Classically, it suffices to prove the local lower bound:

$$\forall\alpha\in\mathbb{R}^+\,,\ \forall\varepsilon>0 \qquad \liminf_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log\mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in]\alpha-\varepsilon,\alpha+\varepsilon[\right] \,\geq\, -\mathcal{J}_{\vec{v}}(\alpha)\,.$$

If $\mathcal{J}_{\vec{v}}(\alpha) = +\infty$, the result is trivial. Otherwise, suppose $\mathcal{J}_{\vec{v}}(\alpha) < +\infty$. The function $\mathcal{I}_{\vec{v}}$ is convex, equal to zero on $[\nu(\vec{v}),+\infty[$, positive on $[0,\nu(\vec{v})[$ and finite on $]\|\vec{v}\|_1\delta,+\infty]$. Then $\mathcal{I}_{\vec{v}}$ is strictly decreasing on $]\|\vec{v}\|_1\delta,\nu(\vec{v})]$, and so is $\mathcal{J}_{\vec{v}}$ (because $\mathcal{I}_{\vec{v}} = \mathcal{J}_{\vec{v}}$ on $]\|\vec{v}\|_1\delta,\nu(\vec{v})]$). Yet $\mathcal{J}_{\vec{v}}(\alpha) < +\infty$ implies that $\alpha \in]\|\vec{v}\|_1\delta,\nu(\vec{v})]$ or $\alpha = \|\vec{v}\|_1\delta$ and $\mathcal{J}_{\vec{v}}(\|\vec{v}\|_1\delta) < +\infty$. Thus we obtain in both cases that $\mathcal{J}_{\vec{v}}(\alpha) < \mathcal{J}_{\vec{v}}(\alpha-\varepsilon/2)$. Then the following inequality, true for $n > 4/\varepsilon^2$,

$$\begin{aligned}\mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in]\alpha-\varepsilon,\alpha+\varepsilon[\right] \,\geq\, & \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq \alpha - \frac{1}{\sqrt{n}}\right] \\ & - \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq \alpha - \frac{\varepsilon}{2} - \frac{1}{\sqrt{n}}\right]\end{aligned}$$

leads to

$$\liminf_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log\mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in]\alpha-\varepsilon,\alpha+\varepsilon[\right] \,\geq\, -\mathcal{J}_{\vec{v}}(\alpha)\,.$$

• **Upper bound**

We have to prove that for all closed subset $\mathcal{F}$ of $\mathbb{R}^+$

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log\mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{F}\right] \,\leq\, -\inf_{\mathcal{F}} \mathcal{J}_{\vec{v}}\,.$$

Let $\mathcal{F}$ be a closed subset of $\mathbb{R}^+$. If $\nu(\vec{v}) \in \mathcal{F}$, the result is obvious. We suppose now that $\nu(\vec{v}) \notin \mathcal{F}$. We consider $\mathcal{F}_1 = \mathcal{F}\cap[0,\nu(\vec{v})]$ and $\mathcal{F}_2 = \mathcal{F}\cap]\nu(\vec{v}),+\infty[$. Let $f_1 = \sup\mathcal{F}_1$ ($f_1 < \nu(\vec{v})$ because

$\mathcal{F}$ is closed) and $f_2 = \inf \mathcal{F}_2$ ($f_2 > \nu(\vec{v})$ for the same reason). We have

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{F}\right]$$
$$\leq \limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \left(\mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1\right] + \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \geq f_2\right]\right).$$

We first take care of the term containing f_1. We have for all positive η and $n \geq 1/\eta^2$

$$\mathbb{P}\left(\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1\right) \leq \mathbb{P}\left(\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1 + \eta - \frac{1}{\sqrt{n}}\right),$$

so

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left(\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1\right) \leq -\mathcal{I}_{\vec{v}}(f_1+\eta)$$

and by sending η to zero, thanks to the definition of $\mathcal{J}_{\vec{v}}$ on $[0,\nu(\vec{v})]$ ($f_1 + \eta \leq \nu(\vec{v})$ for small η), we obtain

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1\right] \leq -\mathcal{J}_{\vec{v}}(f_1).$$

Now we will have a look at the term containing f_2. Thanks to study of the upper large deviations done in the Chapter 3, we know that

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \geq f_2\right] = -\infty.$$

If $\mathcal{J}_{\vec{v}}(f_1) = +\infty$, we have

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{F}\right] \leq -\infty = -\inf_{\mathcal{F}} \mathcal{J}_{\vec{v}},$$

because $\mathcal{J}_{\vec{v}}$ is infinite on $[0,f_1]$ and on $[f_2,\infty[$ so on $\mathcal{F}$. If $\mathcal{J}_{\vec{v}}(f_1) < +\infty$, we have

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{F}\right] \leq -\mathcal{J}_{\vec{v}}(f_1) = -\inf_{\mathcal{F}} \mathcal{J}_{\vec{v}},$$

because $\mathcal{J}_{\vec{v}}$ is non-increasing when it is finite. So the upper bound is proved.

5. Large deviation principle for ϕ

5.1. Large deviation principle for ϕ in straight boxes. First, we shall show that in the straight case, ϕ and τ share the same rate function. Since this function has already been studied, and since the upper deviations of ϕ have been studied in [**55**], the construction of the rate function of ϕ is the main work to do in order to show the large deviation principle for ϕ in straight boxes.

PROPOSITION 5.1. Suppose that $F(0) < 1 - p_c(d)$, and F admits an exponential moment:

$$\exists \gamma > 0 \qquad \int e^{\gamma x} dF(x) < \infty.$$

Let A be a straight hyperrectangle. Suppose that the height function h satisfies

$$\lim_{n\to\infty} h(n) = +\infty \text{ and } \frac{\log h(n)}{n^{d-1}} \xrightarrow[n\to\infty]{} 0.$$

Then, for every λ in $\mathbb{R}^+$,

$$\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\phi(nA,h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right] = \mathcal{I}_{\vec{v}}(\lambda),$$

where $\vec{v} = (0,\ldots,0,1)$.

Proof :
First, notice that

$$\phi(nA, h(n)) \leq \tau(nA, h(n)) \,.$$

Thus, using Lemma 4,

$$\begin{aligned}\limsup_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log\mathbb{P}\left[\phi(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right] \\ \leq \limsup_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log\mathbb{P}\left[\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right] \\ \leq \mathcal{I}_{\vec{v}}(\lambda)\,.\end{aligned}$$

Therefore, we only need to show that

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log\mathbb{P}\left[\phi(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right] \geq \mathcal{I}_{\vec{v}}(\lambda)\,.$$

To shorten the notations, we shall suppose that

$$A \,=\, [0,1]^{d-1} \times \{0\}\,,$$

the general case of a straight hyperrectangle being handled exactly along the same lines. Notice that $\mathcal{H}^{d-1}(nA) = n^{d-1}$. We shall use the following notation:

$$\phi_n \,=\, \phi(nA, h(n))\,.$$

The idea of the proof is the following. Define E_{ϕ_n} to be $E_\phi(nA, h(n))$, the minimal cut for ϕ_n as defined below inequality (5.2). E_{ϕ_n} has a certain intersection with the sides of the cylinder $\mathrm{cyl}(A, h)$. Thanks to Zhang's result, Theorem 12, and after having eventually reduced a little the cylinder, one can prove that the intersection of E_{ϕ_n} with the sides of this reduced cylinder has less than $Cn^{d-1}/n^{1/3}$ edges with very high probability (here C is a constant). This shows that (with very high probability) ϕ_n is larger than the minimum of a collection of random variables $(\tau_F)_{F\in I_n}$, where F designs a possible *trace* of E_{ϕ_n}, i.e., its intersection with the reduced cylinder, and where I_n is the set of all the possible choices for F. Since E_{ϕ_n} itself has less than Cn^{d-1} edges, and since it is connected (in the dual sense), a trivial bound for the cardinal of I_n is roughly

$$\mathrm{card}(I_n) \,\leq\, h(n)(C' n^{2d-3})^{Cn^{d-1}/n^{1/3}}\,.$$

The important point here is that $\log\mathrm{card}(I_n)$ is small compared to n^{d-1}. Having done this, a subadditive argument using symmetries can be performed to show that in fact the smallest τ_F (in distribution) is essentially $\tau(nA, h(n))$, which has $\mathcal{I}_{\vec{v}}$ as a rate function.

In the sequel, we shall suppose that n is large enough to ensure that

$$\log h(n) \,\leq\, n^{d-1}\,.$$

Let β, C_1 and C_2 be as in Theorem 16. Let $L \geq \beta$ be a real number to be chosen later. From Theorem 16, we know that:

$$\mathbb{P}(\mathrm{card}(E_{\phi_n}) \geq Ln^{d-1}) \,\leq\, C_1 e^{-C_2 L n^{d-1}}\,.$$

Define

$$\begin{aligned}\phi_{L,n} \,=\, \min\{V(E) \,|\, E \text{ is a } (B(nA, h(n)), T(nA, h(n)))\text{-cut in } \mathrm{cyl}(nA, h(n)) \\ \text{and } \mathrm{card}(E) \leq Ln^{d-1}\}\,.\end{aligned}$$

Thus,

$$\mathbb{P}(\phi_n \leq \lambda n^{d-1}) \,\leq\, \mathbb{P}(\phi_{L,n} \leq \lambda n^{d-1}) + C_1 e^{-C_2 L n^{d-1}}\,. \tag{5.10}$$

We shall now concentrate on the first summand in the right-hand side of the last inequality. Let $\psi(n) = \lceil n^{1/3} \rceil$. For any k in $\{1, \dots, \psi(n)\}$, define

$$A_{n,k} = [k, n-k]^{d-1} \times \{0\} \,,$$
$$B_{n,k} = [k, n-k]^{d-1} \times [-h(n), h(n)] \,,$$

and

$$S_{n,k} = \partial \left([k, n-k]^{d-1} \right) \times [-h(n), h(n)] \,.$$

In order to perform the announced subadditive argument, we shall need to patch together two cuts of neighbouring boxes which share a same trace in the intersection of these boxes. It is not so trivial to show that one obtains a cut doing so. This is why we shall impose a kind of "connection trace" on the sides of the box, which remembers if a vertex of the side is connected to the top or the bottom of the cylinder once the cut E_{ϕ_n} has been removed. Let us precise the needed definitions. If X is a subset of vertices of a subgraph G of $\mathbb{Z}^d$, we denote by $C_G(X)$ the union of all the connected components of G intersecting X. If v is a vertex of G, we write $C_G(v)$ instead of $C_G(\{v\})$. We shall say that a function x from some $S_{n,k}$ to $\{0, 1, 2\}$ is a *weak connection function* for a trace F in $S_{n,k}$ if for every u and v in $S_{n,k}$,

$$u \in C_{S_{n,k} \smallsetminus F}(v) \;\Rightarrow\; x(u) = x(v) \,.$$

If E cuts $B(A_{n,k}, h(n))$ from $T(A_{n,k}, h(n))$ in $B_{k,n}$, we define x_E, *the connection function of* E *(in* $B_{k,n}$*)* as follows:

$$\forall u \in \mathrm{cyl}(A_{n,k}, h(n)) \qquad x_E(u) = \begin{cases} 1 \text{ if } u \in C_{B_{k,n} \setminus E}(T(A_{n,k}, h(n))) \,, \\ 0 \text{ if } u \in C_{B_{k,n} \setminus E}(B(A_{n,k}, h(n))) \,, \\ 2 \text{ else} \,. \end{cases}$$

Clearly, $\tilde{x}_E$, the restriction of x_E to $S_{n,k}$ is a weak connection function for $E \cap S_{n,k}$. Then, define the following set of "good" couples (F, x) of a trace F and a weak connection function x:

$$I_n = \bigcup_{k=1}^{\psi(n)} \bigcup_{h=-h(n)}^{h(n)-Ln^{d-1}} \left\{ \begin{array}{c} (F, x) \text{ s.t. } F \subset \mathbb{E}^d \cap \partial \left([k, n-k]^{d-1} \right) \times [h, h + Ln^{d-1}], \\ \mathrm{card}(F) \leq \frac{Ln^{d-1}}{\psi(n)},\ x \text{ is a weak connection function for } F \text{ in } S_{n,k} \end{array} \right\} .$$

If F satisfies the conditions in the above definition, then there are at most $2^d Ln^{d-1}/\psi(n)$ distinct connected components in $S_{n,k} \smallsetminus F$. Thus, for a fixed F, there are at most $3^{2^d Ln^{d-1}/\psi(n)}$ distinct weak configuration functions x such that (F, x) belongs to I_n. Thus, there is a constant C_3, which depends only on d, such that

$$(2h(n) + 1) \;\leq\; \mathrm{card}(I_n) \;\leq\; 2h(n)\psi(n)(C_3 n^{2d-3})^{Ln^{d-1}/\psi(n)} \,. \tag{5.11}$$

On the other hand, define, for (F, x) in I_n and k such that $F \subset S_{n,k}$,

$$\begin{aligned} \mathcal{C}_{F,x} \;=\; & \Big\{ E \subset \mathbb{E}^d \,|\, E \text{ is a } (B(A_{n,k}, h(n)), T(A_{n,k}, h(n)))\text{-cut in } B_{n,k} \,, \\ & E \cap S_{n,k} = F,\ \mathrm{card}(E) \leq Ln^{d-1} \text{ and } \tilde{x}_E = x \Big\} \,, \end{aligned}$$

and

$$\tau_{(F,x)} \;=\; \min \{ V(E) \,|\, E \in \mathcal{C}_{F,x} \} \,.$$

We claim that

$$\phi_{L,n} \;\geq\; \min_{(F,x) \in I_n} \tau_{(F,x)} \,. \tag{5.12}$$

To see why (5.12) is true, notice that for any k in $\{1, \dots, \psi(n)\}$, $E_{\phi_{L,n}} \cap B_{n,k}$ cuts $B(A_{n,k}, h(n)$ from $T(A_{n,k}, h(n))$ in $B_{n,k}$, and has less than Ln^{d-1} edges. $E_{\phi_{L,n}}$ is connected in the dual sense (see the proof of Lemma 12 in [**59**]), and has less than Ln^{d-1} edges. Then there is an h such that $E_{\phi_{L,n}}$ is included in $[0, n]^{d-1} \times [h, h + Ln^{d-1}]$. Thus, there is an h such that $E_{\phi_{L,n}} \cap B_{n,k}$

is included in $[k, n-k]^{d-1} \times [h, h+Ln^{d-1}]$. Furthermore, since $S_{n,1}, \ldots, S_{n,\psi(n)}$ are pairwise disjoint, there is at least one k in $\{1, \ldots, \psi(n)\}$ such that

$$\mathrm{card}(E_{\phi_{L,n}} \cap S_{n,k}) \leq \frac{\mathrm{card}(E_{\phi_{L,n}})}{\psi(n)} \leq \frac{Ln^{d-1}}{\psi(n)} .$$

Thus, denoting $F = E_{\phi_{L,n}} \cap S_{n,k}$ and $x = \tilde{x}_{E_{\phi_{L,n}} \cap S_{n,k}}$, this shows that $\phi_{L,n} \geq \tau_{(F,x)}$, and claim (5.12) is proved.

Now, we need to show that $\min_{(F,x)\in I_n} \tau_{(F,x)}/n^{d-1}$ has lower large deviations given by $\mathcal{I}_{\vec{v}}$. First, notice that

$$\begin{aligned} \mathbb{P}(\phi_{L,n} \leq \lambda n^{d-1}) &\leq \mathbb{P}(\min_{(F,x)\in I_n} \tau_{(F,x)} \leq \lambda n^{d-1}) \\ &\leq \sum_{(F,x)\in I_n} \mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1}) . \end{aligned} \tag{5.13}$$

Since, according to inequality (5.11), $\log \mathrm{card}(I_n)$ is small compared to n^{d-1}, we shall be done if we can show that, uniformly in $(F,x) \in I_n$, the probability of deviation $\mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1})$ is asymptotically of order at most $\exp\left(-\mathcal{I}_{\vec{v}}(\lambda) n^{d-1}\right)$. We shall do this using a subadditivity argument. From now on, we fix (F,x) in I_n and k such that $F \subset S_{n,k}$. The notations and rigorous proofs are a little cumbersome, but everything can be guessed in two stages, looking at Figures 5 and 6.

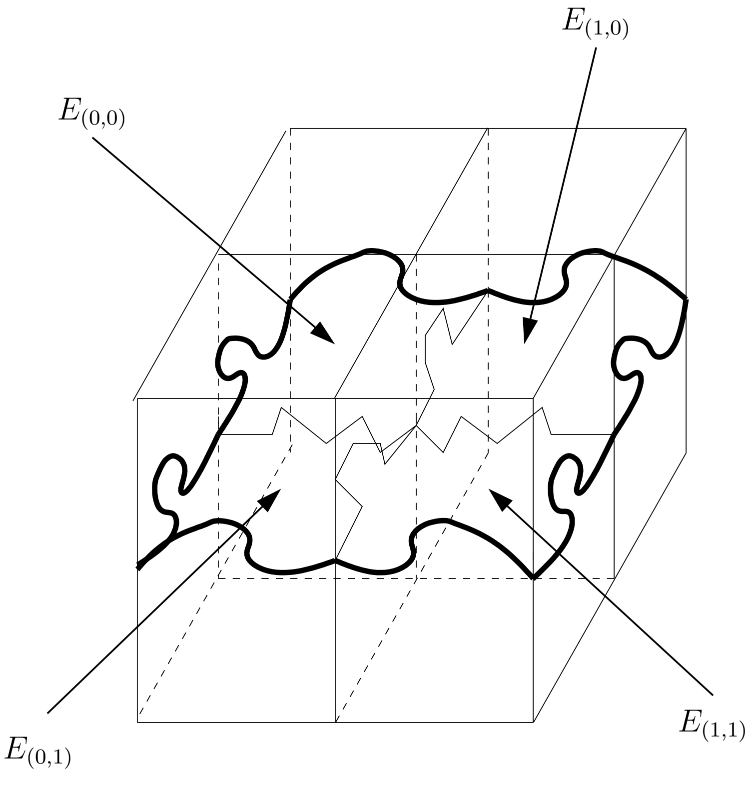

FIGURE 5. Patching E_b for $b \in \{0,1\}^{d-1}$ when $d=3$.

Let N be an integer such that for every $N' \geq N$, $h(2(n-2k)N') \geq h(n)$. Define, for $i = 1, \ldots, d-1$, the following hyperplanes:

$$H_i = \mathbb{R}^{i-1} \times \{n-k\} \times \mathbb{R}^{d-i} .$$

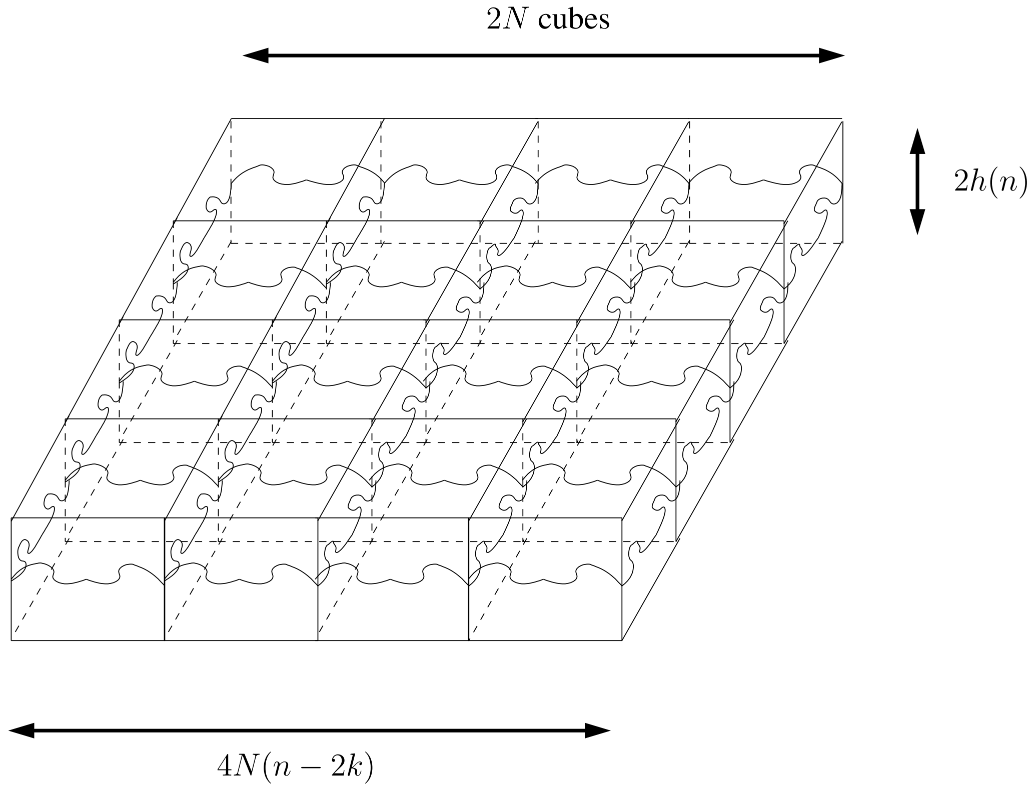

FIGURE 6. Patching cuts with the same perimeter.

We define σ_i to be the affine orthogonal reflection relative to H_i, and tr_i to be the following translation along coordinate i:

$$\mathrm{tr}_i(z) \;=\; z + 2(n-2k)\mathbf{e}_i \;,$$

where $(\mathbf{e}_1, ..., \mathbf{e}_d)$ is the canonical orthonormal basis of $\mathbb{R}^d$. For any $b \in \{-2N, \dots, 2N-1\}^{d-1}$, we define the map σ_b as follows. For every i in $\{1, \dots, d-1\}$, let $a_i = \lfloor b_i/2 \rfloor$ and $c_i = b_i - 2a_i$. Then, we denote by σ_b the (commutative) product of translations and reflections $\prod_{i=1}^{d-1} \mathrm{tr}_i^{a_i} \circ \prod_{i=1}^{d-1} \sigma_i^{c_i}$, where $\sigma_i^{c_i}$ (respectively $\mathrm{tr}_i^{a_i}$) is the c_i-th iterate of σ_i (respectively the a_i-th iterate of tr_i). Finally, we define also, for any set of vertices or set of edges X,

$$\sigma_N(X) \;=\; \bigcup_{b \in \{-2N,\dots,2N-1\}^{d-1}} \sigma_b(X) \;,$$

and

$$\tilde{\sigma}_N(X) \;=\; \sigma_N(X) \cap S_N \;,$$

where

$$S_N \;=\; \partial\left([k-2N(n-2k), k+2N(n-2k)]^{d-1}\right) \times [-h(n), h(n)] \;.$$

The following lemma should be intuitive looking at Figures 5 and 6.

LEMMA 6. *Let (F, x) be fixed in I_n. Suppose that for every $b \in \{-2N, \dots, 2N-1\}^{d-1}$, the set E_b cuts $B(\sigma_b(A_{n,k}), h(n))$ from $T(\sigma_b(A_{n,k}), h(n))$ in $\sigma_b(B_{n,k})$,*

$$E_b \cap \sigma_b(S_{n,k}) \;=\; \sigma_b(F) \;,$$

and

$$\tilde{x}_{E_b} \circ \sigma_b \;=\; x \;.$$

Then,

- *(i) For every $b \in \{-2N,\dots,2N-1\}^{d-1}$, the set of edges $\sigma_b(E_{(0,\dots,0)})$ separates $B(\sigma_b(A_{n,k}),h(n))$ from $T(\sigma_b(A_{n,k}),h(n))$ in $\sigma_b(B_{n,k})$, has configuration function $x\circ\sigma_b^{-1}$, and satisfies*
$$\sigma_b(E_{(0,\dots,0)})\cap\sigma_b(S_{n,k}) \;=\; \sigma_b(F)\;.$$
- *(ii) $E=\bigcup_{b\in\{-2N,\dots,2N-1\}^{d-1}}E_b$ cuts $B(\sigma_N(A_{n,k}),h(n)$ from $T(\sigma_N(A_{n,k}),h(n))$ in $\sigma_N(B_{n,k})$, and:*
$$E\cap S_N \;=\; \tilde\sigma_N(F)\;.$$

Proof :
The fact that E cuts indeed $B(\sigma_N(A_{n,k},h(n))$ from $T(\sigma_N(A_{n,k}),h(n))$ in $\sigma_N(B_{n,k})$ is the only non-trivial point to show. Let b and b' be two members of $\{-2N,\dots,2N-1\}^{d-1}$. The hypotheses on the cuts E_b and E'_b ensure that x_{E_b} and $x_{E_{b'}}$ coincide on $\sigma_b(B_{n,k})\cap\sigma_{b'}(B_{n,k})$. Thus, we can extend all the functions $(x_b)_{b\in\{-2N,\dots,2N-1\}^{d-1}}$ in a single function x on $\sigma_N(B_{n,k})$. This implies that for every two neighbours u and v in $\sigma_N(B_{n,k})$, if $\langle u,v\rangle\notin E$, then $x(u)=x(v)$. Thus, x is constant on each connected component of $\sigma_N(B_{n,k})\smallsetminus E$. Since in each box $\sigma_b(B_{n,k})$, E_b cuts $B(\sigma_b(A_{n,k}),h(n))$ from $T(\sigma_b(A_{n,k}),h(n))$, we have that x takes the value 1 on $B(\sigma_N(A_{n,k}),h(n))$, and the value 0 on $T(\sigma_N(A_{n,k}),h(n))$. Thus, these two sets are disconnected in $\sigma_N(B_{n,k})\smallsetminus E$.

■

Now, for every $b\in\mathbb{Z}^{d-1}$, define
$$\begin{aligned}\mathcal{C}_{F,x,b} \;=\; \big\{&E\subset\mathbb{E}^d\,|\,E \text{ is a } (B(\sigma_b(A_{n,k}),h(n)),T(\sigma_b(A_{n,k}),h(n)))\text{-cut in } \sigma_b(B_{n,k})\,,\\ &E\cap\sigma_b(S_{n,k})=\sigma_b(F),\ \operatorname{card}(E)\le Ln^{d-1}\text{ and } \tilde x_E\circ\sigma_b=x\big\}\;,\end{aligned}$$
and
$$\tau_{(F,x,b)}=\min\{V(E)\,|\,E\in\mathcal{C}_{F,x,b}\}\;.$$
For every N, let E_N denote the set of the edges e in $\sigma_N(B_{n,k})$ such that at least one endpoint of e belongs to S_N. Define $M(N)=N+\psi(N)$ and, for $N'\in\{N,M(N)\}$,
$$\tau_{N'} \;=\; \tau(\sigma_{N'}(A_{n,k}),h(N'))\;.$$
If E is a $(B(\sigma_N(A_{n,k}),h(n)),T(\sigma_N(A_{n,k}),h(n)))$-cut in $\sigma_N(A_{n,k})$, then $E\cup E_{N'}$ clearly cuts $\sigma_N(A_{n,k})_1^{h(n)}$ from $\sigma_N(A_{n,k})_2^{h(n)}$. Thus, part (ii) of Lemma 6 gives us that
$$\sum_{b\in\{-2M(N),\dots,2M(N)-1\}^{d-1}}\tau_{(F,x,b)}+\min_{N'\in\{N,\dots,M(N)\}}\sum_{e\in E_{N'}}t(e)\;\ge\;\min_{N'\in\{N,\dots,M(N)\}}\tau_{N'}\;.$$
Notice that some edges are counted twice on the left-hand side of the preceding inequality. From part (i) of Lemma 6, we know that the random variables $(\tau_{(F,x,b)})_{b\in\{-2M(N),\dots,2M(N)-1\}^{d-1}}$ are identically distributed, with the same distribution as $\tau_{(F,x)}$. Using the FKG inequality,
$$\begin{aligned}&\mathbb{P}(\tau_{(F,x)}\le\lambda n^{d-1})^{(4M(N))^{d-1}}\\ &\quad=\prod_{b\in\{-2M(N),\dots,2M(N)-1\}^{d-1}}\mathbb{P}(\tau_{(F,x,b)}\le\lambda n^{d-1})\\ &\quad\le\mathbb{P}(\forall b\in\{-2M(N),\dots,2M(N)-1\}^{d-1},\ \tau_{(F,x,b)}\le\lambda n^{d-1})\\ &\quad\le\mathbb{P}\left(\sum_{b\in\{-2M(N),\dots,2M(N)-1\}^{d-1}}\tau_{(F,x,b)}\le\lambda n^{d-1}(4M(N))^{d-1}\right)\\ &\quad\le\mathbb{P}\left(\min_{N'\in\{N,\dots,M(N)\}}\tau_{N'}-\min_{N'\in\{N,\dots,M(N)\}}\sum_{e\in E_{N'}}t(e)\le\lambda n^{d-1}(4M(N))^{d-1}\right)\;.\end{aligned}$$

Let $\varepsilon > 0$ be a fixed positive real number.

$$\begin{aligned}
\mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1}) \quad \leq \quad & \Bigg(\mathbb{P}\Bigg(\min_{N' \in \{N,\dots,M(N)\}} \tau_{N'} \leq (\lambda+\varepsilon) n^{d-1} (4M(N))^{d-1}\Bigg) \\
& + \mathbb{P}\Bigg(\min_{N' \in \{N,\dots,M(N)\}} \sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\Bigg)\Bigg)^{\frac{1}{(4M(N))^{d-1}}} . \qquad (5.14)
\end{aligned}$$

Now we shall let N go to infinity. Using Lemma 4 and the fact that $\lim_{N\to\infty} \psi(N)/N = 0$, we get, for N large enough,

$$\begin{aligned}
& \mathbb{P}\Bigg(\min_{N' \in \{N,\dots,M(N)\}} \tau_{N'} \leq (\lambda+\varepsilon) n^{d-1} (4M(N))^{d-1}\Bigg) \\
& \quad \leq \sum_{N'=N}^{M(N)} \mathbb{P}\Bigg(\tau_{N'} \leq (\lambda+\varepsilon) n^{d-1} (4M(N))^{d-1}\Bigg) \\
& \quad \leq \psi(N) \max_{N' \in \{N,\dots,M(N)\}} \mathbb{P}\Bigg(\tau_{N'} \leq (\lambda+\varepsilon) n^{d-1} (4M(N))^{d-1}\Bigg) \\
& \quad \leq \psi(N) \max_{N' \in \{N,\dots,M(N)\}} \mathbb{P}\left(\tau_{N'} \leq \left(\lambda + 2\varepsilon - \frac{1}{\sqrt{4(n-2k)N'}}\right) n^{d-1} (4N')^{d-1}\right) .
\end{aligned}$$

Thus,

$$\begin{aligned}
& \liminf_{N\to\infty} -\frac{1}{(4N(n-2k))^{d-1}} \log \mathbb{P}\Bigg(\min_{N' \in \{N,\dots,M(N)\}} \tau_{N'} \leq (\lambda+\varepsilon) n^{d-1} (4M(N))^{d-1}\Bigg) \\
& \geq \liminf_{N\to\infty} -\frac{1}{(4N(n-2k))^{d-1}} \times \\
& \qquad \max_{N' \in \{N,\dots,M(N)\}} \log \mathbb{P}\left(\tau_{N'} \leq \left(\lambda + 2\varepsilon - \frac{1}{\sqrt{4(n-2k)N'}}\right) n^{d-1} (4N')^{d-1}\right) \\
& \geq \mathcal{I}_{\vec{v}}\left((\lambda+2\varepsilon)\left(\frac{n}{n-2k}\right)^{d-1}\right) . \qquad (5.15)
\end{aligned}$$

Now, we use the fact that F possesses an exponential moment, and that the sets $E_{N'}$ are disjoint. Using Chebyshev inequality, there are positive constants C_4 and C_5, depending only on F and d, such that

$$\begin{aligned}
& \mathbb{P}\Bigg(\min_{N' \in \{N,\dots,M(N)\}} \sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\Bigg) \\
& \qquad = \prod_{N' \in \{N,\dots,M(N)\}} \mathbb{P}\Bigg(\sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\Bigg) , \\
& \qquad \leq \left(e^{C_4 h(n) n^{d-2} M(N)^{d-2} - C_5 n^{d-1} (4M(N))^{d-1}}\right)^{\psi(N)} .
\end{aligned}$$

Thus,

$$\liminf_{N\to\infty} \frac{-1}{(4N(n-2k))^{d-1}} \log \mathbb{P}\left(\min_{N' \in \{N,\dots,M(N)\}} \sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\right) = +\infty .$$

Therefore, inequalities (5.14) and (5.15) imply:

$$-\frac{1}{n^{d-1}}\log\mathbb{P}(\tau_{(F,x)}\leq\lambda n^{d-1})\;\geq\;\frac{(n-2k)^{d-1}}{n^{d-1}}\mathcal{I}_{\vec{v}}\left((\lambda+2\varepsilon)\left(\frac{n}{n-2k}\right)^{d-1}\right)\,.$$

We choose $\varepsilon=\frac{1}{n}$, and replace λ by $\lambda-\frac{1}{\sqrt{n}}$ to get

$$\begin{aligned}-\frac{1}{n^{d-1}}\log&\mathbb{P}\left(\tau_{(F,x)}\leq\left(\lambda-\frac{1}{\sqrt{n}}\right)n^{d-1}\right)\\&\geq\;\frac{(n-2k)^{d-1}}{n^{d-1}}\mathcal{I}_{\vec{v}}\left(\left(\lambda-\frac{1}{\sqrt{n}}+\frac{2}{n}\right)\left(\frac{n}{n-2k}\right)^{d-1}\right)\,.\end{aligned}$$

Since $k\leq\psi(n)$ and $\psi(n)$ is small compared to $\sqrt{n}$, for n large enough,

$$\left(\lambda-\frac{1}{\sqrt{n}}+\frac{2}{n}\right)\left(\frac{n}{n-2k}\right)^{d-1}<\lambda\,.$$

Since $\mathcal{I}_{\vec{v}}$ is non-increasing,

$$-\frac{1}{n^{d-1}}\log\mathbb{P}\left(\tau_{(F,x)}\leq\left(\lambda-\frac{1}{\sqrt{n}}\right)n^{d-1}\right)\geq\frac{(n-2k)^{d-1}}{n^{d-1}}\mathcal{I}_{\vec{v}}\left(\lambda\right)\,.$$

Using inequalities (5.11) and (5.13),

$$\liminf_{n\to\infty}-\frac{1}{n^{d-1}}\log\mathbb{P}(\phi_{L,n}\leq\lambda n^{d-1})\geq\mathcal{I}_{\vec{v}}(\lambda)\,.$$

And thus, from inequality (5.10),

$$\liminf_{n\to\infty}-\frac{1}{n^{d-1}}\log\mathbb{P}(\phi_{n}\leq\lambda n^{d-1})\geq\min\{\mathcal{I}_{\vec{v}}(\lambda),C_2L\}\,.$$

Letting L tend to infinity finishes the proof of Proposition 5.1.

∎

Proposition 5.1 implies that

$$\limsup_{n\to\infty}\frac{-1}{\mathcal{H}^{d-1}(nA)}\log\mathbb{P}(\phi(nA,h(n))\leq(\nu(\vec{v})-\varepsilon))\;>\;0\,,$$

under the condition on h:

$$\lim_{n\to\infty}h(n)\;=\;+\infty\quad\text{and}\quad\lim_{n\to\infty}\frac{\log h(n)}{n^{d-1}}\;=\;0\,.$$

As explained previously, combining this property with the upper large deviations estimates of Chapter 2, we can prove the a.s. convergence of $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ towards $\nu(\vec{v_0})$ under relevant hypotheses on h thanks to a Borel-Cantelli lemma.

REMARK 23. The subadditive argument does not work in the "non-straight" case. It is perhaps important to note that in this case, it is not obvious at all to know in advance for which F the random variable τ_F has the "minimal" distribution. It is natural to conjecture that this "minimal" F is a hyperrectangle, but we do not know how to prove this for all dimensions. When $d=2$, though, we are able to solve this problem and to show that if $h(n)/n$ converges towards $\tan(\alpha)$ for an α in $[0,\pi/2]$, and if $\vec{v}=(\cos\theta,\sin\theta)=\vec{v_\theta}$ is orthogonal to $A=A_\theta$, then $\phi(nA_\theta,h(n))/n$ converges towards $\min\{\nu(\vec{v_{\widetilde{\theta}}})/\cos(\widetilde{\theta}-\theta)\,|\,|\widetilde{\theta}-\theta|\leq\alpha\}$. A similar method gives an analog result for the lower large deviations. This will be done rigorously in a forthcoming paper.

The only thing we have to prove to obtain Theorem 15 is that for all $\lambda>\nu(\vec{v})$,

$$\lim_{n\to\infty}\frac{1}{\mathcal{H}^{d-1}(nA)}\log\mathbb{P}\left[\frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)}\geq\lambda\right]\;=\;-\infty\,. \tag{5.16}$$

Equation (5.16) has been proved in Chapter 2 of the thesis, under the hypothesis of the existence of one exponential moment for the law of the capacities. Then we can write exactly the same proof for Theorem 15 as for Theorem 12 (see section 4.6), since we know that $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ converges a.s., so in probability, towards $\nu(\vec{v_0})$. This ends the proof of the large deviation principle for ϕ in straight cylinders.

5.2. Large deviation principle for ϕ in small boxes. In this section, we shall prove Corollary 2.2, i.e., under the assumption that $\lim_{n\to\infty} h(n)/n = 0$, the sequence

$$\left(\frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)}, n\in\mathbb{N}\right)$$

satisfies the same large deviation principle as $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA), n\in\mathbb{N})$.

We will use a result of exponential equivalence. For (X_n) and (Y_n) two sequences of random variables defined on the same probability space $(\Omega,\mathcal{A},\mathbb{P})$, and for a given speed function $v(n)$ which goes to infinity with n, we say that (X_n) and (Y_n) are exponentially equivalent with regard to $v(n)$ if and only if for all positive ε we have

$$\limsup_{n\to\infty} \frac{1}{v(n)} \log \mathbb{P}\left(|X_n - Y_n| \geq \varepsilon\right) \,=\, -\infty\,.$$

We will use the following classical result in large deviations theory (see [**24**], Theorem 4.2.13):

THEOREM 17. *Let (X_n) and (Y_n) be two sequences of random variables defined on the same probability space $(\Omega,\mathcal{A},\mathbb{P})$. If (X_n) satisfies a large deviation principle of speed $v(n)$ with a good rate function, and if (X_n) and (Y_n) are exponentially equivalent with regard to $v(n)$, then (Y_n) satisfies the same large deviation principle as (X_n).*

We will prove that the sequences $(\phi(nA,h(n))/\mathcal{H}^{d-1}(nA))$ and $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA))$ are exponentially equivalent with regard to $\mathcal{H}^{d-1}(nA)$ under the assumptions that there exist exponential moment of the law of capacity of all orders and for height functions h satisfying $\lim_{n\to\infty} h(n)/n = 0$.

We take a hyperrectangle A. We will use the same notations as in section 3.3. Let E_1^+ be the set of the edges that belong to $\mathcal{E}_1^+$ defined as

$$\mathcal{E}_1^+ \,=\, \mathcal{V}(\mathrm{cyl}(\partial(nA),h(n)),\zeta)\cap \mathrm{cyl}(nA,h(n))\,.$$

We have for all n

$$\phi(nA,h(n)) \,\leq\, \tau(nA,h(n)) \,\leq\, \phi(nA,h(n)) + V(E_1^+)\,.$$

Thus for all $\varepsilon>0$, for all $n\geq p$, we obtain

$$\mathbb{P}\left(\left|\frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)} - \frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)}\right| \geq \varepsilon\right) \,\leq\, \mathbb{P}\left(V(E_1^+)\geq \varepsilon\mathcal{H}^{d-1}(nA)\right)\,.$$

We know that there exists a constant C^+ such that

$$\mathrm{card}(E_1^+) \leq C^+ n^{d-2}h(n)\,,$$

so for all $\varepsilon>0$, for all $\gamma>0$, for a family (t_k) of independent variables with the same law as the capacities of the edges, we have

$$\begin{aligned}
\mathbb{P}\left[V(E_1^+)\geq \varepsilon\mathcal{H}^{d-1}(nA)\right] \,&\leq\, \mathbb{P}\left[\sum_{k=1}^{C^+n^{d-2}h(n)} t_k \geq \varepsilon\mathcal{H}^{d-1}(nA)\right]\\
&\leq\, \mathbb{E}(e^{\gamma t})^{C^+n^{d-2}h(n)} \exp\left(-\gamma\varepsilon\mathcal{H}^{d-1}(nA)\right)\\
&\leq\, \exp\left(-\mathcal{H}^{d-1}(nA)\left(\gamma\varepsilon - C^+\frac{n^{d-2}h(n)}{\mathcal{H}^{d-1}(nA)}\log\mathbb{E}(e^{\gamma t})\right)\right)\,.
\end{aligned}$$

For a fixed $R>0$, we can choose γ large enough to have $\gamma\varepsilon \geq 2R$, and also there exists n_2 such that for all $n \geq n_2$ we have

$$C^+ \frac{n^{d-2}h(n)}{\mathcal{H}^{d-1}(nA)} \log \mathbb{E}(e^{\gamma t}) \;\leq\; R\,,$$

so for all $R>0$

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[V(E_1^+) \geq \varepsilon \mathcal{H}^{d-1}(nA)\right] \;\leq\; -R$$

and then

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[V(E_1^+) \geq \varepsilon \mathcal{H}^{d-1}(nA)\right] \;=\; -\infty\,.$$

We obtain immediately that $(\phi(nA,h(n))/\mathcal{H}^{d-1}(nA))$ and $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA))$ are exponentially equivalent with regard to $\mathcal{H}^{d-1}(nA)$, and so by Theorem 17 $(\phi(nA,h(n))/\mathcal{H}^{d-1}(nA))$ satisfies the same large deviation principle as $(\tau(nA,h(n))/\mathcal{H}^{d-1}(nA))$.

6. Study of $\mathcal{J}_{\vec{v}}(\delta\|\vec{v}\|_1)$

This study is not needed in this paper, but it allows us to better understand the behaviour of the function $\mathcal{J}_{\vec{v}}$ near the point $\delta\|v\|_1$. As we said before, the behaviour of $\mathcal{I}_{\vec{v}}$ near $\delta\|v\|_1$ is not clear, that is the reason why we defined the function $\mathcal{J}_{\vec{v}}$. The problem comes from the corrective term $1/\sqrt{n}$ we added in the definition of $\mathcal{I}_{\vec{v}}$ to work with subadditive objects, it is not related to the behaviour of $\tau(nA,h(n))$ itself. We will show the following proposition

PROPOSITION 6.1. For all unit vector $\vec{v}$, we have:

$$\mathcal{J}_{\vec{v}}(\delta\|\vec{v}\|_1) \;=\; \lim_{\lambda\to\delta\|\vec{v}\|_1\,,\,\lambda>\delta\|\vec{v}\|_1} \mathcal{I}_{\vec{v}}(\lambda) \;\leq\; -\|\vec{v}\|_1 \log \mathbb{P}(t(e)=\delta)\,.$$

Proof :
Let A be a non degenerate hyperrectangle orthogonal to $\vec{v}$, and $h:\mathbb{N}\to\mathbb{R}^+$ such that

$$\lim_{n\to\infty} h(n) \;=\; +\infty\,.$$

We fix a $\varepsilon>0$, then for all n large enough we have $1/\sqrt{n} \leq \varepsilon/2$ and

$$\frac{\mathcal{N}(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \;\leq\; \|\vec{v}\|_1 + \frac{\varepsilon}{2\delta}\,.$$

The definition of $\mathcal{N}(nA,h(n))$ itself implies that there exists a set of $\mathcal{N}(nA,h(n))$ edges E_{min} that cuts $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\mathrm{cyl}(nA,h(n))$. We deduce from this remark that

$$\begin{aligned}
\mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq \delta\|\vec{v}\|_1 + \varepsilon - \frac{1}{\sqrt{n}}\right] \;&\geq\; \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \leq \delta\|\vec{v}\|_1 + \frac{\varepsilon}{2}\right]\\
&\geq\; \mathbb{P}\left[\frac{V(E_{min})}{\mathcal{H}^{d-1}(nA)} \leq \delta\|\vec{v}\|_1 + \frac{\varepsilon}{2}\right]\\
&\geq\; \mathbb{P}\left[\forall e\in E_{min}\,,\; t(e)=\delta\right]\\
&\geq\; \mathbb{P}(t=\delta)^{\mathcal{N}(nA,h(n))}\,.
\end{aligned}$$

We obtain immediately that for all $\varepsilon>0$,

$$\mathcal{I}_{\vec{v}}(\delta\|\vec{v}\|_1+\varepsilon) \;\leq\; -\|\vec{v}\|_1 \log \mathbb{P}(t=\delta)\,,$$

and so that

$$\mathcal{J}_{\vec{v}}(\delta\|\vec{v}\|_1) \leq -\|\vec{v}\|_1 \log \mathbb{P}(t=\delta)\,.$$

■

It is much more difficult to obtain a lower bound on $\mathcal{J}_{\vec{v}}(\delta\|\vec{v}\|_1)$. However, we can obtain it in the case where $d=3$, $\delta>0$ and $\vec{v}_0=(0,0,1)$. We will prove the proposition:

PROPOSITION 6.2. Let $d=3$. We suppose that F is such that $\delta>0$. Let $\vec{v}_0=(0,0,1)$ (so $\|\vec{v}_0\|_1=1$). Then we have

$$\mathcal{J}_{\vec{v}_0}(\delta) \,=\, -\log\mathbb{P}(t=\delta)\,.$$

Proof :
Let $d=3$. Using the proposition 6.1, we only have to prove that

$$\mathcal{J}_{\vec{v}_0}(\delta) \,\geq\, -\log\mathbb{P}(t=\delta)\,.$$

We suppose that $\delta>0$. We consider a straight cylinder $\mathrm{cyl}(nA,h(n))$, i.e., a cylinder of the form $\prod_{i=1}^{d-1}[na_i,nb_i]\times[c-h(n),c+h(n)]$. Let us define

$$\mathcal{P}_\varepsilon \,=\, \mathbb{P}\left[\frac{\tau(nA,h(n))}{\mathcal{H}^{d-1}(nA)}\leq \delta+\varepsilon-\frac{1}{\sqrt{n}}\right]\,.$$

Then

$$\mathcal{P}_\varepsilon \,=\, \mathbb{P}\left[\exists E\subset\mathbb{E}^d \,\middle|\, \begin{array}{c} E\ cuts\ (nA)_1^{h(n)}\ from\ (nA)_2^{h(n)}\ in\ \mathrm{cyl}(nA,h(n)) \\ and\ V(E)\leq\left(\delta+\varepsilon-\frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\end{array}\right]\,.$$

Such a set of edges E cannot contain more than $\mathcal{M}(n,A)$ edges, for

$$\mathcal{M}(n,A) \,=\, \left\lfloor\frac{1}{\delta}\left(\delta+\varepsilon-\frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right\rfloor\,.$$

We deduce that

$$\begin{aligned}\mathcal{P}_\varepsilon \leq \sum_{p=\mathcal{N}(nA,h(n))}^{\mathcal{M}(n,A)} &\mathrm{card}\left(\left\{E\subset\mathbb{E}^d \,\middle|\, \begin{array}{l} E\ cuts\ (nA)_1^{h(n)}\ from\ (nA)_2^{h(n)}\ in\ \mathrm{cyl}(nA,h(n)) \\ and\ \mathrm{card}(E)=p\end{array}\right\}\right)\\ &\times\mathbb{P}\left[\sum_{i=1}^p t_i\leq\left(\delta+\varepsilon-\frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right]\,,\end{aligned}$$

where (t_i) are independent and distributed as the capacities of the edges. For all fixed $\eta>0$, for all n large enough we have

$$\mathcal{N}(nA,h(n)) \,\geq\, (1-\eta)\mathcal{H}^{d-1}(nA)\,,$$

whence, by taking η small enough, for all $\varepsilon'>\varepsilon$ there exists $n(\delta,\varepsilon,\varepsilon')$ such that for all $n\geq n(\delta,\varepsilon,\varepsilon')$, for all $p\in\{\mathcal{N}(nA,h(n)),...,\mathcal{M}(n,A)\}$, we have

$$\frac{1}{p}\left(\delta+\varepsilon-\frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\leq\delta+\varepsilon'\,.$$

Thus for all $n\geq n(\delta,\varepsilon,\varepsilon')$ and all positive γ we have

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^p t_i\leq\left(\delta+\varepsilon-\frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right] &\leq\, \mathbb{P}\left[\frac{\sum_{i=1}^p t_i}{p}\leq\frac{\left(\delta+\varepsilon-\frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)}{p}\right]\\ &\leq\, \mathbb{P}\left[\frac{\sum_{i=1}^p t_i}{p}\leq\delta+\varepsilon'\right]\\ &\leq\, e^{\gamma\varepsilon' p}\mathbb{E}(e^{-\gamma(t-\delta)})^p\,.\end{aligned}$$

In the previous inequality, we remark that $t-\delta \geq 0$ a.s., so the expectation is well defined. Using lemma 14 in [**31**] stated by Grimmett and Gielis we know that there exists a constant $K(>1)$ such that

$$\operatorname{card}\left(\left\{E\subset\mathbb{E}^d\,\middle|\,\begin{array}{l} E\ cuts\ (nA)_1^{h(n)}\ from\ (nA)_2^{h(n)}\ in \\ \operatorname{cyl}(nA,h(n))\ and\ \operatorname{card}(E)=p\end{array}\right\}\right)\leq \mathcal{H}^{d-1}(nA)K^{p-\mathcal{N}(nA,h(n))}$$

(here the factor $\mathcal{H}^{d-1}(nA)$ corresponds to the number of possible origin for groups of wall as introduced in [**31**]). Then we obtain that

$$\begin{aligned}\mathcal{P}_\varepsilon &\leq \mathcal{H}^{d-1}(nA)\sum_{p=\mathcal{N}(nA,h(n))}^{\mathcal{M}(n,A)} K^{p-\mathcal{N}(nA,h(n))}e^{\gamma\varepsilon' p}\mathbb{E}(e^{-\gamma(t-\delta)})^p \\ &\leq \left(\frac{\varepsilon}{\delta}-\eta\right)K^{(\varepsilon/\delta-\eta)\mathcal{H}^{d-1}(nA)}e^{\gamma\varepsilon'(1+\varepsilon/2)\mathcal{H}^{d-1}(nA)}\mathbb{E}\big(e^{-\gamma(t-\delta)}\big)^{(1-\eta)\mathcal{H}^{d-1}(nA)}\,.\end{aligned}$$

We conclude that

$$\mathcal{I}_{\vec{v}_0}(\delta+\varepsilon)\;\geq\;-\gamma\varepsilon'(1+\varepsilon/2)-(1-\eta)\log\mathbb{E}(e^{-\gamma(t-\delta)})-\left(\frac{\varepsilon}{\delta}-\eta\right)\log K\,.$$

We know that

$$\lim_{\gamma\to\infty}\mathbb{E}(e^{-\gamma(t-\delta)})\;=\;\mathbb{P}(t=\delta)\,,$$

and by sending ε, η and ε' to 0 we obtain that

$$\mathcal{J}_{\vec{v}_0}(\delta)\geq -\log\mathbb{P}(t=\delta)\,.$$

This ends the proof of the proposition 6.2.

■

It is really difficult to obtain a lower bound on $\mathcal{J}_{\vec{v}}(\delta\|\vec{v}\|_1)$ in general. Indeed, if $\delta=0$, we have to consider an infinite number of cutsets (of arbitrarily high cardinality), and this is a major issue in the proof. For general $\vec{v}$, we cannot use the estimates of Gielis and Grimmett [**31**]. We tried to use instead estimates of Cerf and Kenyon (lemma 5.8 in [**17**]), but the upper bound we can obtain involves the entropy of $\vec{v}$, which quantifies the number of cutsets of almost minimal cardinality among those whose boundary is fixed along an hyperrectangle oriented towards the direction given by $\vec{v}$. Thus, the upper and lower bounds are not the same, and we have no idea of the real behaviour of $\mathcal{J}_{\vec{v}}(\delta\|\vec{v}\|_1)$ for $\vec{v}$ in general, even if $\delta>0$.

Part 3

The case of dimension two

CHAPTER 6

Law of large numbers, lower and upper large deviations for the maximal flow from the top to the bottom of a tilted cylinder in dimension two

This chapter from section 1 to section 4 is a joint work with Raphaël Rossignol.

Equip the edges of the lattice $\mathbb{Z}^2$ with i.i.d. random capacities of distribution function F. When $F(0) < 1/2$, we prove a law of large numbers and a large deviation principle from below for the maximal flow crossing a rectangle in $\mathbb{R}^2$ and we investigate the order of the upper large deviations of this variable, when the side lengths of the rectangle go to infinity. This extends and improves previous large deviations results of [**34**] obtained for straight boxes.

Contents

1. Introduction

The model of maximal flow in a randomly porous medium with independent and identically distributed capacities has been introduced by [**20**] and [**41**]. The purpose of this model is to understand the behaviour of the maximum amount of flow that can cross the medium from one part to another.

All the precise definitions will be given in section 2, but let us draw the general picture in dimension d. The random medium is represented by the lattice $\mathbb{Z}^d$. We see each edge as a microscopic pipe which the fluid can flow through. To each edge e, we attach a nonnegative capacity $t(e)$

which represents the amount of fluid (or the amount of fluid per unit of time) that can effectively go through the edge e. Capacities are then supposed to be random, identically and independently distributed with common distribution function F. Let A be some hyper-rectangle in $\mathbb{R}^d$ and n an integer. The portion of medium that we will look at is a box B_n of basis nA and of height $2h(n)$, which nA splits into two boxes of equal volume. The boundary of B_n is thus split into two parts, A_n^1 and A_n^2. There are two protagonists in this play, two types of flows through B_n: the maximal flow τ_n for which the fluid can enter the box through A_n^1 and leave it through A_n^2, and the maximal flow ϕ_n for which the fluid enters B_n only through its bottom side and leaves it through its top side. The first quality of τ_n is that it is (almost) a subadditive quantity, whereas ϕ_n is not. The main question now is: "How do ϕ_n and τ_n behave when n is large ?".

Existing results for ϕ_n and τ_n are essentially of two types: laws of large numbers and large deviations results. It is important to stress that the orientation on A plays an important role in these results. More precisely, the first results were obtained for "straight" boxes, i.e., when A is of the form $\prod_{i=1}^{d-1}[0, a_i] \times \{0\}$. Especially concerning the study of ϕ_n, this simplifies considerably the task. Here is the state of the art: the law of large numbers for τ_n were proved under mild hypotheses: in [**41**] for straight boxes and in [**52**] (Chapter 5 of the thesis) for general boxes. These results follow neatly from the subadditivity property already alluded to. Suppose that $t(e)$ has finite expectation, $\vec{v}$ denotes a unit vector orthogonal to A, and $h(n)$ goes to infinity. Then there is a function ν defined on S^{d-1} such that:

$$\nu(\vec{v}) = \lim_{n\to\infty} \frac{\mathbb{E}[\tau(nA, h(n))]}{\mathcal{H}^{d-1}(nA)},$$

where $\mathcal{H}^{d-1}(nA)$ is the $(d-1)$-dimensional Hausdorff measure of nA. Moreover, if the origin of the graph belongs to A and if there exists a real M such that all the coordinates of $M\vec{v}$ are rational, then

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}) \qquad \text{a.s. and in } L^1 .$$

The a.s. convergence does also happen for general A and $\vec{v}$ under stronger assumptions on F (see [**52**] or Chapter 5 of the thesis). The law of large numbers for ϕ_n was proved only for straight boxes, with suboptimal assumptions on the height h, the moments of F and on $F(\{0\})$, in [**41**]. In dimension 2, this was first studied in [**34**]. The assumption on $F(\{0\})$ was optimized in [**58**] and [**59**]. A specificity of the lattice $\mathbb{Z}^d$, namely its invariance under reflexions with respect to integer coordinate hyperplanes, implies that the law of large numbers is the same for ϕ_n and τ_n in straight cylinders (provided $\log h(n)$ does not grow too fast).

Concerning large deviations results for τ_n, a full large deviation principle from below was proved in [**52**] (Chapter 5 of the thesis), for general boxes. For ϕ_n, upper large deviations were studied in [**55**] for straight boxes only, and a complementary result is given in Chapter 3 of this thesis in the case where $h(n)/n$ goes to zero as n goes to infinity. Lower large deviations for ϕ_n were studied in [**52**] for straight boxes also (previous works include [**20**], [**56**] and, for dimension 2, [**21**]). In these results, very few is known about the rate functions (see also [**34**]).

Summarizing, τ_n is fairly well studied concerning laws of large numbers and large deviation principles from below in general boxes, but for ϕ_n, nothing is known when the boxes are not straight (except when the height is small compared to n, in which case it behaves like τ_n, cf. [**52**]).

This paper aims at filling this gap, although we can do so only in dimension 2. For instance, suppose that $2h(n)/(nl(A))$ goes to $\tan(\alpha)$ when n goes to infinity, with $\alpha \in [0, \frac{\pi}{2}]$ and $l(A)$ denoting the length of the line segment A. Our main results imply, under some conditions on F, that:

$$\frac{\phi_n}{nl(A)} \xrightarrow[n\to\infty]{a.s} \inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}, \tag{6.1}$$

where we re-encoded the function ν as follows: $\nu_{\widetilde{\theta}} := \nu(\vec{v}(\widetilde{\theta}))$ when $\vec{v}(\widetilde{\theta})$ makes an angle $\widetilde{\theta}$ with $(1,0)$. We shall also obtain a large deviation principle from below in the same spirit. We also investigate the order of the upper large deviations of ϕ_n. Notice that there is no reason for the limit in (6.1) to be identical to ν_θ. Thus, something different happens when the boxes are not straight, and this is, to our opinion, an important contribution of our study. Notice that this fact can already be observed when F is concentrated on one point. For instance, if $t(e) = 1$ deterministically and $2h(n)/(nl(A))$ goes to $\tan(\alpha)$ when n goes to infinity, with $\alpha > \frac{\pi}{4}$, then one may easily compute that $\nu_\theta = |\cos\theta| + |\sin\theta|$, whereas the limit of $\phi_n/(nl(A))$ is $\min\{1/|\cos\theta|, 1/|\sin\theta|\}$.

The paper is organized as follows. In section 2, we give the precise definitions and state the main results of the paper. Section 3 is devoted to the law of large numbers for ϕ_n, whereas the large deviation principle from below for ϕ_n is proved in section 4. The section 5 is devoted to the study of the upper large deviations of ϕ_n.

2. Notations, background and main results

The most important notations are gathered in this section.

2.1. Maximal flow on a graph. First, let us define the notion of a flow on a finite unoriented graph $G = (V, \mathcal{E})$ with set of vertices V and set of edges $\mathcal{E}$. Let $t = (t(e))_{e\in\mathcal{E}}$ be a collection of non-negative real numbers, which are called capacities. It means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time. To each edge e, one may associate two oriented edges, and we shall denote by $\overrightarrow{\mathcal{E}}$ the set of all these oriented edges. Let A and Z be two finite, disjoint, non-empty sets of vertices of G: A denotes the source of the network, and Z the sink. A function θ on $\overrightarrow{\mathcal{E}}$ is called a flow from A to Z with strength $\|\theta\|$ and capacities t if it is antisymmetric, i.e. $\theta_{\overrightarrow{xy}} = -\theta_{\overrightarrow{yx}}$, if it satisfies the node law at each vertex x of $V \smallsetminus (A \cup Z)$:

$$\sum_{y\sim x} \theta_{\overrightarrow{xy}} = 0 \ ,$$

where $y \sim x$ means that y and x are neighbours on G, if it satisfies the capacity constraints:

$$\forall e \in \mathcal{E},\ |\theta(e)| \leq t(e) \ ,$$

and if the "flow in" at A and the "flow out" at Z equal $\|\theta\|$:

$$\|\theta\| = \sum_{a\in A} \sum_{\substack{y\sim a \\ y\notin A}} \theta(\overrightarrow{ay}) = \sum_{z\in Z} \sum_{\substack{y\sim z \\ y\notin A}} \theta(\overrightarrow{yz}) \ .$$

The *maximal flow from A to Z*, denoted by $\phi_t(G, A, Z)$, is defined as the maximum strength of all flows from A to Z with capacities t. We shall in general omit the subscript t when it is understood from the context. The *max-flow min-cut theorem* (see [**12**] for instance) asserts that the maximal flow from A to Z equals the minimal capacity of a cut between A and Z. Precisely, let us say that $E \subset \mathcal{E}$ is a cut between A and Z in G if every path from A to Z borrows at least one edge of E. Define $V(E) = \sum_{e\in E} t(e)$ to be the capacity of a cut E. Then,

$$\phi_t(G, A, Z) = \min\{V(E) \text{ s.t. } E \text{ is a cut between } A \text{ and } Z \text{ in } G\} \ . \tag{6.2}$$

2.2. On the square lattice. We shall always consider G as a piece of $\mathbb{Z}^2$. More precisely, we consider the graph $\mathbb{L} = (\mathbb{Z}^2, \mathbb{E}^2)$ having for vertices $\mathbb{Z}^2$ and for edges $\mathbb{E}^2$, the set of pairs of nearest neighbours for the standard L^1 norm. The notation $\langle x, y\rangle$ corresponds to the edge with endpoints x and y. To each edge e in $\mathbb{E}^2$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. *We suppose that the family $(t(e), e \in \mathbb{E}^2)$ is independent and identically distributed, with a common distribution function F.* More formally, we take the product measure $\mathbb{P} = F^{\otimes\Omega}$ on $\Omega = \prod_{e\in\mathbb{E}^2} [0, \infty[$, and we write its expectation $\mathbb{E}$. If G is a subgraph of $\mathbb{L}$, and A and Z are two subsets of vertices of G, *we shall denote by $\phi(G, A, Z)$ the maximal flow in G from A to Z*, where

G is equipped with capacities t. When B is a subset of $\mathbb{R}^2$, and A and Z are subsets of $\mathbb{Z}^2 \cap B$, we shall denote by $\phi(B, A, Z)$ again the maximal flow $\phi(G, A, Z)$ where G is the induced subgraph of $\mathbb{Z}^2$ with set of vertices $\mathbb{Z}^2 \cap B$.

We denote by $\overrightarrow{e}_1$ (resp. $\overrightarrow{e}_2$) the vector $(1,0) \in \mathbb{R}^2$ (resp. $(0,1)$). Let A be a non-empty line segment in $\mathbb{R}^2$. We shall denote by $l(A)$ its (euclidean) length. All line segments will be supposed to be closed in $\mathbb{R}^2$. We denote by $\vec{v}(\theta)$ the vector of unit euclidean norm orthogonal to $\mathrm{hyp}(A)$, the hyperplane spanned by A, and such that there is $\theta \in [0, \pi[$ such that $\vec{v}(\theta) = (\cos\theta, \sin\theta)$. Define $\vec{v}^{\perp}(\theta) = (\sin\theta, -\cos\theta)$ and denote by a and b the end-points of A such that $(b-a).\vec{v}^{\perp}(\theta) > 0$. For h a positive real number, *we denote by* $\mathrm{cyl}(A,h)$ *the cylinder of basis* A *and height* $2h$, i.e., the set

$$\mathrm{cyl}(A,h) \,=\, \{x + t\vec{v}(\theta) \,|\, x \in A \,,\, t \in [-h,h]\}\,.$$

We define also *the* r*-neighbourhood* $\mathcal{V}(H,r)$ *of a subset* H *of* $\mathbb{R}^d$ as

$$\mathcal{V}(H,r) \,=\, \{x \in \mathbb{R}^d \,|\, d(x,H) < r\}\,,$$

where the distance is the euclidean one ($d(x,H) = \inf\{\|x-y\|_2 \,|\, y \in H\}$).

Now, we define $D(A,h)$ the set of *admissible boundary conditions* on $\mathrm{cyl}(A,h)$ (see Figure 1):

$$D(A,h) = \left\{(k,\tilde{\theta}) \,|\, k \in [0,1] \text{ and } \tilde{\theta} \in \left[\theta - \arctan\left(\frac{2hk}{l(A)}\right), \theta + \arctan\left(\frac{2h(1-k)}{l(A)}\right)\right]\right\} \,.$$

The meaning of an element $\kappa = (k,\tilde{\theta})$ of $D(A,h)$ is the following. We define

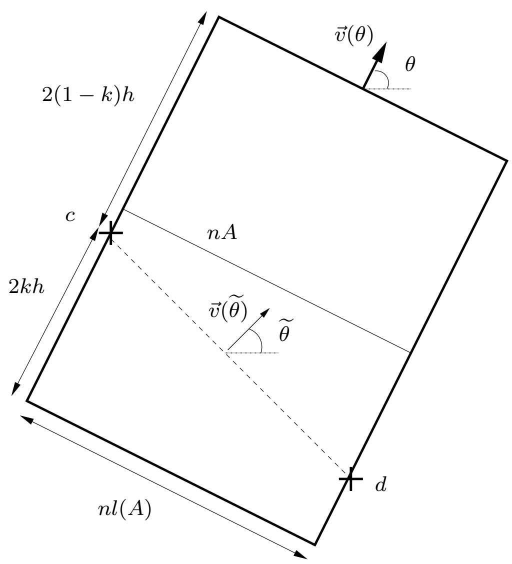

FIGURE 1. An admissible boundary condition $(k, \widetilde{\theta})$.

$$\vec{v}(\widetilde{\theta}) \,=\, (\cos\widetilde{\theta}, \sin\widetilde{\theta}) \qquad \text{and} \qquad \vec{v}^{\perp}(\widetilde{\theta}) \,=\, (\sin\widetilde{\theta}, -\cos\widetilde{\theta})\,.$$

In $\mathrm{cyl}(nA, h(n))$, we may define two points c and d such that c is "at height $2kh$ on the left side of $\mathrm{cyl}(A,h)$", and d is "on the right side of $\mathrm{cyl}(A,h)$" by

$$c = a + (2k-1)h\vec{v}(\theta)\,, \quad (d-c) \text{ is orthogonal to } \vec{v}(\widetilde{\theta}) \quad \text{ and } d \text{ satisfies } \vec{cd} \cdot \vec{v}^{\perp}(\widetilde{\theta}) > 0\,.$$

Then we see that $D(A,h)$ is exactly the set of parameters so that c and d remain "on the sides of $\mathrm{cyl}(A,h)$".

We define also $\mathcal{D}(A,h)$, the set of angles $\tilde{\theta}$ such that there is an admissible boundary condition with angle $\tilde{\theta}$:

$$\mathcal{D}(A,h) = \left[\theta - \arctan\left(\frac{2h}{l(A)}\right), \theta + \arctan\left(\frac{2h}{l(A)}\right)\right] .$$

It will be useful to define the *left side (resp. right side) of* $\mathrm{cyl}(A,h)$: let $\mathrm{left}(A)$ (resp. $\mathrm{right}(A)$) be the set of vertices in $\mathrm{cyl}(A,h) \cap \mathbb{Z}^2$ such that there exists $y \notin \mathrm{cyl}(A,h)$, $\langle x,y\rangle \in \mathbb{E}^d$ and $[x,y[$, the segment that includes x and excludes y, intersects $a + [-h,h].\vec{v}(\theta)$ (resp. $b + [-h,h].\vec{v}(\theta)$).

Now, the set $\mathrm{cyl}(A,h) \smallsetminus (c + \mathbb{R}(d-c))$ has two connected components, which we denote by $\mathcal{C}_1(A,h,k,\tilde{\theta})$ and $\mathcal{C}_2(A,h,k,\tilde{\theta})$. For $i=1,2$, let $A_i^{h,k,\tilde{\theta}}$ be the set of the points in $\mathcal{C}_i(A,h,k,\tilde{\theta}) \cap \mathbb{Z}^2$ which have a nearest neighbour in $\mathbb{Z}^2 \smallsetminus \mathrm{cyl}(A,h)$:

$$A_i^{h,k,\tilde{\theta}} = \{x \in \mathcal{C}_i(A,h,k,\tilde{\theta}) \cap \mathbb{Z}^2 \,|\, \exists y \in \mathbb{Z}^2 \smallsetminus \mathrm{cyl}(A,h) \,,\, \|x-y\|_1 = 1\} .$$

We define *the flow in* $\mathrm{cyl}(A,h)$ *constrained by the boundary condition* $\kappa = (k,\tilde{\theta})$ as:

$$\phi^\kappa(A,h) := \phi(\mathrm{cyl}(A,h), A_1^{h,k,\tilde{\theta}}, A_2^{h,k,\tilde{\theta}}) .$$

A special role is played by the condition $\kappa = (1/2,\theta)$, and we shall denote:

$$\tau(A,h) = \tau(\mathrm{cyl}(A,h), \vec{v}(\theta)) = \phi^{(1/2,\theta)}(A,h) .$$

Let $T(A,h)$ (respectively $B(A,h)$) be the top (respectively the bottom) of $\mathrm{cyl}(A,h)$, i.e.,

$$T(A,h) = \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h) \,,\ \langle x,y\rangle \in \mathbb{E}^d \text{ and } \langle x,y\rangle \text{ intersects } A + h\vec{v}(\theta)\}$$

and

$$B(A,h) = \{x \in \mathrm{cyl}(A,h) \,|\, \exists y \notin \mathrm{cyl}(A,h) \,,\ \langle x,y\rangle \in \mathbb{E}^d \text{ and } \langle x,y\rangle \text{ intersects } A - h\vec{v}(\theta)\} .$$

We shall denote the flow in $\mathrm{cyl}(A,h)$ from the top to the bottom as:

$$\phi(A,h) = \phi(\mathrm{cyl}(A,h), \vec{v}(\theta)) = \phi(\mathrm{cyl}(A,h), T(A,h), B(A,h)) .$$

2.3. Duality. The main reason why dimension 2 is easier to deal with than dimension $d \geq 3$ is duality. Planar duality implies that there are only $O(h^2)$ admissible boundary conditions on $\mathrm{cyl}(A,h)$. Let us go a bit into the details.

The dual lattice $\mathbb{L}^*$ of $\mathbb{L}$ is constructed as follows: place a vertex in the centre of each face of $\mathbb{L}$ and join two vertices in $\mathbb{L}^*$ if and only if the corresponding faces of $\mathbb{L}$ share an edge. To each edge e^* of $\mathbb{L}^*$, we assign the time coordinate $t(e)$, where e is the unique edge of $\mathbb{E}^2$ crossed by e^*. Now, let A be a line segment in $\mathbb{R}^2$. Let G_A be the induced subgraph of $\mathbb{L}$ with set of vertices $\mathrm{cyl}(A,h) \cap \mathbb{Z}^2$. Let G_A^* be the planar dual of G_A in the following sense: G_A^* has set of edges $\{e^* \text{ s.t. } e \in G_A\}$, and set of vertices those vertices which belong to this set of edges. Now, we define $\mathrm{left}^*(A)$ (resp. $\mathrm{right}^*(A)$) as the set of vertices v of G_A^* which have at least one neighbour in $\mathbb{L}^*$ which is not in G_A and such that there exists an edge e^* in G_A^* with $v \in e^*$ and $e^* \cap \mathrm{left}(A) \neq \emptyset$ (resp. $e^* \cap \mathrm{right}(A) \neq \emptyset$).

It is well known that the (planar) dual of a cut between the top and the bottom of $\mathrm{cyl}(A,h)$ is a self-avoiding path from "left" to "right". Furthermore, if the cut is minimal for the inclusion, the dual self-avoiding path has only one vertex on the left boundary of the dual of $A \cap \mathbb{Z}^2$ and one vertex on the right boundary. The following lemma is a formulation in our setting of those classical duality results (see for instance [**34**] p.358 and [**12**], p.47).

LEMMA 7. *Let A be a line segment $\mathbb{R}^2$ and h be a positive real number. If E is a set of edges, let*

$$E^* = \{e^* \,|\, e \in E\} .$$

If E is a cut between $B(A,h)$ and $T(A,h)$, minimal for the inclusion, then E^* is a self-avoiding path from $\mathrm{left}^*(A)$ *to* $\mathrm{right}^*(A)$ *such that exactly one point of E^* belongs to* $\mathrm{left}^*(A)$*, exactly one point of E^* belongs to* $\mathrm{right}^*(A)$*, and these two points are the end-points of the path.*

An immediate consequence of this planar duality is the following.

LEMMA 8. *Let A be any line segment in $\mathbb{R}^2$ and h a positive real number. Then,*

$$\phi(A,h) = \min_{\kappa\in D(A,h)} \phi^\kappa(A,h) \, .$$

Notice that the condition κ belongs to the non-countable set $D(A,h)$, but the graph is discrete so $\phi^\kappa(A,h)$ takes only a finite number of values when $\kappa \in D(A,h)$. Precisely, there is a finite subset $\tilde{D}(A,h)$ of $D(A,h)$, such that:

$$\mathrm{card}(\tilde{D}(A,h)) \leq C_4 h^2 \, , \tag{6.3}$$

for some universal constant C_4, and:

$$\phi(A,h) = \min_{\kappa\in \tilde{D}(A,h)} \phi^\kappa(A,h) \, .$$

2.4. Background and main results. First, let us recall some facts concerning the behaviour of $\tau(nA,h(n))$ when n and $h(n)$ go to infinity. We start with the law of large numbers satisfied by $\tau(nA,h(n))$. We follow exactly the steps of the proof of this result presented in the introduction of the thesis. Let $A=[a,b]$ be a non-empty line segment, $\vec{v}(\theta)=(\cos\theta,\sin\theta)$ a unit vector orthogonal to $\mathrm{hyp}(A)$. Define X_a (resp. X_b) the sum of the variables $t(e)$ for all edges e totally included in the ball $B(a,4)$ (resp. in $B(b,4)$). Patching cuts together, it may be seen that for every fixed height h, $(\tau(nA,h)+X_{na}+X_{nb})_{n\in\mathbb{N}}$ is a subadditive sequence. If the variables $t(e)$ have finite mean, Kingman's subadditive ergodic theorem (see [**42**], p. 884) implies the almost sure convergence and in L^1 of $(\tau(nA,h)+X_{na}+X_{nb})/(nl(A))$ (and therefore of $(\tau(nA,h)/(nl(A)))$ to some constant $\nu_\theta(h)$, when n goes to infinity, under three hypotheses: $\mathbb{E}(t(e))<\infty$, the origin of the graph belongs to A and there exists a real M such that $M\vec{v}(\theta)$ has rational coordinates. Standard techniques like in [**52**] allow to show that this limit depends only on h and θ. Also, if $h(n)$ goes to infinity, arguments as in [**55**] show that $\tau(nA,h(n))/(nl(A))$ converges to some constant $\nu_\theta = \inf_h \nu_\theta(h)$. The techniques of [**52**] allow also to prove that the convergence of $\mathbb{E}[\tau(nA,h(n))]/(nl(A))$ towards ν_θ happens under the hypothesis that $\mathbb{E}(t(e))<\infty$, without any assumption on A and $\vec{v}(\theta)$. We will only use this property during the chapter 6 of the thesis, so we state it clearly:

THEOREM 18. *Let A be a non-empty line segment, and $\vec{v}(\theta)=(\cos\theta,\sin\theta)$ a unit vector orthogonal to* $\mathrm{hyp}(A)$*. If F has finite mean, and $h(n)$ goes to infinity, $\mathbb{E}[\tau(nA,h(n))]/(nl(A))$ converges to some constant ν_θ when n goes to infinity. This limit ν_θ depends on F, d and θ, but not on h and on A itself.*

Finally, let us remark that ν_θ is equal to $\mu(\vec{v}^\perp(\theta))$, where $\mu(.)$ is the time-constant function of first passage percolation as defined in [**40**], (3.10) p. 158. This follows from the duality considerations of section 2.3 and standard first passage percolation techniques (see also Theorem 5.1 in [**34**]) that relate cylinder passage times to unrestricted passage times (as in [**38**], Theorem 4.3.7 for instance).

One consequence of this equality between ν and μ is that $\theta\mapsto\nu_\theta$ is either constant equal to zero, or always non-zero. The former case occurs when $F(0)\geq 1/2$, and the latter when $F(0)<1/2$.

Concerning large deviations results, Lemma 2 and Theorem 2 in [**52**] state this result:

THEOREM 19. *Suppose that F satisfies the condition:*

$$\left\{ \begin{array}{l} F(0) < \frac{1}{2} \,, \\ \exists \gamma > 0, \; \int e^{\gamma x} \, dF(x) < \infty \,. \end{array} \right. \tag{6.4}$$

For every non-empty line-segment A in $\mathbb{R}^2$, with euclidean length $l(A)$, for every sequence of positive real numbers $(h(n))_{n\geq 0}$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, for all λ in $\mathbb{R}^+$, the limit

$$\mathcal{I}_\theta(\lambda) \;=\; \lim_{n\to\infty} \frac{-1}{nl(A)} \log \mathbb{P}\left[\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) nl(A) \right]$$

exists in $[0,+\infty]$ and depends only on $\theta \in [0,\pi[$ such that $(\cos\theta, \sin\theta)$ is orthogonal to A. Moreover, the function $\mathcal{I}_\theta$ has the following properties: it is convex on $\mathbb{R}^+$, infinite on $[0, \delta(|\cos\theta| + |\sin\theta|)[$, where $\delta = \inf\{\lambda \,|\, \mathbb{P}[t(e) \leq \lambda] > 0\}$, finite on $]\delta(|\cos\theta|+|\sin\theta|), +\infty[$, continuous and strictly decreasing on $]\delta(|\cos\theta| + |\sin\theta|), \nu_\theta]$, positive on $]\delta(|\cos\theta| + |\sin\theta|), \nu_\theta[$ and equal to 0 on $[\nu_\theta, +\infty[$.

For simplicity of notations, we define $\mathcal{I}_{\widetilde{\theta}} = +\infty$ on $\mathbb{R}^-_*$, and for all $a \geq 0$,

$$\mathcal{I}_{\widetilde{\theta}}(a^+) \;=\; \lim_{\varepsilon\to 0,\, \varepsilon>0} \mathcal{I}_{\widetilde{\theta}}(a+\varepsilon) \qquad \text{and} \qquad \mathcal{I}_{\widetilde{\theta}}(a^-) \;=\; \lim_{\varepsilon\to 0,\, \varepsilon>0} \mathcal{I}_{\widetilde{\theta}}(a-\varepsilon)\,.$$

We denote by $\mathcal{J}_\theta$ the function defined on $\mathbb{R}^+$ by

$$\mathcal{J}_\theta(\lambda) \;=\; \left\{ \begin{array}{ll} \mathcal{I}_\theta(\lambda^+) & if\ \lambda \leq \nu_\theta \,, \\ +\infty & if\ \lambda > \nu_\theta \,. \end{array} \right.$$

The following large deviation principles have also been proved in [**52**]. Under the assumptions on F and h in Theorem 19, if moreover F admits exponential moment of all orders:

$$\forall \gamma > 0, \qquad \int e^{\gamma x} dF(x) \;<\; +\infty\,,$$

then the sequence $(\tau(nA, h(n))/nl(A))_{n\geq 0}$ satisfies a large deviation principle of speed $nl(A)$ with the good rate function $\mathcal{J}_\theta$. The same large deviation principle is satisfied by $(\phi_n/nl(A))_{n\geq 0}$ under the same hypothesis if $\lim_{n\to\infty} h(n)/n = 0$. Finally, under the assumptions on F and h in Theorem 19 and the added assumption that $\lim_{n\to\infty} \log h(n)/n = 0$, the sequence $(\phi_n/nl(A))_{n\geq 0}$ satisfies the same large deviation principle if A is horizontal, i.e., of the form $[a,b] \times \{0\}$.

We recall that for all $n \in \mathbb{N}$, we have defined

$$\mathcal{D}(nA, h(h)) = \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)} \right), \theta + \arctan\left(\frac{2h(n)}{nl(A)} \right) \right] \,.$$

We may now state our main results.

THEOREM 20 (Law of Large Numbers). *Let A be a non-empty line-segment in $\mathbb{R}^2$, with euclidean length $l(A)$. Let $\theta \in [0,\pi[$ be such that $(\cos\theta, \sin\theta)$ is orthogonal to A and $(h(n))_{n\geq 0}$ be a sequence of positive real numbers such that:*

$$\left\{ \begin{array}{l} h(n) \xrightarrow[n\to\infty]{} +\infty \,, \\ \frac{\log h(n)}{n} \xrightarrow[n\to\infty]{} 0 \,. \end{array} \right. \tag{6.5}$$

Define:

$$\overline{\mathcal{D}} = \limsup_{n\to\infty} \mathcal{D}(nA, h(n)) = \bigcap_{N\geq 1} \bigcup_{n\geq N} \mathcal{D}(nA, h(n)) \,,$$

and,

$$\underline{\mathcal{D}} = \liminf_{n\to\infty} \mathcal{D}(nA, h(n)) = \bigcup_{N\geq 1} \bigcap_{n\geq N} \mathcal{D}(nA, h(n)) \,.$$

Suppose that the following conditions on F are satisfied:

$$\begin{cases} F(0) < \frac{1}{2} \, , \\ \exists \varepsilon > 0, \ \int x^{2+\varepsilon} \, dF(x) < \infty \, . \end{cases} \tag{6.6}$$

Then,

$$\liminf_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} = \inf \left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \,|\, \widetilde{\theta} \in \overline{\mathcal{D}} \right\} \qquad a.s.$$

and

$$\limsup_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} = \inf \left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \,|\, \widetilde{\theta} \in \underline{\mathcal{D}} \right\} \qquad a.s.$$

COROLLARY 2.1. *Under the hypotheses of Theorem 20, suppose there is some $\alpha \in [0, \frac{\pi}{2}]$ such that:*

$$\frac{2h(n)}{nl(A)} \xrightarrow[n\to\infty]{} \tan\alpha \, .$$

Then,

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} = \inf \left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \,|\, \widetilde{\theta} \in [\theta - \alpha, \theta + \alpha] \right\} \qquad a.s.$$

It has already been remarked in [**56**] (see the discussion after Theorem 2) that the condition on h is the good one to have positive speed when one allows edge capacities to be null with positive probability. Also, as observed before, the condition on $F(0)$ is optimal if one wants to have positive speed.

REMARK 24. Notice that Theorem 20 is consistent with the law of large numbers obtained in the “straight case”, i.e. when $\theta \in \{0, \pi/2\}$ (cf. Corollary 4.2 in [**34**]). Indeed, it is known that ν satisfies the weak triangle inequality (see section 4.5 of the Chapter 5 of the thesis), and for symmetry reasons, it implies that when $\theta \in \{0, \pi/2\}$, the function $\widetilde{\theta} \mapsto \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta} - \theta)$ is minimum for $\widetilde{\theta} = \theta$ and thus, Theorem 20 implies that $\phi(nA, h(n))/(nl(A))$ converges to ν_0, the limit of $\tau(nA, h(n))/(nl(A))$, when $\mathrm{cyl}(nA, h(n))$ is a straight cylinder. In fact, the same phenomenon occurs for any θ such that there is a symmetry axis of direction θ for the lattice $\mathbb{Z}^2$. These directions in $[0, \pi[$ are of course $\{0, \pi/4, \pi/2, 3\pi/4\}$. We do not give more details here, because we will use the same ideas in a more complicated case in Lemma 16 in section 4.2.1 to prove the equivalent remark 25 for the large deviation principle.

Also, Corollary 2.1 is consistent with the fact that for general boxes, when $h(n)$ is small with respect to n, $\phi(nA, h(n))/(nl(A))$ and $\tau(nA, h(n))/(nl(A))$ have the same limit.

THEOREM 21 (Lower Large Deviation Principle). *Let A be a non-empty line-segment in $\mathbb{R}^2$, and $(h(n))_{n\geq 0}$ be a sequence of positive real numbers satisfying condition (6.5) and such that*

$$\lim_{n\to\infty} \frac{2h(n)}{nl(A)} = \tan\alpha$$

exists in $[0, +\infty]$. Let $\mathcal{D} = [\theta - \alpha, \theta + \alpha]$, and

$$\eta_{\theta,h} = \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \, .$$

Suppose that the conditions (6.4) on F are satisfied:

$$\begin{cases} F(0) < \frac{1}{2} \, , \\ \exists \gamma > 0, \ \int e^{\gamma x} \, dF(x) < \infty \, . \end{cases}$$

Define the rate function $\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\mathcal{K}(\lambda) = \begin{cases} \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta} - \theta)^+) & \textit{if } \lambda \leq \eta_{\theta,h} \, , \\ +\infty & \textit{if } \lambda > \eta_{\theta,h} \, . \end{cases} \tag{6.7}$$

Then the sequence

$$\left(\frac{\phi(nA, h(n))}{nl(A)} \right)_{n\in\mathbb{N}}$$

satisfies a large deviation principle of speed $nl(A)$ with the good rate function $\mathcal{K}$.

Moreover, if we define

$$\delta_{\theta,h} \;=\; \delta \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{|\cos\widetilde{\theta}| + |\sin\widetilde{\theta}|}{\cos(\widetilde{\theta}-\theta)} \,.$$

where $\delta = \inf\{\lambda \,|\, \mathbb{P}(t(e) \leq \lambda) > 0\}$, the good rate function $\mathcal{K}$ has the following properties: it is infinite on $[0, \delta_{\theta,h}[\cup]\eta_{\theta,h}, +\infty[$, finite on $]\delta_{\theta,h}, \eta_{\theta,h}]$, positive on $[\delta_{\theta,h}, \eta_{\theta,h}[$ and equal to 0 at $\eta_{\theta,h}$, and strictly decreasing when it is finite, in the sense that if $\mathcal{K}(\lambda) < \infty$, for all $\varepsilon > 0$, $\mathcal{K}(\lambda - \varepsilon) > \mathcal{K}(\lambda)$.

REMARK 25. We will prove in Lemma 16 in section 4.2.1 that when $\theta \in \{0, \pi/2\}$, we have $\mathcal{K}(\lambda) = \mathcal{I}_\theta(\lambda^+)$, and so Theorem 21 is consistent with the large deviation principle obtained in [**52**] in the case of straight cylinders.

THEOREM 22 (Upper large deviations). *Let A be a non-empty line-segment in $\mathbb{R}^2$, with euclidean length $l(A)$. Let $\theta \in [0, \pi[$ be such that $(\cos\theta, \sin\theta)$ is orthogonal to the hyperplane spanned by A and $(h(n))_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} h(n) = +\infty$. Suppose that the conditions (6.4) on F are satisfied:*

$$\begin{cases} F(0) < \frac{1}{2} \,, \\ \exists \gamma > 0, \ \int e^{\gamma x} \, dF(x) < \infty \,. \end{cases}$$

and that

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} \;=\; \eta_{\theta,h}$$

exists a.s. Then for all $\lambda > \eta_{\theta,h}$, we have

$$\liminf_{n\to\infty} \frac{-1}{nl(A)h(n)} \log \mathbb{P}\left[\phi(nA, h(n)) \geq \lambda nl(A)\right] \;>\; 0 \,. \tag{6.8}$$

The upper large deviations are thus of volume order.

REMARK 26. For the reasons given in Chapter 3, we were not able to adapt the proof of the large deviation principle from above we obtained in Chapter 2 for straight cylinders, even in dimension two.

We shall often use two abbreviations: *làglàd* for “limite à gauche, limite à droite”, meaning that a function admits, on every point of its domain, a limit (eventually infinite) from the left and a limit from the right. We shall also use *l.s.c* for “lower semi-continuous”.

3. Law of large numbers

In this section, we shall prove Theorem 20. So we suppose that A is a non-empty line segment in $\mathbb{R}^2$. To shorten the notations, we shall write $D_n = D(nA, h(n))$, the set of all admissible conditions for $(nA, h(n))$:

$$D_n = \left\{ (k, \tilde{\theta}) \,|\, k \in [0,1] \text{ and } \tilde{\theta} \in \left[\theta - \arctan\left(\frac{2h(n)k}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)(1-k)}{nl(A)}\right) \right] \right\} \,,$$

and

$$\mathcal{D}_n = \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)}{nl(A)}\right) \right] \,.$$

Also, we shall use:

$$\phi_n = \phi(nA, h(n)), \quad \text{ and } \quad \phi_n^\kappa = \phi^\kappa(nA, h(n)) \,.$$

Furthermore, we shall denote by E_{ϕ_n} a cut whose capacity achieves the minimum in the dual definition (6.2) of ϕ_n.

3.1. Sketch of the proof. First, recall that from Lemma 8,

$$\phi_n = \min_{\kappa \in D_n} \phi_n^\kappa \,.$$

Step 1. A concentration result for ϕ_n (section 3.2) and a Borel-Cantelli argument allow us to reduce the study of the LLN of ϕ_n to the study of the convergence of $\mathbb{E}(\phi_n)/nl(A)$ as n tends to infinity.

Step 2. A deviation result for ϕ_n^κ (section 3.2 again) shows that the mean of ϕ_n is equivalent to $\min_{\kappa \in D_n} \mathbb{E}(\phi_n^\kappa)$.

Step 3. Now, for a fixed $\kappa = (k, \tilde{\theta})$, ϕ_n^κ is roughly bounded from above by a variable $\tau(nA', h'(n))$, for some small enough h' and a line segment A' having angle $\tilde{\theta}$ (see Figure 2). This allow us to show in section 3.3 that

$$\limsup_{n\to\infty} \frac{\phi_n}{nl(A)} \le \inf_{\widetilde{\theta} \in \underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}$$

and

$$\liminf_{n\to\infty} \frac{\phi_n}{nl(A)} \le \inf_{\widetilde{\theta} \in \overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \,.$$

Step 4. On the other hand, by a subadditive argument (see Figure 3), we show in section 3.4 that

$$\liminf_{n\to\infty} \inf_{\kappa \in D_n} \frac{\phi_n^\kappa}{nl(A)} \ge \inf_{\widetilde{\theta} \in \overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}$$

and

$$\liminf_{n\to\infty} \inf_{\kappa \in D_n} \frac{\phi_n^\kappa}{nl(A)} \ge \inf_{\widetilde{\theta} \in \overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \,.$$

3.2. Concentration properties of the maximal flows. The following proposition, due to Kesten, allows to control the size of the minimal cut, and is of fundamental importance in the study of First Passage Percolation.

PROPOSITION 3.1 (Proposition 5.8 in [**40**]). Suppose that $F(0) < \frac{1}{2}$. Then, there are constants a, C_1 and C_2, depending only on F, such that:

$$\mathbb{P}\left(\begin{array}{c} \exists \text{ a self-avoiding path } \gamma \text{ in } \mathbb{L}^*, \text{ starting at } (\frac{1}{2}, \frac{1}{2}), \\ \text{with } \mathrm{card}(\gamma) \ge m \text{ and } \sum_{e^* \in \gamma} t(e^*) \le am \end{array} \right) \le C_1 e^{-C_2 m} \,.$$

Thanks to Proposition 3.1 and general concentration and moment inequalities due to [**14, 13**], we obtain the following concentration results for the maximal flows ϕ_n and ϕ_n^κ.

PROPOSITION 3.2. Let A be a non-empty line segment.
i) Suppose that F satisfies (6.6) for some $\varepsilon \in]0,1[$. Then, there is a positive constant C_ε, depending only on F and A such that, for every n:

$$\mathbb{E}(|\phi_n - \mathbb{E}(\phi_n)|^{2+\varepsilon}) \le C_\varepsilon n^{1+\frac{\varepsilon}{2}} \,. \tag{6.9}$$

ii) Suppose that $F(0) < 1/2$ and

$$\int_{[0,+\infty[} x^2 \, dF(x) \, < \, +\infty \,.$$

Then, there are positive constants K_1 and K_2, depending only on F, such that, for every n and every $\eta \in]0,1[$:

$$\max_{\kappa \in D_n} \mathbb{P}(\phi_n^\kappa < \mathbb{E}(\phi_n^\kappa)(1-\eta)) \le K_1 e^{-K_2 \eta \min_\kappa \mathbb{E}(\phi_n^\kappa)} + K_1 e^{-K_2(1-\eta) \min_\kappa \mathbb{E}(\phi_n^\kappa)} \,. \tag{6.10}$$

An immediate consequence of Proposition 3.2 (through Borel-Cantelli's lemma) is that to prove Theorem 20, it is sufficient to prove it with ϕ_n replaced by $\mathbb{E}(\phi_n)$.

Proof of Proposition 3.2: First, we prove (6.9). We shall make use of the "ϕ-Sobolev inequalities" obtained in Lemma 3 of [**13**]. To this end, we need some notation. We order the edges in $\mathrm{cyl}(nA, h(n)) \cap \mathbb{L}$ as $e_1, \dots, e_{m_n}$.

We keep the notation t for the collection of random capacities and we take t' an independent collection of capacities with the same law as t. For each edge $e_i \in \mathrm{cyl}(A, h)$, we denote by $t^{(i)}$ the collection of capacities obtained from t by replacing $t(e_i)$ by $t'(e_i)$, and leaving all other coordinates unchanged. Define:

$$V_+ = \mathbb{E}\left[\sum_{i=1}^{m_n} (\phi_n(t) - \phi_n(t^{(i)}))_+^2 \middle| t \right] ,$$

where $\phi_n(t)$ is the maximal flow through $\mathrm{cyl}(nA, h)$ when capacities are given by t, and $u_+ = \max(u, 0)$. Similarly, define:

$$V_- = \mathbb{E}\left[\sum_{i=1}^{m_n} (\phi_n(t) - \phi_n(t^{(i)}))_-^2 \middle| t \right] ,$$

where $u_- = \min(u, 0)$. Observe that:

$$\phi_n(t^{(i)}) - \phi_n(t) \leq (t'(e_i) - t(e_i)) \, 1\!\!1_{e_i \in E_{\phi_n}} ,$$

and

$$\mathbb{E}[(t'(e_i) - t(e_i))_+^2 | t] \;\leq\; \mathbb{E}[t'(e_i)^2 | t] \;=\; M_2 ,$$

where we defined $M_r = \mathbb{E}(t(e)^r)$. Thus,

$$V_- \leq M_2 \,\mathrm{card}(E_{\phi_n}) \; . \tag{6.11}$$

Recall the following version of Efron-Stein's inequality (see for instance the discussion after Proposition 1 in [**14**]),

$$\mathrm{Var}(\phi_n) \leq \mathbb{E}(V_-) \; .$$

Thus,

$$\mathrm{Var}(\phi_n) \leq M_2 \mathbb{E}(\mathrm{card}(E_{\phi_n})) = O(n) \; , \tag{6.12}$$

where the estimation is a simple consequence of Proposition 3.1 (that kind of bound was first shown by Kesten, see Theorem 1 in [**39**]). Now, let $\varepsilon \in]0, 1[$ be such that $\int x^{2+\varepsilon} \; dF(x) < \infty$. Define $q = 2 + \varepsilon$ and $\alpha = q - 1 = 1 + \varepsilon$. Then, Lemma 3 in [**13**] states that:

$$\mathbb{E}[(\phi_n - \mathbb{E}(\phi_n))_+^q] \leq \mathbb{E}[(\phi_n - \mathbb{E}(\phi_n))_+^\alpha]^{q/\alpha} + \alpha \mathbb{E}[V_+ (\phi_n - \mathbb{E}(\phi_n))_+^\varepsilon] \; , \tag{6.13}$$

and:

$$\mathbb{E}[(\phi_n - \mathbb{E}(\phi_n))_-^q] \leq \mathbb{E}[(\phi_n - \mathbb{E}(\phi_n))_-^\alpha]^{q/\alpha} + \alpha \mathbb{E}[V_- (\phi_n - \mathbb{E}(\phi_n))_-^\varepsilon] \; . \tag{6.14}$$

Since $\alpha < 2$, and using (6.12),

$$\mathbb{E}[|\phi_n - \mathbb{E}(\phi_n)|^\alpha]^{q/\alpha} \leq \mathrm{Var}(\phi_n)^{q/2} = O(n^{1+\frac{\varepsilon}{2}}) \; . \tag{6.15}$$

Now, let us bound $\mathbb{E}[V_- (\phi_n - \mathbb{E}(\phi_n))_-^\varepsilon]$. First,

$$\mathbb{E}[V_- (\phi_n - \mathbb{E}(\phi_n))_-^\varepsilon] \leq \mathbb{E}[V_- |\phi_n - \mathbb{E}(\phi_n)|^\varepsilon] \leq M_2 \mathbb{E}[\mathrm{card}(E_{\phi_n}) \cdot |\phi_n - \mathbb{E}(\phi_n)|^\varepsilon] \; . \tag{6.16}$$

Then, using Hölder's inequality,

$$\mathbb{E}[\mathrm{card}(E_{\phi_n}) \cdot |\phi_n - \mathbb{E}(\phi_n)|^\varepsilon] \leq \mathbb{E}[\mathrm{card}(E_{\phi_n})^{\frac{2}{2-\varepsilon}}]^{\frac{2-\varepsilon}{2}} \, \mathrm{Var}(\phi_n)^{\frac{\varepsilon}{2}} \; . \tag{6.17}$$

Now, we claim that it is a consequence of Proposition 3.1 that for every $\beta > 0$, there is some constant K_4, depending only on F, A and β such that:

$$\mathbb{E}(\mathrm{card}(E_{\phi_n})^\beta) \leq K_4 \left(n^\beta + \mathbb{E}[|\phi_n - \mathbb{E}(\phi_n)|^\beta]\right) . \tag{6.18}$$

We shall show this claim later. Noticing that $\frac{2}{2-\varepsilon} < 2$, and using (6.12), it implies that:

$$\mathbb{E}[\mathrm{card}(E_{\phi_n})^{\frac{2}{2-\varepsilon}}]^{\frac{2-\varepsilon}{2}} = O(n) .$$

This equation, together with inequalities (6.15), (6.16) and (6.17) shows that:

$$\mathbb{E}[(\phi_n - \mathbb{E}(\phi_n))_-^q] = O(n^{1+\frac{\varepsilon}{2}}) . \tag{6.19}$$

Now, let us bound $\mathbb{E}[V_+(\phi_n - \mathbb{E}(\phi_n))_+^\varepsilon]$. First, since we cannot bound V_+ efficiently, but only V_-, we need a small trick. Notice, since $0 < \varepsilon < 1$, that for any real numbers a and b:

$$(a+b)_+^\varepsilon \leq a_+^\varepsilon + b_+^\varepsilon .$$

For any i,

$$\begin{aligned}
&\mathbb{E}[(\phi_n(t) - \phi_n(t^{(i)}))_+^2(\phi_n - \mathbb{E}(\phi_n))_+^\varepsilon] \\
&\qquad \leq \mathbb{E}[(\phi_n(t) - \phi_n(t^{(i)}))_+^{2+\varepsilon}] + \mathbb{E}[(\phi_n(t) - \phi_n(t^{(i)}))_+^2(\phi_n(t^{(i)}) - \mathbb{E}(\phi_n))_+^\varepsilon] , \\
&\qquad = \mathbb{E}[(\phi_n(t) - \phi_n(t^{(i)}))_+^{2+\varepsilon}] + \mathbb{E}[(\phi_n(t) - \phi_n(t^{(i)}))_-^2(\phi_n(t) - \mathbb{E}(\phi_n))_+^\varepsilon] ,
\end{aligned}$$

where we have used the fact that t and $t^{(i)}$ have the same distribution. Thus,

$$\mathbb{E}[V_+(\phi_n - \mathbb{E}(\phi_n))_+^\varepsilon] \leq \sum_{i=1}^{m_n} \mathbb{E}[(\phi_n(t) - \phi_n(t^{(i)}))_+^{2+\varepsilon}] + \mathbb{E}[V_-|\phi_n - \mathbb{E}(\phi_n)|^\varepsilon] . \tag{6.20}$$

As we obtained (6.11), we get that:

$$\sum_{i=1}^{m_n} \mathbb{E}[(\phi_n(t) - \phi_n(t^{(i)}))_+^{2+\varepsilon}] \leq M_{2+\varepsilon}\mathbb{E}[\mathrm{card}(E_{\phi_n})] .$$

From this inequality, (6.20) and (6.18), we get:

$$\mathbb{E}[V_+(\phi_n - \mathbb{E}(\phi_n))_+^\varepsilon] = O(n^{1+\frac{\varepsilon}{2}}) . \tag{6.21}$$

Inequalities (6.13), (6.15), (6.19) and (6.21) imply that $\mathbb{E}[|\phi_n - \mathbb{E}(\phi_n)|^{2+\varepsilon}] = O(n^{1+\frac{\varepsilon}{2}})$. Thus, to prove (6.9) it remains to show that claim (6.18) is true. Let a be as in Proposition 3.1 and define:

$$m_0 = \frac{1}{a}\max\{2\mathbb{E}(\phi_n), n + \mathbb{E}(\phi_n)\} .$$

Notice that for every $m \geq m_0$,

$$\mathbb{P}(\phi_n \geq am) \leq \mathbb{P}(|\phi_n - \mathbb{E}(\phi_n)| > am/2) . \tag{6.22}$$

Now, using Proposition 3.1 and (6.22), for any $\beta > 0$,

$$\begin{aligned}
\mathbb{E}(\mathrm{card}(E_{\phi_n})^\beta) &= \int_0^\infty \mathbb{P}(\mathrm{card}(E_{\phi_n}) > t^{1/\beta})\, dt , \\
&\leq \int_0^\infty [C_1 e^{-C_2 t^{1/\beta}} + \mathbb{P}(\phi_n > a t^{1/\beta})]\, dt , \\
&\leq K_1(\beta) + m_0^\beta + \int_{m_0^\beta}^\infty \mathbb{P}(|\phi_n - \mathbb{E}(\phi_n)| > \frac{a}{2} t^{1/\beta})\, dt , \\
&\leq K_1(\beta) + m_0^\beta + \frac{2^\beta}{a^\beta}\mathbb{E}(|\phi_n - \mathbb{E}(\phi_n)|^\beta) ,
\end{aligned}$$

where $K_1(\beta)$ is some positive constant depending only on β, C_1 and C_2. Noticing that $\mathbb{E}(\phi_n) = O(n)$, we see that $m_0 = O(n)$. Thus, claim (6.18) is true.

Now, we turn to the proof of (6.10). This will be a consequence of Corollary 3 in [**14**]. Let us denote by $E_{\phi_n^\kappa}$ a cut achieving the minimum in the definition of $\phi_n^\kappa(t)$. From Proposition 3.1, we know that there are constants a, C_1 and C_2, depending only on F such that, for every κ and m:

$$\mathbb{P}(\mathrm{card}(E_{\phi_n^\kappa}) \geq m \text{ and } \phi_n^\kappa \leq am) \leq C_1 e^{-C_2 m} \,. \tag{6.23}$$

Let η be in $]0,1[$. Define:

$$\psi_n^\kappa \,=\, \min \left\{ \begin{array}{c} V(E) \text{ s.t. } \mathrm{card}(E) \leq (1-\eta)\mathbb{E}(\phi_n^\kappa)/a \\ \text{and } E \text{ cuts } T(nA,h(n)) \text{ from } B(nA,h(n)) \text{ in } \mathrm{cyl}(nA,h(n)) \end{array} \right\} \,,$$

and:

$$V_-^\kappa = \mathbb{E}\left[\sum_{i=1}^{m_n} (\psi_n^\kappa(t) - \psi_n^\kappa(t^{(i)}))_-^2 \middle| t \right] \,.$$

As one obtains (6.11), we can get:

$$V_-^\kappa \leq M_2 \,\mathrm{card}(E_{\psi_n^\kappa}) \leq M_2 \frac{1-\eta}{a} \mathbb{E}(\phi_n^\kappa) \,. \tag{6.24}$$

Notice that $\mathbb{E}(\phi_n^\kappa) \leq \mathbb{E}(\psi_n^\kappa)$. Thus, Corollary 3 in [**14**] and inequality (6.24) imply that, for every $\eta \in]0,1[$,

$$\mathbb{P}(\psi_n^\kappa < \mathbb{E}(\psi_n^\kappa)(1-\eta)) \leq \exp\left(-\frac{a\eta}{4M_2(1-\eta)} \mathbb{E}(\phi_n^\kappa) \right) \leq \exp\left(-\frac{a\eta}{4M_2} \mathbb{E}(\phi_n^\kappa) \right) \,. \tag{6.25}$$

Now, using (6.25) and (6.23),

$$\begin{aligned} &\mathbb{P}(\phi_n^\kappa < \mathbb{E}(\phi_n^\kappa)(1-\eta)) \\ &\quad \leq \mathbb{P}(\psi_n^\kappa < \mathbb{E}(\phi_n^\kappa)(1-\eta)) + \mathbb{P}\left(\mathrm{card}(E_{\phi_n^\kappa}) \leq \frac{1-\eta}{a} \mathbb{E}(\phi_n^\kappa) \text{ and } \phi_n^\kappa < \mathbb{E}(\phi_n^\kappa)(1-\eta) \right) \,, \\ &\quad \leq e^{-\frac{a\eta}{4M_2}\mathbb{E}(\phi_n^\kappa)} + C_1 e^{-\frac{C_2}{a}\mathbb{E}(\phi_n^\kappa)(1-\eta)} \,. \end{aligned}$$

This finishes the proof of Proposition 3.2.

□

We remark that such a concentration inequality for the variable τ instead of ϕ allows us to deduce from the theorem 18 that the sequence $(\tau(nA,h(n))/(nl(A)), n \in \mathbb{N})$ converges a.s. to ν_θ, whatever the segment A and the direction $\vec{v}(\theta)$ we consider.

3.3. Upper bound. From now on, we suppose that the conditions (6.6) on F and (6.5) on h are satisfied.

This is the easier part of the work. We consider a line segment A, of orthogonal unit vector $\vec{v}(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [0,\pi[$, and a function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$. Recall that $D_n = D(nA,h(n))$. Let $\widetilde{\theta} \in \underline{\mathcal{D}} = \liminf_{n\to\infty} \mathcal{D}_n$, and we only consider n large enough to have $\widetilde{\theta} \in \mathcal{D}_n$. For each such n, we choose a k_n by a deterministic way such that $\kappa_n = (k_n, \widetilde{\theta}) \in D_n$ (the set $\{k \in [0,1] \,|\, (k,\widetilde{\theta}) \in D_n\}$ is closed and non empty, so we can for example take its infimum). What we want to do is to compare $\phi_n^{\kappa_n}$ with the minimal weight of a cutset in a flat cylinder inside $\mathrm{cyl}(nA,h(n))$ and oriented toward the direction $\widetilde{\theta}$. The following definitions can seem a little bit complicated, but Figure 2 is more explicit. We choose two functions $h', \zeta : \mathbb{N} \to \mathbb{R}^+$ such that

$$\lim_{n\to\infty} h'(n) \,=\, \lim_{n\to\infty} \zeta(n) \,=\, +\infty \,,$$

and

$$\lim_{n\to\infty} \frac{h'(n)}{\zeta(n)} \,=\, 0 \,, \quad \lim_{n\to\infty} \frac{\zeta(n)}{n} \,=\, 0 \;\text{ and }\; \lim_{n\to\infty} \frac{\zeta(n)}{h(n)} \,=\, 0 \,. \tag{6.26}$$

Let $\vec{v}(\widetilde{\theta}) = (\cos\widetilde{\theta}, \sin\widetilde{\theta})$, $\vec{v}^\perp(\widetilde{\theta}) = (\sin\widetilde{\theta}, -\cos\widetilde{\theta})$. In $\mathrm{cyl}(nA,h(n))$, we denote by x_n and y_n the two points corresponding to the boundary conditions κ_n, such that $\overrightarrow{x_n y_n} \cdot \vec{v}^\perp(\widetilde{\theta}) > 0$. Then we

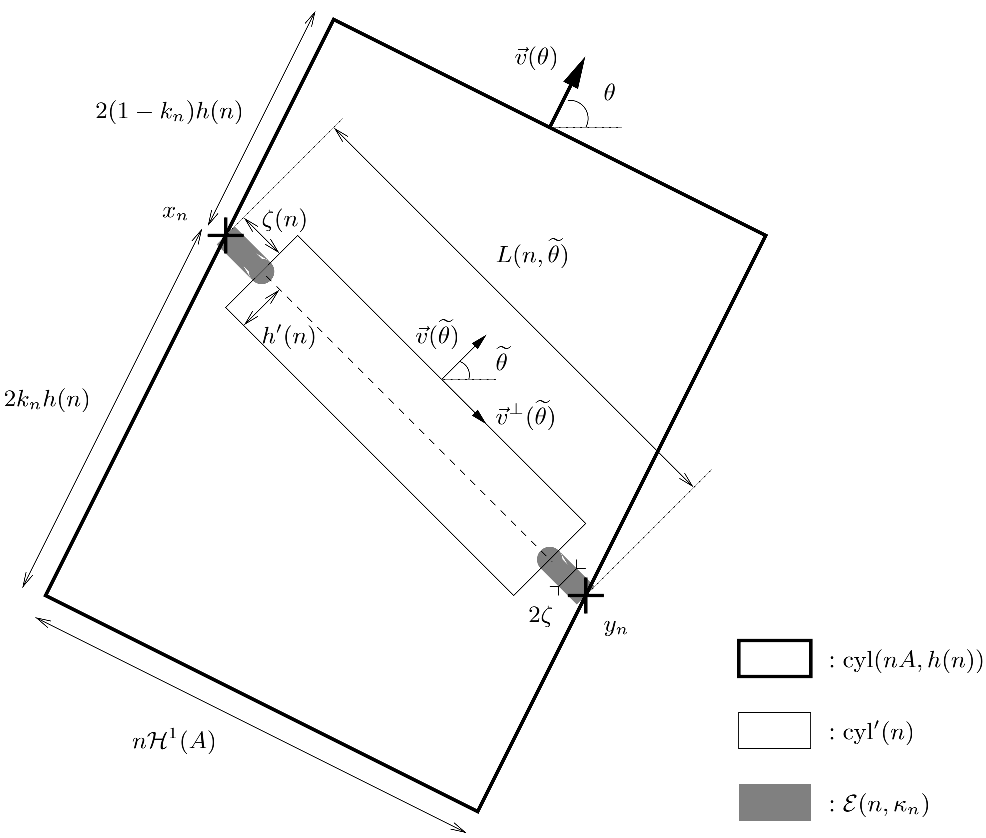

FIGURE 2. The cylinders $\mathrm{cyl}(nA,h(n))$ and $\mathrm{cyl}'(n)$.

define

$$\mathrm{cyl}'(n) \,=\, \mathrm{cyl}([x_n+\zeta(n)\vec{v}^{\perp}(\widetilde{\theta}), y_n-\zeta(n)\vec{v}^{\perp}(\widetilde{\theta})], h'(n))\,.$$

For all n sufficiently large, thanks to the condition (6.26), $\mathrm{cyl}'(n) \subset \mathrm{cyl}(nA,h(n))$, and so we only consider such large n. We want to compare $\phi_n^{\kappa_n}$ and $\tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))$. If we consider a cutset in $\mathrm{cyl}'(n)$, we have to add edges near x_n and y_n to obtain a cutset in $\mathrm{cyl}(nA,h(n))$ of boundary conditions κ_n. So we define

$$\mathcal{E}(n,\kappa_n) \,=\, \mathcal{V}\left([x_n,x_n+\zeta(n)\vec{v}^{\perp}(\widetilde{\theta})]\cup[y_n-\zeta(n)\vec{v}^{\perp}(\widetilde{\theta}),y_n],\zeta\right)\bigcap \mathrm{cyl}(nA,h(n))\,,$$

where ζ is a fixed constant bigger than 4, and we denote by $E(n,\kappa_n)$ the set of the edges included in $\mathcal{E}(n,\kappa_n)$. If $E(\mathrm{cyl}'(n))$ is a cutset in $\mathrm{cyl}'(n)$ of fixed boundary condition $(1/2,\widetilde{\theta})$, then $E(\mathrm{cyl}'(n))\cup E(n,\kappa_n)$ is a cutset in $\mathrm{cyl}(nA,h(n))$ of boundary condition κ_n. We obtain:

$$\phi_n^{\kappa_n} \,\leq\, \tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))+V(E(n,\kappa_n))\,, \tag{6.27}$$

and so,

$$\forall \widetilde{\theta}\in\mathcal{D}_n \qquad \phi_n \,\leq\, \phi_n^{\kappa_n} \,\leq\, \tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))+V(E(n,\kappa_n))\,. \tag{6.28}$$

If we denote by $L(n,\widetilde{\theta})$ the distance between x_n and y_n, we have:

$$L(n,\widetilde{\theta}) \,=\, \frac{nl(A)}{\cos(\widetilde{\theta}-\theta)}\,.$$

Moreover, there exists a constant C_5 such that:

$$\mathrm{card}(E(n,\kappa_n)) \,\leq\, C_5\zeta(n)\,,$$

and since the set of edges $E(n,\kappa_n)$ is deterministic,

$$\mathbb{E}[E(n,\kappa_n)] \le C_5\zeta(n)\mathbb{E}(t)\,.$$

So

$$\forall\widetilde{\theta}\in\underline{\mathcal{D}}\qquad \frac{\mathbb{E}(\phi_n)}{nl(A)} \le \frac{L(n,\widetilde{\theta})-2\zeta(n)}{nl(A)}\times\frac{\mathbb{E}[\tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))]}{L(n,\widetilde{\theta})-2\zeta(n)}+\frac{C_5\mathbb{E}(t)\zeta(n)}{nl(A)}\,.$$

According to Theorem 18, the right-hand side of the previous equation converges towards the constant $\nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$ when n goes to infinity, and thus:

$$\limsup_{n\to\infty}\frac{\mathbb{E}(\phi_n)}{nl(A)} \le \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,. \tag{6.29}$$

Notice now that the same arguments work for $\tilde{\theta}\in\overline{\mathcal{D}}=\limsup_{n\to\infty}\mathcal{D}_n$ as soon as we restrict ourselves to values of n such that $\tilde{\theta}\in\mathcal{D}_n$. More precisely, for all $\widetilde{\theta}\in\overline{\mathcal{D}}$, for all large n such that $\widetilde{\theta}\in\mathcal{D}_n$, we have

$$\frac{\mathbb{E}(\phi_n)}{nl(A)} \le \frac{L(n,\widetilde{\theta})-2\zeta(n)}{nl(A)}\times\frac{\mathbb{E}[\tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))]}{L(n,\widetilde{\theta})-2\zeta(n)}+\frac{C_5\mathbb{E}(t)\zeta(n)}{nl(A)}\,.$$

Thus,

$$\liminf_{n\to\infty}\frac{\mathbb{E}(\phi_n)}{nl(A)} \le \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,. \tag{6.30}$$

3.4. Lower bound. Here what we would like to do is the symmetric construction of the one done in the previous section: we would like to consider a large cylinder tilted in the direction $\widetilde{\theta}$, and to put $\mathrm{cyl}(nA,h(n))$ inside. The point is that we cannot replace $h(n)$ by a smaller function, as we did in the study of the upper bound. This is the reason why we will consider numerous translated of $\mathrm{cyl}(nA,h(n))$ inside our big cylinder. Once again, Figure 3 is more explicit than the following definitions. We consider n and N in $\mathbb{N}$ and take N a lot bigger than n. We choose functions $\zeta',h'':\mathbb{N}\to\mathbb{R}^+$ such that

$$\lim_{n\to\infty}\zeta'(n) = \lim_{n\to\infty}h''(n) = +\infty\,,$$

and

$$\lim_{n\to\infty}\frac{h(n)}{\zeta'(n)} = 0\,. \tag{6.31}$$

Keeping the same notations as in section 3.3, we define

$$\mathrm{cyl}''(N) = \mathrm{cyl}\left([0,N\vec{v}^{\perp}(\widetilde{\theta})],h''(N)\right)\,.$$

We consider $\kappa=(k,\widetilde{\theta})\in D_n$. We will translate $\mathrm{cyl}(nA,h(n))$ numerous times in $\mathrm{cyl}''(N)$. The condition κ defines two points x_n and y_n on the boundary of $\mathrm{cyl}(nA,h(n))$ (see section 3.3). As in section 3.3, we denote by $L(n,\widetilde{\theta})$ the distance between x_n and y_n, and we have

$$L(n,\widetilde{\theta}) = \frac{nl(A)}{cos(\widetilde{\theta}-\theta)}\,.$$

We define

$$z_i = \left(\zeta'(n)+(i-1)L(n,\widetilde{\theta})\right)\vec{v}^{\perp}(\widetilde{\theta})\,,$$

for $i=1,\dots,\mathcal{N}$, where

$$\mathcal{N} = \left\lfloor\frac{N-2\zeta'(n)}{L(n,\widetilde{\theta})}\right\rfloor\,.$$

Of course we consider only N large enough to have $\mathcal{N}\ge 2$. For $i=1,\dots,\mathcal{N}$, we denote by $\widetilde{B_i}$ the image of $\mathrm{cyl}(nA,h(n))$ by the translation of vector $\overrightarrow{x_nz_i}$. For N sufficiently large, thanks to

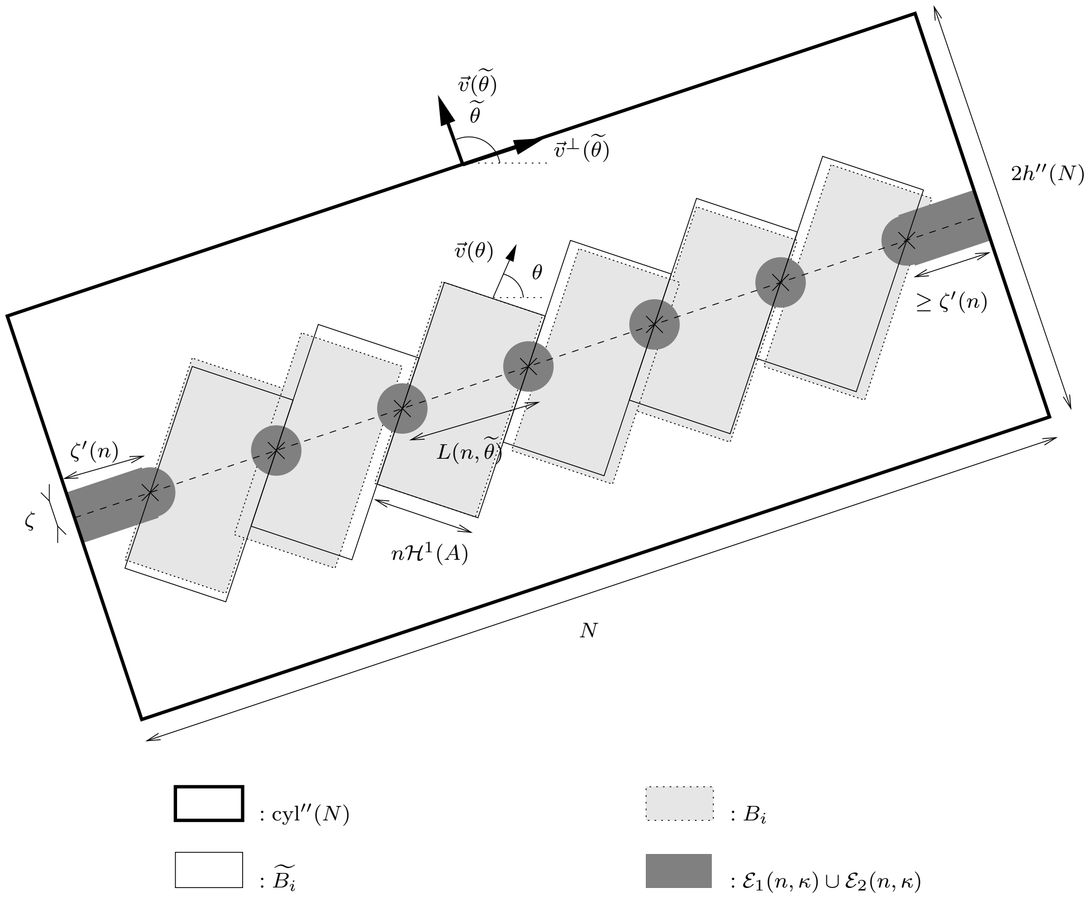

FIGURE 3. The cylinders $\mathrm{cyl}''(N)$ and B_i, for $i = 1, ..., \mathcal{N}$.

condition (6.31), we know that $\widetilde{B_i} \subset \mathrm{cyl}''(N)$ for all i. We can translate $\widetilde{B_i}$ again by a vector of norm strictly smaller than 1 to obtain an integer translate of $\mathrm{cyl}(nA, h(n))$ (i.e., a translate by a vector whose coordinates are in $\mathbb{Z}^2$) that we will call B_i. Now we want to glue together cutsets of boundary condition κ in the different B_i. We define:

$$\mathcal{E}_1(n,\kappa) \,=\, \left(\bigcup_{i=1}^{\mathcal{N}} \mathcal{V}(z_i,\zeta) \right) \bigcap \mathrm{cyl}''(N)\,,$$

where ζ is still a fixed constant bigger than 4, and:

$$\mathcal{E}_2(n,\kappa) \,=\, \mathcal{V}\left([0, \zeta'(n)\vec{v}^{\perp}(\widetilde{\theta})] \cup [z_{\mathcal{N}}, N\vec{v}^{\perp}(\widetilde{\theta})], \zeta \right) \bigcap \mathrm{cyl}''(N)\,.$$

Let $E_1(n,\kappa)$ (respectively $E_2(n,\kappa)$) be the set of the edges included in $\mathcal{E}_1(n,\kappa)$ (respectively $\mathcal{E}_2(n,\kappa)$). Then, still by gluing cutsets together, we obtain:

$$\tau(\mathrm{cyl}''(N), \vec{v}(\widetilde{\theta})) \,\leq\, \sum_{i=1}^{\mathcal{N}} \phi^{\kappa}(B_i, \vec{v}(\theta)) + V(E_1(n,\kappa) \cup E_2(n,\kappa))\,. \tag{6.32}$$

On one hand, there exists a constant C_6 (independent of κ) such that:

$$\mathrm{card}(E_1(n,\kappa) \cup E_2(n,\kappa)) \,\leq\, C_6 \left(\mathcal{N} + \zeta'(n) + L(n,\widetilde{\theta}) \right)\,,$$

and since the sets $E_1(n,\kappa)$ and $E_2(n,\kappa)$ are deterministic, we deduce:

$$\mathbb{E}[V(E_1(n,\kappa) \cup E_2(n,\kappa))] \,\leq\, C_6 \mathbb{E}(t) \left(\mathcal{N} + \zeta'(n) + L(n,\widetilde{\theta}) \right)\,.$$

On the other hand, the variables $(\phi^\kappa(B_i))_{i=1,\dots,\mathcal{N}}$ are identically distributed, with the same law as ϕ_n^κ (because we only consider integer translates), so (6.32) leads to

$$\mathbb{E}[\tau(\mathrm{cyl}''(N),\vec{v}(\widetilde{\theta}))] \;\leq\; \mathcal{N}\mathbb{E}[\phi_n^\kappa] + C_6\mathbb{E}(t)\left(\mathcal{N}+\zeta'(n)+L(n,\widetilde{\theta})\right)\,.$$

Dividing by N and sending N to infinity, we get, thanks to Theorem 18:

$$\nu_{\widetilde{\theta}} \;\leq\; \frac{\mathbb{E}[\phi_n^\kappa]}{L(n,\widetilde{\theta})} + \frac{C_6\mathbb{E}(t)}{L(n,\widetilde{\theta})}\,,$$

and so:

$$\frac{\mathbb{E}[\phi_n^\kappa]}{nl(A)} \;\geq\; \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} - \frac{C_6\mathbb{E}(t)}{nl(A)}\,.$$

Since C_6 is independent of κ,

$$\inf_{\kappa\in D_n}\frac{\mathbb{E}[\phi_n^\kappa]}{nl(A)} \;\geq\; \inf_{\widetilde{\theta}\in\mathcal{D}_n}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} - \frac{C_6\mathbb{E}(t)}{nl(A)}\,.$$

First, we affirm:

$$\liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \;\geq\; \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,, \tag{6.33}$$

and thus:

$$\liminf_{n\to\infty}\inf_{\kappa\in D_n}\frac{\mathbb{E}[\phi_n^\kappa]}{nl(A)} \;\geq\; \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,. \tag{6.34}$$

We also claim that:

$$\limsup_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \;\geq\; \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,, \tag{6.35}$$

and therefore:

$$\limsup_{n\to\infty}\inf_{\kappa\in D_n}\frac{\mathbb{E}[\phi_n^\kappa]}{nl(A)} \;\geq\; \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}}\frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,. \tag{6.36}$$

Let us prove inequality (6.33). In fact, we will state a more general result, that will be useful for us later:

LEMMA 9. *Let $\theta\in[0,\pi[$, and f be a lower semi-continuous function from $[\theta-\pi/2,\theta+\pi/2]$ to $\mathbb{R}^+\cup\{+\infty\}$. Then we have*

$$\liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}) \;\geq\; \inf_{\widetilde{\theta}\in ad(\overline{\mathcal{D}})} f(\widetilde{\theta})\,,$$

where $\mathrm{ad}(\overline{\mathcal{D}})$ is the adherence of $\overline{\mathcal{D}}$.

Proof :
We consider a positive ε. For all n, since f is lower semi-continuous and $\mathcal{D}_n$ is compact, there exists $\widetilde{\theta}_n\in\mathcal{D}_n$ such that $f(\widetilde{\theta}_n)=\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta})$. Up to extracting a subsequence, we can suppose that the sequence $(\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}))_{n\geq 0}$ converges towards $\liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta})$, and so:

$$\lim_{n\to\infty} f(\widetilde{\theta}_n) \;=\; \liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta})\,.$$

The sequence $(\widetilde{\theta}_n)_{n\geq 0}$ (in fact the previous subsequence) takes values in the compact $[\theta-\pi/2,\theta+\pi/2]$, so up to extracting a second subsequence we can suppose that $(\widetilde{\theta}_n)_{n\geq 0}$ converges towards a limit $\widetilde{\theta}_\infty$ in this compact. Since f is lower semi-continuous,

$$f(\widetilde{\theta}_\infty) \;\leq\; \lim_{n\to\infty} f(\widetilde{\theta}_n) \;=\; \liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta})\,,$$

and we just have to prove that $\widetilde{\theta}_\infty$ belongs to $\mathrm{ad}(\overline{\mathcal{D}})$. Indeed, for all positive ε, $\widetilde{\theta}_n \in [\widetilde{\theta}_\infty-\varepsilon, \widetilde{\theta}_\infty+\varepsilon]$ for an infinite number of n. We remember that all the $\mathcal{D}_n$ are closed intervals centered at θ. If $\widetilde{\theta}_\infty = \theta$, the result is obvious. We suppose that $\widetilde{\theta}_\infty > \theta$ for example, and thus, for ε small enough, $\widetilde{\theta}_\infty - \varepsilon > \theta$. Then $[\theta, \widetilde{\theta}_\infty - \varepsilon]$ is included in an infinite number of $\mathcal{D}_n$, so $\widetilde{\theta}_\infty - \varepsilon$ belongs to $\overline{\mathcal{D}}$, and then $\widetilde{\theta}_\infty$ belongs to $\mathrm{ad}(\overline{\mathcal{D}})$. The same holds if $\widetilde{\theta}_\infty < \theta$. This ends the proof of Lemma 9.

■

We use Lemma 9 with $f(\widetilde{\theta}) = \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$. Here f is lower semi-continuous, because $\widetilde{\theta} \to \nu_{\widetilde{\theta}}$ is continuous (it satisfies the weak triangle inequality, see section 4.5 of the Chapter 5 of the thesis). Moreover we know that f is finite and continuous on $]\theta-\pi/2, \theta+\pi/2[$, infinite at $\theta+\pi/2$ and $\theta-\pi/2$ and

$$\lim_{\widetilde{\theta}\to\theta+\pi/2} f(\widetilde{\theta}) = \lim_{\widetilde{\theta}\to\theta-\pi/2} f(\widetilde{\theta}) = +\infty\,,$$

so we can even say in this case:

$$\inf_{\widetilde{\theta}\in\mathrm{ad}(\overline{\mathcal{D}})} f(\widetilde{\theta}) = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} f(\widetilde{\theta})\,,$$

and we obtain inequality (6.33).

Let us now prove inequality (6.35). We state again a more general result:

LEMMA 10. *Let $\theta \in [0,\pi[$, and f be a lower semi-continuous function from $[\theta-\pi/2, \theta+\pi/2]$ to $\mathbb{R}^+\cup\{+\infty\}$. Then we have*

$$\limsup_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}) \geq \inf_{\widetilde{\theta}\in\mathrm{ad}(\underline{\mathcal{D}})} f(\widetilde{\theta})\,,$$

where $\mathrm{ad}(\underline{\mathcal{D}})$ is the adherence of $\underline{\mathcal{D}}$.

Proof :
We denote $\mathrm{ad}(\underline{\mathcal{D}})$ by $[\theta-\alpha, \theta+\alpha]$. For all integer $p\geq 1$, there exists $n_p \geq n_{p-1}$ ($n_0 = 1$) such that:

$$\theta+\alpha+1/p \notin \mathcal{D}_{n_p} \qquad \text{and} \qquad \theta-\alpha-1/p \notin \mathcal{D}_{n_p}\,,$$

thus

$$\mathcal{D}_{n_p} \subset]\theta-\alpha-1/p, \theta+\alpha+1/p[\,,$$

then

$$\begin{aligned}\limsup_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}) &\geq \limsup_{p\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_{n_p}} f(\widetilde{\theta}) \\ &\geq \limsup_{p\to\infty} \inf_{\widetilde{\theta}\in[\theta-\alpha-1/p,\theta+\alpha+1/p]} f(\widetilde{\theta})\,.\end{aligned}$$

The function f is l.s.c. and $[\theta-\alpha-1/p, \theta+\alpha+1/p]$ is compact, so for all integers p there exists $\widetilde{\theta}_p \in [\theta-\alpha-1/p, \theta+\alpha+1/p]$ such that $f(\widetilde{\theta}_p) = \inf_{\widetilde{\theta}\in[\theta-\alpha-1/p,\theta+\alpha+1/p]} f(\widetilde{\theta})$. Up to extraction, we can suppose that $(\widetilde{\theta}_p)_{p\geq1}$ converges towards a limit $\widetilde{\theta}_\infty$, that belongs obviously to $[\theta-\alpha, \theta+\alpha]$. Finally, because f is l.s.c.,

$$\inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} f(\widetilde{\theta}) \leq f(\widetilde{\theta}_\infty) \leq \limsup_{p\to\infty} f(\widetilde{\theta}_p) \leq \limsup_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta})\,,$$

so Lemma 10 is proved.

■

As previously, we use Lemma 10 with $f(\widetilde{\theta}) = \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$. Again, we have:

$$\inf_{\widetilde{\theta}\in\mathrm{ad}(\underline{\mathcal{D}})} f(\widetilde{\theta}) = \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} f(\widetilde{\theta})\,,$$

and equation (6.35) is proved.

3.5. End of the proof. First, we show that $\mathbb{E}(\phi_n)$ and $\min_\kappa \mathbb{E}(\phi_n^\kappa)$ are of the same order.

LEMMA 11. *Let A be a line segment in $\mathbb{R}^2$. Suppose that conditions (6.5) and (6.6) are satisfied. Then, there is a sequence of real numbers $(\beta_n)_{n\geq 1}$ which goes to 1 as n goes to infinity and such that:*

$$\mathbb{E}(\phi_n) \geq \beta_n \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa) \ .$$

Proof :
Recall from (6.3) and Lemma 8 that there is a finite subset $\tilde{D}_n$ of D_n, such that:

$$\mathrm{card}(\mathcal{D}_n) \leq C_4 h(n)^2 \ ,$$

for some constant C_4 and every n, and

$$\phi_n = \min_{\kappa\in\tilde{D}_n} \phi_n^\kappa \ . \tag{6.37}$$

Thus, letting u be a positive real number,

$$\begin{aligned}
\mathbb{P}(\min_{\kappa\in\tilde{D}_n} \phi_n^\kappa \geq \min_{\kappa\in\tilde{D}_n} \mathbb{E}(\phi_n^\kappa) - u) &= 1 - \mathbb{P}(\exists\kappa\in\tilde{D}_n,\ \phi_n^\kappa < \min_{\kappa\in\tilde{D}_n} \mathbb{E}(\phi_n^\kappa) - u) \ , \\
&\geq 1 - |\tilde{D}_n| \max_{\kappa\in\tilde{D}_n} \mathbb{P}(\phi_n^\kappa < \min_{\kappa\in\tilde{D}_n} \mathbb{E}(\phi_n^\kappa) - u) \ , \\
&\geq 1 - C_4 h(n)^2 \max_{\kappa\in\tilde{D}_n} \mathbb{P}(\phi_n^\kappa < \mathbb{E}(\phi_n^\kappa) - u) \ .
\end{aligned}$$

Now, Proposition 3.2 implies that for η in $]0,1[$,

$$\mathbb{P}(\min_{\kappa\in D_n} \phi_n^\kappa \geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)(1-\eta)) \geq 1 - 2C_4K_1h(n)^2 e^{-K_2\min\{\eta,(1-\eta)\}\min_{\kappa\in D_n}\mathbb{E}(\phi_n^\kappa)} \ .$$

Now, let η_n be a positive sequence in $]0,1/2[$, to be chosen later.

$$\begin{aligned}
\mathbb{E}(\min_{\kappa\in D_n} \phi_n^\kappa) &= \int_0^{+\infty} \mathbb{P}(\min_{\kappa\in D_n} \phi_n^\kappa \geq t)\, dt \ , \\
&\geq \int_0^{\min_{\kappa\in D_n}\mathbb{E}(\phi_n^\kappa)} \mathbb{P}\left(\min_{\kappa\in D_n} \phi_n^\kappa \geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa) - u\right) du \ , \\
&\geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa) \int_{\eta_n}^{(1-\eta_n)} \mathbb{P}\left(\min_{\kappa\in D_n} \phi_n^\kappa \geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)(1-\eta)\right) d\eta \ , \\
&\geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)(1-2\eta_n)\left(1 - 2C_4K_1h(n)^2 e^{-K_2\eta_n\min_{\kappa\in D_n}\mathbb{E}(\phi_n^\kappa)}\right) \ .
\end{aligned}$$

Thanks to inequality (6.34), we know that there is a strictly positive constant $C(A)$ such that:

$$\liminf_{n\to\infty} \frac{\min_{\kappa\in\tilde{D}_n}\mathbb{E}(\phi_n^\kappa)}{n} \geq C(A) \ .$$

Thus, choose η_n so that:

$$\eta_n \xrightarrow[n\to\infty]{} 0 \ ,$$

and:

$$\frac{n\eta_n}{\log(h(n))} \xrightarrow[n\to\infty]{} \infty \ .$$

This is always possible since by assumption (6.5), $\log h(n)$ is small compared to n. Then, if we define:

$$\beta_n := (1-2\eta_n)\left(1 - 2C_4K_1h(n)^2 e^{-K_2\eta_n\min_{\kappa\in D_n}\mathbb{E}(\phi_n^\kappa)}\right) \ ,$$

the sequence $(\beta_n)_{n\in\mathbb{N}}$ goes to 1 as n goes to infinity. This finishes the proof of Lemma 11.

■

Theorem 20 follows from Lemma 11 and inequalities (6.29), (6.34), (6.30) and (6.36) (using also Borel-Cantelli's Lemma, as already noted, through Proposition 3.2).

Obviously, the condition

$$\inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} := \eta_{\theta,h}\,, \tag{6.38}$$

necessary and sufficient for the convergence a.s. of $(\phi_n/(nl(A)))_{n\geq 0}$, is closely linked to the asymptotic behaviour of $h(n)/n$. Indeed we know that

$$\mathcal{D}_n = [\theta-\alpha_n, \theta+\alpha_n]\,,$$

where $\alpha_n = \arctan\left(\frac{2h(n)}{nl(A)}\right)$. If $\lim_{n\to\infty} 2h(n)/(nl(A))$ exists in $\mathbb{R}^+\cup\{+\infty\}$, and we denote it by $\tan\alpha$ ($\alpha\in[0,\pi/2]$), then $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ are equal to $[\theta-\alpha,\theta+\alpha]$ or $]\theta-\alpha,\theta+\alpha[$, and we obtain that $\eta_{\theta,h}$ exists and

$$\eta_{\theta,h} = \inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,.$$

As previously, we do not care keeping $\theta+\alpha$ and $\theta-\alpha$ in the infimum. We then obtain the corollary 2.1. Obviously, if there exists a $\widetilde{\theta}_0$ such that

$$\frac{\nu_{\widetilde{\theta}_0}}{\cos(\widetilde{\theta}_0-\theta)} = \inf_{\widetilde{\theta}\in[\theta-\pi/2,\theta+\pi/2]} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}$$

and if

$$\liminf_{n\to\infty} \frac{2h(n)}{nl(A)} \geq |\tan(\widetilde{\theta}_0-\theta)|\,,$$

then $\eta_{\theta,h}$ also exists (and equals $\nu_{\widetilde{\theta}_0}/\cos(\widetilde{\theta}_0-\theta)$) and is the limit of $(\phi_n/(nl(A)))_{n\in\mathbb{N}}$ almost surely, even if $\lim_{n\to\infty} h(n)/n$ does not exist.

REMARK 27. In dimension $d\geq 3$, if we denote by $\vec{v}$ a unit vector orthogonal to a non-degenerate hyperrectangle A and by $\overrightarrow{\mathcal{D}_n(A)}$ the set of all admissible directions for the cylinder $\mathrm{cyl}(nA,h(n))$, i.e., the set of the vectors $\vec{v}'$ in S^{d-1} such that there exists a hyperplane $\mathcal{P}$ orthogonal to $\vec{v}'$ that intersects $\mathrm{cyl}(nA,h(n))$ only on its "vertical faces", and if $\lim_{n\to\infty} h(n)/n$ exists (thus $\overrightarrow{\mathcal{D}(A)} = \mathrm{ad}(\underline{\overrightarrow{\mathcal{D}(A)}}) = \mathrm{ad}(\overline{\overrightarrow{\mathcal{D}(A)}})$ exists), we conjecture that

$$\lim_{n\to\infty} \frac{\phi(nA,h(n))}{n^{d-1}\mathcal{H}^{d-1}(A)} = \inf_{\vec{v}'\in\overrightarrow{\mathcal{D}(A)}} \frac{\nu(\vec{v}')}{|\vec{v}\cdot\vec{v}'|} \qquad \text{a.s.}\,,$$

under assumptions (6.6) on F and if $h(n)$ goes to infinity with n in such a way that we have $\lim_{n\to\infty}\log h(n)/n^{d-1} = 0$. We could not prove this conjecture, because we are not able to prove that $\phi(nA,h(n))$ behaves asymptotically like $\min_{\kappa\in K}\phi^{\kappa}(nA,h(n))$, where K is the set of the flat boundary conditions, i.e., the boundary conditions given by the intersection of a hyperplane with the vertical faces of $\mathrm{cyl}(nA,h(n))$.

4. Lower large deviation principle

In this section, we shall prove Theorem 21. After a technical lemma about the function $\widetilde{\theta}\mapsto\mathcal{I}_{\widetilde{\theta}}(\lambda)$ in section 4.1.1, we shall show in section 4.1.2 the lower bounds:

$$\liminf_{n\to\infty} \frac{1}{nl(A)} \log\mathbb{P}[\phi_n\leq\lambda nl(A)] \geq -\inf_{\widetilde{\theta}\in\mathrm{ad}(\underline{\mathcal{D}})} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}\left(\lambda\cos(\widetilde{\theta}-\theta)^-\right)\,, \tag{6.39}$$

and

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log\mathbb{P}[\phi_n\leq\lambda nl(A)] \geq -\inf_{\widetilde{\theta}\in\mathrm{ad}(\overline{\mathcal{D}})} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}\left(\lambda\cos(\widetilde{\theta}-\theta)^-\right)\,. \tag{6.40}$$

In section 4.1.3, we shall prove the upper bounds:

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}[\phi_n \leq \lambda nl(A)] \,\leq\, - \inf_{\widetilde{\theta}\in\mathrm{ad}(\overline{\mathcal{D}})} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}\left(\lambda \cos(\widetilde{\theta}-\theta)^+\right) \,, \tag{6.41}$$

and

$$\liminf_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}[\phi_n \leq \lambda nl(A)] \,\leq\, - \inf_{\widetilde{\theta}\in\mathrm{ad}(\underline{\mathcal{D}})} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}\left(\lambda \cos(\widetilde{\theta}-\theta)^+\right) \,. \tag{6.42}$$

These inequalities are the building blocks of the full large deviation principle from below, proved in section 4.2.

4.1. Lower large deviations. From now on, we suppose that conditions (6.4) on F and (6.5) on h are satisfied (so the functions $\mathcal{I}_{\widetilde{\theta}}$ are well defined).

4.1.1. *Technical lemma.* We state here a property which comes from the weak triangle inequality for ν (see section 4.5 in the Chapter 5 of the thesis):

LEMMA 12. *Let (abc) be a non degenerate triangle in $\mathbb{R}^2$ and let v_a, v_b, v_c be the exterior normal unit vectors to the sides $[bc]$, $[ac]$, $[bc]$. We denote by $(\cos\widetilde{\theta}_i, \sin\widetilde{\theta}_i)$ the coordinates of v_i, and by $l(ij)$ the length of the side $[i,j]$ for i, j in $\{a,b,c\}$. If the angles $\widehat{cab}$ and $\widehat{abc}$ have values strictly smaller than $\pi/2$, then for all $\lambda \geq 0$, for all $\alpha\in[0,1]$, we have*

$$l(ab)\mathcal{I}_{\widetilde{\theta}_c}\left(\frac{\lambda}{l(ab)}^+\right) \,\leq\, l(ac)\mathcal{I}_{\widetilde{\theta}_b}\left(\alpha\frac{\lambda}{l(ac)}^+\right) + l(bc)\mathcal{I}_{\widetilde{\theta}_a}\left((1-\alpha)\frac{\lambda}{l(bc)}^+\right) \,.$$

Proof :
This proof follows the one of proposition 11.6 in [**19**]. We consider the cylinder

$$\mathrm{cyl}_c(n) \,=\, \mathrm{cyl}(n[ab], n)$$

of dimensions $nl(ab)\times 2n$ oriented towards the direction $\widetilde{\theta}_c$, and we define $\tau_c(n) = \tau(\mathrm{cyl}_c(n))$ (implicitly, for the direction defined by $\widetilde{\theta}_c$). Exactly as in section 3.3, we choose two functions $\zeta, h' : \mathbb{N}\to\mathbb{R}^+$ such that

$$\lim_{n\to\infty} h'(n) \,=\, \lim_{n\to\infty} \zeta(n) \,=\, +\infty \,,$$

and

$$\lim_{n\to\infty} \frac{\zeta(n)}{n} \,=\, 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{h'(n)}{\zeta(n)} \,=\, 0 \,.$$

We construct smaller cylinders oriented towards the directions $\widetilde{\theta}_b$ and $\widetilde{\theta}_a$ inside $\mathrm{cyl}_c(n)$ (see figure 4). We define

$$\mathrm{cyl}_b(n) \,=\, \mathrm{cyl}\left([na+\zeta(n)(\sin\widetilde{\theta}_b, -\cos\widetilde{\theta}_b), nc-\zeta(n)(\sin\widetilde{\theta}_b, -\cos\widetilde{\theta}_b)], h'(n)\right) \,,$$

$$\mathrm{cyl}_a(n) \,=\, \mathrm{cyl}\left([nb+\zeta(n)(\sin\widetilde{\theta}_a, -\cos\widetilde{\theta}_a), nc-\zeta(n)(\sin\widetilde{\theta}_a, -\cos\widetilde{\theta}_a)], h'(n)\right) \,,$$

respectively oriented towards the direction $\widetilde{\theta}_b$ and $\widetilde{\theta}_a$, and $\tau_b(n) = \tau(\mathrm{cyl}_b(n))$ and $\tau_a(n) = \tau(\mathrm{cyl}_a(n))$. The dimensions of $\mathrm{cyl}_b(n)$ (respectively $\mathrm{cyl}_a(n)$) are $(nl(ac)-2\zeta(n))\times 2h'(n)$ (respectively $(nl(bc)-2\zeta(n))\times 2h'(n)$), and for n large enough $\mathrm{cyl}_b(n)$ and $\mathrm{cyl}_a(n)$ are included in $\mathrm{cyl}_c(n)$ (we only consider such large n), because $\widehat{cab}$ and $\widehat{abc}$ are strictly smaller than $\pi/2$. To glue together a cutset in $\mathrm{cyl}_b(n)$ and a cutset in $\mathrm{cyl}_a(n)$ to obtain a cutset in $\mathrm{cyl}_c(n)$ we have to add some edges. We finally define, for a constant $\zeta\geq 4$,

$$\mathcal{E}_3(n,a,b,c) = \mathcal{V}\left(\begin{array}{c} [na, na+\zeta(n)(\sin\widetilde{\theta}_b, -\cos\widetilde{\theta}_b)] \cup [nc-\zeta(n)(\sin\widetilde{\theta}_b, -\cos\widetilde{\theta}_b), nc] \\ \cup [nc, nc-\zeta(n)(\sin\widetilde{\theta}_a, -\cos\widetilde{\theta}_a)] \cup [nb+\zeta(n)(\sin\widetilde{\theta}_a, -\cos\widetilde{\theta}_a), nb] \end{array}, \zeta\right),$$

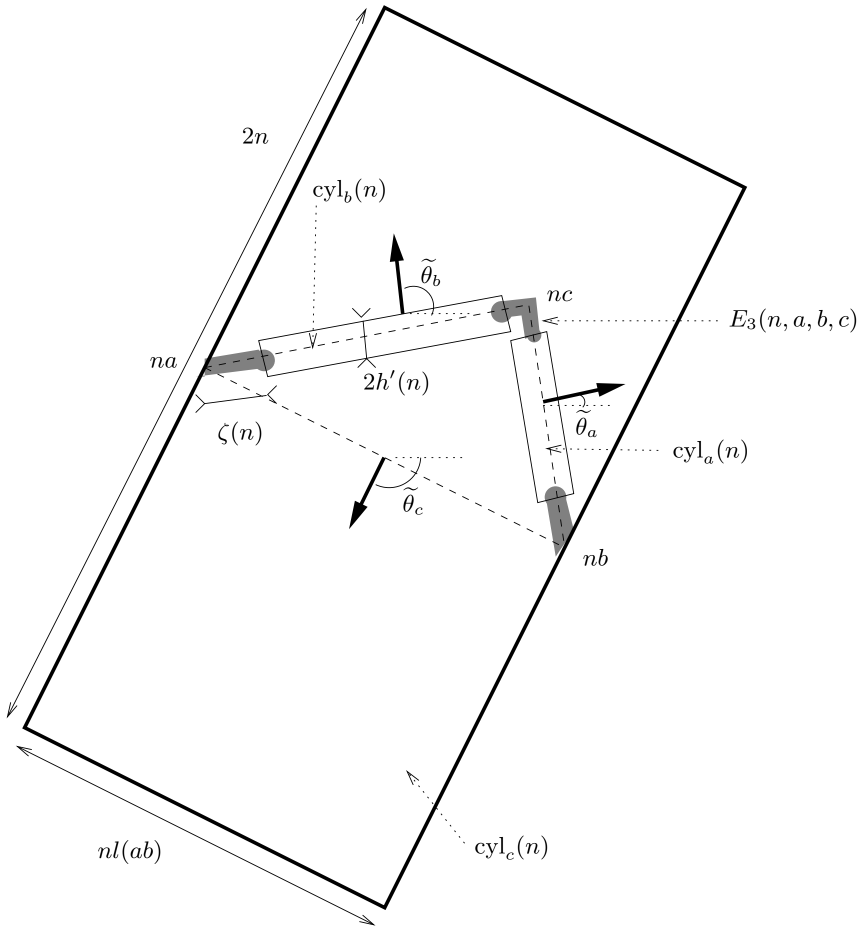

FIGURE 4. The cylinders $\mathrm{cyl}_c(n)$, $\mathrm{cyl}_b(n)$ and $\mathrm{cyl}_a(n)$.

and we denote by $E_3(n,a,b,c)$ the set of the edges included in $\mathcal{E}_3(n,a,b,c)$. There exists a constant C_7 such that

$$\mathrm{card}(E_3(n,a,b,c)) \,\leq\, C_7\zeta(n)\,.$$

Obviously (see figure 4), we have

$$\tau_c(n) \,\leq\, \tau_b(n) + \tau_a(n) + V(E_3(n,a,b,c))\,. \tag{6.43}$$

Then for all $\lambda \geq 0$, for all positive η, for all large n, for all $\alpha \in [0,1]$, by the FKG inequality we have

$$\begin{aligned}
\mathbb{P}\Bigg[\frac{\tau_c(n)}{nl(ab)} \leq & \lambda + 3\eta - \frac{1}{\sqrt{nl(ab)}}\Bigg] \\
&\geq \mathbb{P}\Bigg[\frac{\tau_c(n)}{nl(ab)} \leq \lambda + 2\eta\Bigg] \\
&\geq \mathbb{P}\left[\tau_b(n) \leq \alpha(\lambda+\eta)nl(ab)\right] \times \mathbb{P}\left[\tau_a(n) \leq (1-\alpha)(\lambda+\eta)nl(ab)\right] \\
&\qquad \times \mathbb{P}\left[V(E_3(n,a,b,c)) \leq \eta nl(ab)\right] \\
&\geq \mathbb{P}\left[\tau_b(n) \leq \alpha(\lambda+\eta)nl(ab) - \frac{1}{\sqrt{nl(ac)}}\right]
\end{aligned}$$

$$\times\,\mathbb{P}\left[\tau_a(n) \le (1-\alpha)(\lambda+\eta)nl(ab) - \frac{1}{\sqrt{nl(bc)}}\right]$$
$$\times\,\mathbb{P}\left[t(e) \le \frac{\eta nl(ab)}{C_7\zeta(n)}\right]^{C_7\zeta(n)} .$$

We take the logarithm of the previous inequality, divide it by $-n$, and send n to infinity. Since $\zeta(n)/n$ converges to zero when n goes to infinity, we obtain that

$$(6.44)\quad l(ab)\mathcal{I}_{\widetilde{\theta}_c}(\lambda+3\eta) \le l(ac)\mathcal{I}_{\widetilde{\theta}_b}\left(\alpha(\lambda+\eta)\frac{l(ab)}{l(ac)}\right) + l(bc)\mathcal{I}_{\widetilde{\theta}_a}\left((1-\alpha)(\lambda+\eta)\frac{l(ab)}{l(bc)}\right) .$$

Sending η to zero, we obtain the desired inequality.

■

We state next a property of continuity:

LEMMA 13. *For all $\lambda \ge 0$, we define $g_\lambda : [\theta-\pi/2, \theta+\pi/2] \to \mathbb{R}^+ \cup \{+\infty\}$ by*

$$\forall \widetilde{\theta} \in [\theta-\pi/2,\theta+\pi/2], \qquad g_\lambda(\widetilde{\theta}) = \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+) .$$

Then g_λ is lower semi-continuous, and g_λ is continuous on

$$H_\lambda^> = \left\{\widetilde{\theta}\,|\,\lambda > \delta\frac{|\cos\widetilde{\theta}|+|\sin\widetilde{\theta}|}{\cos(\widetilde{\theta}-\theta)}\right\} .$$

REMARK 28. For all $\lambda \ge 0$, we have $g_\lambda(\theta+\pi/2) = g_\lambda(\theta-\pi/2) = +\infty$, because $\mathcal{I}_{\widetilde{\theta}}(0^+) > 0$ as soon as $\nu_{\widetilde{\theta}} > 0$, and it is the case for all $\widetilde{\theta}$ since $F(0) < 1/2$.

Proof :
The proof is based on the same ideas as the one of lemma 12, so we will use part of it. We consider two angles $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ such that $\widetilde{\theta}_1 - \widetilde{\theta}_2 = \hat{\varepsilon}$ (positive or negative) and $|\hat{\varepsilon}| = \varepsilon$ is small. Let (abc) be the right triangle such that, using the same notations as in the previous proof, $l(ab) = 1$, $\widetilde{\theta}_c = \widetilde{\theta}_1 + \pi$, $\widetilde{\theta}_b = \widetilde{\theta}_2$ and $\widetilde{\theta}_a = \widetilde{\theta}_2 - \pi/2$, and so $b\hat{a}c = \varepsilon$, $a\hat{c}b = \pi/2$ and $a\hat{b}c < \pi/2$. Obviously we are confronted with a particular case of triangle (abc) studied in lemma 12. We do exactly the same construction as in the previous proof, and we start again from equation (6.44). Here we have constructed (abc) such that $l(ab) = 1$, $l(ac) = \cos\varepsilon$ and $l(bc) = \sin\varepsilon$, and by invariance of the graph by a rotation of angle $\pi/2$, we know that the functions $\mathcal{I}_{\widetilde{\theta}_2}$ and $\mathcal{I}_{\widetilde{\theta}_2-\pi/2}$ (respectively $\mathcal{I}_{\widetilde{\theta}_1}$ and $\mathcal{I}_{\widetilde{\theta}_1+\pi}$) are equal. We can rewrite equation (6.44) the following way:

$$(6.45)\qquad \mathcal{I}_{\widetilde{\theta}_1}(\lambda+3\eta) \le (\cos\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left(\alpha\frac{\lambda+\eta}{\cos\varepsilon}\right) + (\sin\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left((1-\alpha)\frac{\lambda+\eta}{\sin\varepsilon}\right) .$$

We want to make appear the factor $\cos(\widetilde{\theta}_1-\theta)$, so for all $\lambda \ge 0$ and for all small η we deduce from (6.45) that for all ε small enough,

$$\begin{aligned}
&\mathcal{I}_{\widetilde{\theta}_1}(\lambda\cos(\widetilde{\theta}_1-\theta)+3\eta)\\
&\quad\le (\cos\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left(\alpha\frac{\lambda\cos(\widetilde{\theta}_1-\theta)+\eta}{\cos\varepsilon}\right) + (\sin\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left((1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)+\eta}{\sin\varepsilon}\right)\\
&\quad\le (\cos\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left(\alpha(\lambda\cos(\widetilde{\theta}_2-\theta)+\eta/2)\right) + (\sin\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left((1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)+\eta}{\sin\varepsilon}\right) .
\end{aligned}$$

If $\lambda > 0$ we choose $\alpha \in]\max(2/3, 1-\eta/(12\lambda)), 1[$ (remember that λ is fixed and we can choose η small in comparison with λ), then $\alpha(\lambda\cos(\widetilde{\theta}_2-\theta)+\eta/2) \ge \lambda\cos(\widetilde{\theta}_2-\theta)+\eta/4$. This equation is satisfied for all $1 > \alpha \ge 1/2$ if $\lambda = 0$. We stress here the fact that how large must be α depends on λ and η, but not on ε. With a such fixed big α, we obtain that

$$\begin{aligned}&\mathcal{I}_{\widetilde{\theta}_1}(\lambda\cos(\widetilde{\theta}_1-\theta)+3\eta)\\&\quad\le\ (\cos\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left(\lambda\cos(\widetilde{\theta}_2-\theta)+\eta/4\right)+(\sin\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left((1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)+\eta}{\sin\varepsilon}\right).\end{aligned}$$

We send $\widetilde{\theta}_2$ to $\widetilde{\theta}_1$, i.e. ε to zero by fixing $\widetilde{\theta}_1$. Since $(1-\alpha)(\lambda\cos(\widetilde{\theta}_1-\theta)+\eta)$ is fixed and positive, we know that for small ε we obtain

$$(1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)+\eta}{\sin\varepsilon}\ >\ \nu_{\max}\ =\ \max_{\theta\in[0,\pi]}\nu_\theta\,,$$

and so for all $\widetilde{\theta}$ we have

$$\mathcal{I}_{\widetilde{\theta}}\left((1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)+\eta}{\sin\varepsilon}\right)\ =\ 0\,.$$

We send finally η to zero and obtain

$$\mathcal{I}_{\widetilde{\theta}_1}(\lambda\cos(\widetilde{\theta}_1-\theta)^+)\ \le\ \liminf_{\eta\to0}\liminf_{\hat{\varepsilon}\to0}\mathcal{I}_{\widetilde{\theta}_1+\hat{\varepsilon}}\left(\lambda\cos(\widetilde{\theta}_1+\hat{\varepsilon}-\theta)+\eta/4\right)\,.$$

We know that the limit $\lim_{\eta\to0}\mathcal{I}_{\widetilde{\theta}_1+\hat{\varepsilon}}(\lambda\cos(\widetilde{\theta}_1+\hat{\varepsilon}-\theta)+\eta/4)$ is an increasing limit for all fixed $\hat{\varepsilon}$, so we get:

$$\mathcal{I}_{\widetilde{\theta}_1}(\lambda\cos(\widetilde{\theta}_1-\theta)^+)\ \le\ \liminf_{\hat{\varepsilon}\to0}\mathcal{I}_{\widetilde{\theta}_1+\hat{\varepsilon}}(\lambda\cos(\widetilde{\theta}_1+\hat{\varepsilon}-\theta)^+)\,. \tag{6.46}$$

We will now fix $\widetilde{\theta}_2$ and send $\widetilde{\theta}_1$ to $\widetilde{\theta}_2$. Starting again from (6.44), for all $\beta>0$, for all $\lambda>0$, for all $\widetilde{\theta}_2\in]\theta-\pi/2,\theta+\pi/2[$, for all η small enough and ε small (in particular such that $\widetilde{\theta}_1\in]\theta-\pi/2,\theta+\pi/2[$ too), we obtain

$$\begin{aligned}&\mathcal{I}_{\widetilde{\theta}_1}(\lambda\cos(\widetilde{\theta}_1-\theta)+\beta)\\&\quad\le\ \mathcal{I}_{\widetilde{\theta}_1}(\lambda\cos(\widetilde{\theta}_1-\theta))\\&\quad\le\ (\cos\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left(\alpha\frac{\lambda\cos(\widetilde{\theta}_1-\theta)-2\eta}{\cos\varepsilon}\right)+(\sin\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left((1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)-2\eta}{\sin\varepsilon}\right)\\&\quad\le\ (\cos\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left(\alpha(\lambda\cos(\widetilde{\theta}_2-\theta)-3\eta)\right)+(\sin\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left((1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)-2\eta}{\sin\varepsilon}\right).\end{aligned}$$

Exactly as previously, for $\lambda<1$ but sufficiently close to 1 (how close depending on λ and η but not on ε), we have

$$\begin{aligned}&\mathcal{I}_{\widetilde{\theta}_1}(\lambda\cos(\widetilde{\theta}_1-\theta)+\beta)\\&\quad\le\ (\cos\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left(\lambda\cos(\widetilde{\theta}_2-\varepsilon)-4\eta\right)+(\sin\varepsilon)\mathcal{I}_{\widetilde{\theta}_2}\left((1-\alpha)\frac{\lambda\cos(\widetilde{\theta}_1-\theta)-2\eta}{\sin\varepsilon}\right).\end{aligned}$$

We send first β to zero, then $\widetilde{\theta}_1$ to $\widetilde{\theta}_2$ (thus ε to zero), and finally η to zero to obtain as for (6.46) that

$$\mathcal{I}_{\widetilde{\theta}_2}(\lambda\cos(\widetilde{\theta}_2-\theta)^-)\ \ge\ \limsup_{\hat{\varepsilon}\to0}\mathcal{I}_{\widetilde{\theta}_2+\hat{\varepsilon}}(\lambda\cos(\widetilde{\theta}_2+\hat{\varepsilon}-\theta)^+)\,. \tag{6.47}$$

This inequality remains valid for $\lambda=0$ or $\cos(\widetilde{\theta}_2-\theta)=0$, since for convenience we decided that $\mathcal{I}_{\widetilde{\theta}_2}(0^-)=+\infty$. From (6.46) and (6.47), we conclude that for all $\lambda\ge0$:

$$\frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+)\ \le\ \liminf_{\hat{\varepsilon}\to0}g_\lambda(\widetilde{\theta}+\hat{\varepsilon})\ \le$$

$$\leq \limsup_{\hat{\varepsilon}\to 0} g_\lambda(\widetilde{\theta}+\hat{\varepsilon}) \leq \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^-)\,.$$

Lemma 13 follows, since we know that:

$$\forall\widetilde{\theta}\in H_\lambda^> \qquad \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+) = \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^-)\,.$$

■

4.1.2. *Lower bound.* We remember equation (6.28). Then for all $\widetilde{\theta}\in\mathcal{D}_n$, for all $\lambda>0$, for all positive small ε, by the FKG inequality,

$$\begin{aligned}
\mathbb{P}[\phi_n\leq\lambda l(A)n] &\geq \mathbb{P}[\{\tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))\leq(\lambda-\varepsilon)l(A)n\}\cap\{V(E(n,\kappa_n))\leq\varepsilon l(A)n\}]\\
&\geq \mathbb{P}\left[\frac{\tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))}{L(n,\widetilde{\theta})-2\zeta(n)}\leq\frac{(\lambda-\varepsilon)l(A)n}{L(n,\widetilde{\theta})-2\zeta(n)}\right]\times\mathbb{P}\left[\forall e\in E(n,\kappa_n)\,,\,t(e)\leq\frac{\varepsilon l(A)n}{C_5\zeta(n)}\right]\\
&\geq \mathbb{P}\left[\frac{\tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))}{L(n,\widetilde{\theta})-2\zeta(n)}\leq(\lambda-\varepsilon)\cos(\widetilde{\theta}-\theta)\right]\times\mathbb{P}\left[t(e)\leq\frac{\varepsilon l(A)n}{C_5\zeta(n)}\right]^{\mathrm{card}(E(n,\kappa_n))}\\
&\geq \mathbb{P}\left[\frac{\tau(\mathrm{cyl}'(n),\vec{v}(\widetilde{\theta}))}{L(n,\widetilde{\theta})-2\zeta(n)}\leq(\lambda-\varepsilon)\cos(\widetilde{\theta}-\theta)-\frac{1}{\sqrt{L(n,\widetilde{\theta})-2\zeta(n)}}\right]\\
&\qquad\times\mathbb{P}\left[t(e)\leq\frac{\varepsilon l(A)n}{C_5\zeta(n)}\right]^{C_5\zeta(n)}\,.
\end{aligned}$$

Thanks to the hypothesis $\lim_{n\to\infty}\zeta(n)/n=0$ and Theorem 19, for all $\widetilde{\theta}\in\mathcal{D}_n$, and $\lambda>\varepsilon>0$,

$$\liminf_{n\to\infty}\frac{1}{nl(A)}\log\mathbb{P}\left[\frac{\phi_n}{nl(A)}\leq\lambda\right] \geq \frac{-1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}\big((\lambda-\varepsilon)\cos(\widetilde{\theta}-\theta)\big)\,.$$

Sending ε to zero (remember that $\mathcal{I}_{\widetilde{\theta}}$ is làglàd) and taking the infimum in $\widetilde{\theta}$,

$$\liminf_{n\to\infty}\frac{1}{nl(A)}\log\mathbb{P}[\phi_n\leq\lambda nl(A)] \geq -\inf_{\widetilde{\theta}\in\underline{\mathcal{D}}}\frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}\left(\lambda\cos(\widetilde{\theta}-\theta)^-\right)\,, \tag{6.48}$$

and

$$\limsup_{n\to\infty}\frac{1}{nl(A)}\log\mathbb{P}[\phi_n\leq\lambda nl(A)] \geq -\inf_{\widetilde{\theta}\in\overline{\mathcal{D}}}\frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}\left(\lambda\cos(\widetilde{\theta}-\theta)^-\right)\,. \tag{6.49}$$

These inequalities remain valid for $\lambda=0$, since $\mathcal{I}_{\widetilde{\theta}}(0^-)=+\infty$, so equations (6.48) and (6.49) are satisfied for all $\lambda\geq 0$.

We will transform a little bit inequalities (6.48) and (6.49) to make it more useful for us in the proof of the large deviation principle below. Actually, let us prove that:

$$\inf_{\widetilde{\theta}\in\mathcal{D}}\frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}\left(\lambda\cos(\widetilde{\theta}-\theta)^-\right) = \inf_{\widetilde{\theta}\in\mathrm{ad}(\mathcal{D})}\frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}\left(\lambda\cos(\widetilde{\theta}-\theta)^-\right)\,, \tag{6.50}$$

where $\mathcal{D}$ is an interval of $[\theta-\pi/2,\theta+\pi/2]$ which is centered at θ and symmetric with respect to θ (representing $\overline{\mathcal{D}}$ or $\underline{\mathcal{D}}$ here). As we did previously, we define

$$H_\lambda^* = \left\{\widetilde{\theta}\,|\,\lambda*\delta\frac{|\cos\widetilde{\theta}|+|\sin\widetilde{\theta}|}{\cos(\widetilde{\theta}-\theta)}\right\}\,,$$

where $*$ represents $<$, $>$, $\leq$, $\geq$ or $=$, and for simplicity of notations we define also:

$$\widetilde{g}_\lambda(\widetilde{\theta}) = \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}\left(\lambda\cos(\widetilde{\theta}-\theta)^-\right)\,.$$

The function $\widetilde{g}_\lambda$ is infinite on $H_\lambda^{\leq}$, and finite, continuous and equal to g_λ on $H_\lambda^{>}$. If $\mathcal{D}$ is included in $H_\lambda^{\leq}$, then $\mathrm{ad}(\mathcal{D})$ too because $H_\lambda^{\leq}$ is closed, and then:

$$\inf_{\mathcal{D}} \widetilde{g}_\lambda \;=\; +\infty \;=\; \inf_{\mathrm{ad}(\mathcal{D})} \widetilde{g}_\lambda\,.$$

Otherwise, $\mathcal{D}\cap H_\lambda^{>}$ is non empty, so $\inf_{\mathcal{D}} \widetilde{g}_\lambda$ is finite. If $\mathrm{ad}(\mathcal{D})\neq\mathcal{D}$ (otherwise the result is obvious), then $\mathcal{D}$ is open since it is symmetric with respect to θ, and we denote by $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ the two points of $\mathrm{ad}(\mathcal{D})\smallsetminus\mathcal{D}$. Either $\widetilde{g}_\lambda$ is continuous at $\widetilde{\theta}_1$ (respectively $\widetilde{\theta}_2$), or $\widetilde{g}_\lambda(\widetilde{\theta}_1)$ (respectively $\widetilde{g}_\lambda(\widetilde{\theta}_2)$) is infinite, so

$$\inf_{\mathcal{D}} \widetilde{g}_\lambda \;=\; \inf_{\mathrm{ad}(\mathcal{D})} \widetilde{g}_\lambda\,,$$

and equation (6.50) is proved. Inequalities (6.48) and (6.49) are equivalent to (6.39) and (6.40).

4.1.3. *Upper bound.* We start again from equation (6.32). Then for all $\kappa\in D_n$, for all $\widetilde{\lambda}\geq\varepsilon>0$, for all large N, we have by the FKG inequality

$$\begin{aligned}
&\mathbb{P}\left[\tau(\mathrm{cyl}''(N),\vec{v}(\widetilde{\theta}))\leq\left(\widetilde{\lambda}-\frac{1}{\sqrt{N}}\right)N\right]\\
&\quad\geq\;\mathbb{P}\left[\phi_n^\kappa\leq\left(\widetilde{\lambda}-\varepsilon\right)\frac{N}{\mathcal{N}}\right]^{\mathcal{N}}\times\mathbb{P}\left[V(E_1(n,\kappa)\cup E_2(n,\kappa))\leq\frac{\varepsilon}{2}N\right]\\
&\quad\geq\;\mathbb{P}\left[\frac{\phi_n^\kappa}{nl(A)}\leq\frac{\widetilde{\lambda}-\varepsilon}{\cos(\widetilde{\theta}-\theta)}\right]^{\mathcal{N}}\times\mathbb{P}\left[t(e)\leq\frac{\varepsilon N}{2C_6(\mathcal{N}+\zeta'(n)+L(n,\widetilde{\theta}))}\right]^{C_6(\mathcal{N}+\zeta'(n)+L(n,\widetilde{\theta}))}\\
&\quad\geq\;\mathbb{P}\left[\frac{\phi_n^\kappa}{nl(A)}\leq\frac{\widetilde{\lambda}-\varepsilon}{\cos(\widetilde{\theta}-\theta)}\right]^{\mathcal{N}}\times\mathbb{P}\left[t(e)\leq\frac{\varepsilon l(A)n}{4C_6}\right]^{C_6(\mathcal{N}+\zeta'(n)+L(n,\widetilde{\theta}))}\,.
\end{aligned}$$

We take the logarithm of the previous inequality, divide it by $-N$, and send N to infinity to obtain that:

$$\mathcal{I}_{\widetilde{\theta}}(\widetilde{\lambda})\;\leq\;\frac{-1}{L(n,\widetilde{\theta})}\log\mathbb{P}\left[\frac{\phi_n^\kappa}{nl(A)}\leq\frac{\widetilde{\lambda}-\varepsilon}{\cos(\widetilde{\theta}-\theta)}\right]-\frac{C_6}{L(n,\widetilde{\theta})}\log\mathbb{P}\left[t(e)\leq\frac{\varepsilon l(A)n}{4C_6}\right]\,.$$

For n large enough,

$$\mathbb{P}\left[t(e)\leq\frac{\varepsilon l(A)n}{4C_6}\right]\geq\frac{1}{2}\,,$$

and thus,

$$\frac{1}{nl(A)}\log\mathbb{P}\left[\frac{\phi_n^\kappa}{nl(A)}\leq\frac{\widetilde{\lambda}-\varepsilon}{\cos(\widetilde{\theta}-\theta)}\right]\;\leq\;-\frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(\widetilde{\lambda})+\frac{K}{n}\,,$$

where $K=C_6\log 2/l(A)$. We set $\lambda=(\widetilde{\lambda}-\varepsilon)/\cos(\widetilde{\theta}-\theta)$ (so $\lambda\geq0$), and let ε go to zero to conclude that for all $\lambda\geq0$ and $\kappa\in D_n$,

$$\mathbb{P}\left[\frac{\phi_n^\kappa}{nl(A)}\leq\lambda\right]\;\leq\;\exp-\left[nl(A)\left(\frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+)+\frac{K}{n}\right)\right]\,. \tag{6.51}$$

We come back now to the study of ϕ_n itself. We have seen that $\phi_n=\inf_{\kappa\in D_n}\phi_n^\kappa$. We also noticed that ϕ_n^κ takes only a finite number of values when $\kappa\in D_n$, thus one may restrict ourselves to a finite subset $\tilde{D}_n$ of D_n such that $\mathrm{card}(\tilde{D}_n)\leq C_4h(n)^2$. Therefore,

$$\begin{aligned}
\mathbb{P}[\phi_n\leq\lambda nl(A)]\;&=\;\mathbb{P}\left[\exists\kappa\in\widetilde{D}'_n\,|\,\phi_n^\kappa\leq\lambda nl(A)\right]\\
&\leq\;\sum_{\kappa\in\widetilde{D}'_n}\mathbb{P}[\phi_n^\kappa\leq\lambda nl(A)]\\
&\leq\;C_4h(n)^2\times\max_{\kappa\in D_n}\mathbb{P}[\phi_n^\kappa\leq\lambda nl(A)]
\end{aligned}$$

$$\leq C_4 h(n)^2 \exp\left[-nl(A)\left(\frac{K}{n} + \inf_{\widetilde{\theta}\in\mathcal{D}_n} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+)\right)\right] .$$

If we suppose that $\lim_{n\to\infty} \log(h(n))/n = 0$, we obtain that:

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}[\phi_n \leq \lambda nl(A)] \leq -\liminf_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+) , \tag{6.52}$$

and

$$\liminf_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}[\phi_n \leq \lambda nl(A)] \leq -\limsup_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+) , \tag{6.53}$$

We can now apply Lemma 9 and 10 with $f = g_\lambda$, that we know to be l.s.c. thanks to Lemma 13, and so (6.52) and (6.53) lead to (6.41) and (6.42).

4.1.4. *Discussion.* Combining the results of the two previous parts, we obtain that for all $\lambda \geq 0$, if we define

$$\square_n = \frac{1}{nl(A)} \mathbb{P}[\phi_n \leq \lambda nl(A)] ,$$

we have

$$\begin{cases} -\inf_{\widetilde{\theta}\in\mathrm{ad}(\underline{\mathcal{D}})} \dfrac{\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^-)}{\cos(\widetilde{\theta}-\theta)} \leq \liminf_{n\to\infty} \square_n \leq -\inf_{\widetilde{\theta}\in\mathrm{ad}(\underline{\mathcal{D}})} \dfrac{\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+)}{\cos(\widetilde{\theta}-\theta)} , \\ -\inf_{\widetilde{\theta}\in\mathrm{ad}(\overline{\mathcal{D}})} \dfrac{\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^-)}{\cos(\widetilde{\theta}-\theta)} \leq \limsup_{n\to\infty} \square_n \leq -\inf_{\widetilde{\theta}\in\mathrm{ad}(\overline{\mathcal{D}})} \dfrac{\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+)}{\cos(\widetilde{\theta}-\theta)} . \end{cases}$$

This is not completely satisfying. In the case were $\lim_{n\to\infty} h(n)/n$ exists, then there exists some closed set $\mathcal{D}$ centered at θ such that $\mathrm{ad}(\underline{\mathcal{D}}) = \mathrm{ad}(\overline{\mathcal{D}}) = \mathcal{D}$, one could hope to obtain that

$$\lim_{n\to\infty} \square_n = -\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+) = -\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^-) , \tag{6.54}$$

at least for:

$$\lambda \neq \delta \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{|\cos\widetilde{\theta}| + |\sin\widetilde{\theta}|}{\cos(\widetilde{\theta}-\theta)} := \delta_{\theta,h} .$$

Obviously, if for all $\widetilde{\theta} \in \mathcal{D}$ we have $\mathcal{I}_{\widetilde{\theta}}(\lambda^+) = \mathcal{I}_{\widetilde{\theta}}(\lambda^-)$ (this can be true even if $\lambda = \delta(|\cos\widetilde{\theta}| + |\sin\widetilde{\theta}|)/\cos(\widetilde{\theta}-\theta)$ for some $\widetilde{\theta}$), then equation (6.54) holds. On the opposite, if $\mathcal{I}_{\widetilde{\theta}_0}(\delta(|\cos\widetilde{\theta}_0| + |\sin\widetilde{\theta}_0|)^+) < \infty$ for some $\widetilde{\theta}_0 \in \mathcal{D}$ such that $\delta_{\theta,h} = \delta(|\cos\widetilde{\theta}_0| + |\sin\widetilde{\theta}_0|)/\cos(\widetilde{\theta}_0-\theta)$, then

$$-\infty = -\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\delta_{\theta,h}\cos(\widetilde{\theta}-\theta)^-) < -\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)} \mathcal{I}_{\widetilde{\theta}}(\delta_{\theta,h}\cos(\widetilde{\theta}-\theta)^+) .$$

This is the reason why we think that the behaviour of $\square_n$ for $\lambda = \delta_{\theta,h}$ is not clear. Although we could not prove (6.54), equations (6.39) and (6.41) are sufficient to prove the large deviation principle.

4.2. Large deviation principle for ϕ_n. From now on, we suppose that $\lim_{n\to\infty} h(n)/n$ exists in $\mathbb{R}^+ \cup \{+\infty\}$, and we denote it by $\tan\alpha$ ($\alpha \in [0, \pi/2]$). We define

$$\mathcal{D} = [\theta-\alpha, \theta+\alpha] ,$$

and we know that $(\phi_n/(nl(A)))_{n\in\mathbb{N}}$ converges towards

$$\eta_{\theta,h} = \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} .$$

We remark that $\eta_{\theta,h} = \inf_{\widetilde{\theta}\in\mathcal{D}} \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$ is positive since $F(0) < 1/2$. We also know that:

$$\mathrm{ad}(\overline{\mathcal{D}}) \;=\; \mathrm{ad}(\underline{\mathcal{D}}) \;=\; \mathcal{D}\,.$$

Recall that the rate function $\mathcal{K}:\mathbb{R}^+\to\mathbb{R}^+\cup\{+\infty\}$ is defined by:

$$\mathcal{K}(\lambda) \;=\; \begin{cases} \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+) & \text{if } \lambda\le\eta_{\theta,h}\,,\\ +\infty & \text{if } \lambda>\eta_{\theta,h}\,. \end{cases} \tag{6.55}$$

4.2.1. *Properties of* $\mathcal{K}$. We first stress here the fact that $\mathcal{K}$ depends on θ and h (via α). We should have denoted the function $\mathcal{K}$ by $\mathcal{K}_{\theta,h}$, but we decided to omit the indices θ and h to make the reading of this paper easier.

We remember all the properties of $\mathcal{I}_{\widetilde{\theta}}$ we know (see Theorem 19). Let:

$$\widetilde{\mathcal{K}}(\lambda) \;=\; \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(\lambda\cos(\widetilde{\theta}-\theta)^+)\,,$$

i.e., on $[0,\eta_{\theta,h}]$ we have $\mathcal{K}=\widetilde{\mathcal{K}}$, and on $]\eta_{\theta,h},+\infty[$ we have $\mathcal{K}=+\infty$. Let us define

$$\delta_{\theta,h} \;=\; \delta\times\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{|\cos\widetilde{\theta}|+|\sin\widetilde{\theta}|}{\cos(\widetilde{\theta}-\theta)}\,,$$

the infimum of the values that $\phi_n/(nl(A))$ can take. Then obviously $\widetilde{\mathcal{K}}$ is infinite on $[0,\delta_{\theta,h}[$ and finite on $]\delta_{\theta,h},+\infty[$. It is also obvious that $\widetilde{K}$ is null on $[\eta_{\theta,h},+\infty[$. Since for all $\widetilde{\theta}$, $\lambda\to\mathcal{I}_{\widetilde{\theta}}(\lambda^+)$ is non increasing, so is $\widetilde{\mathcal{K}}$ on $\mathbb{R}^+$. There are only two non-obvious properties of $\mathcal{K}$ we have to prove: $\mathcal{K}$ is a good rate function and $\mathcal{K}$ is strictly decreasing when it is finite. We will prove them now:

LEMMA 14. *The function $\mathcal{K}$ is lower semi-continuous and coercive on $\mathbb{R}^+$, i.e., for all $t\ge0$, the set $\{\lambda\,|\,\mathcal{K}(\lambda)\le t\}$ is compact.*

Proof :
In fact it is sufficient to prove that for all $t\ge0$, the set $\{\lambda\,|\,\widetilde{\mathcal{K}}(\lambda)\le t\}$ is closed, because we know that

$$\forall t\ge0\qquad \{\lambda\,|\,\mathcal{K}(\lambda)\le t\} \;=\; \{\lambda\,|\,\widetilde{\mathcal{K}}(\lambda)\le t\}\cap[0,\eta(\theta,h)]\,.$$

Let $(\lambda_n)_{n\ge0}$ be a sequence of $\{\lambda\,|\,\widetilde{\mathcal{K}}(\lambda)\le t\}$, converging towards some λ_0. For each fixed λ in $\mathbb{R}^+$, since the function g_λ is lower semi-continuous and $\mathcal{D}$ is compact, there exists $\widetilde{\theta}_\lambda$ such that

$$\widetilde{\mathcal{K}}(\lambda) \;=\; g_\lambda(\widetilde{\theta}_\lambda)\,.$$

The sequence $(\widetilde{\theta}_{\lambda_n})_{n\ge0}$ takes values in the compact $\mathcal{D}$, so up to extracting a subsequence, we can suppose that it converges towards a limit $\widetilde{\theta}_0\in\mathcal{D}$. For all positive ε, for all large n we have $\lambda_n\le\lambda_0+\varepsilon$, and so, since $\mathcal{I}_{\widetilde{\theta}}$ is non increasing for all $\widetilde{\theta}$, we obtain for all large n that

$$g_{(\lambda_0+\varepsilon)}(\widetilde{\theta}_{\lambda_n}) \;\le\; g_{\lambda_n}(\widetilde{\theta}_{\lambda_n}) \;\le\; t\,.$$

Since $g_{(\lambda_0+\varepsilon)}$ is l.s.c. and a subsequence $(\widetilde{\theta}_{\psi(n)})_{n\ge0}$ of $(\widetilde{\theta}_{\lambda_n})_{n\ge0}$ converges towards $\widetilde{\theta}_0$, we obtain:

$$g_{(\lambda_0+\varepsilon)}(\widetilde{\theta}_0) \;\le\; \liminf_{n\to\infty} g_{(\lambda_0+\varepsilon)}(\widetilde{\theta}_{\psi(n)}) \;\le\; t\,.$$

This inequality is satisfied for all positive ε, and $\widetilde{\theta}_0\in\mathcal{D}$, so

$$\widetilde{\mathcal{K}}(\lambda_0) \;\le\; g_{\lambda_0}(\widetilde{\theta}_0) \;=\; \lim_{\varepsilon\to\infty\,,\,\varepsilon>0} g_{(\lambda_0+\varepsilon)}(\widetilde{\theta}_0) \;\le\; t\,.$$

This ends the proof of Lemma 14.

■

LEMMA 15. *For all $\lambda\in\mathbb{R}^+$ such that $\mathcal{K}(\lambda)<\infty$, for all positive ε, we have*

$$\mathcal{K}(\lambda) \;<\; \mathcal{K}(\lambda-\varepsilon)\,.$$

Proof :
Let $\lambda \in \mathbb{R}^+$ such that $\mathcal{K}(\lambda) < \infty$. If $\mathcal{K}(\lambda - \varepsilon) = +\infty$, the result is obvious, so we suppose that $\mathcal{K}(\lambda - \varepsilon) < \infty$. Thanks to Lemma 13, for every fixed λ, $\widetilde{\theta} \mapsto g_\lambda(\widetilde{\theta})$ is l.s.c. Since $\mathcal{D}$ is compact, $\inf_{\widetilde{\theta}\in\mathcal{D}} g_\lambda(\widetilde{\theta})$ is reached at some $\theta_\lambda \in \mathcal{D}$. Notice also that for every fixed $\widetilde{\theta}$, $\lambda \mapsto g_\lambda(\widetilde{\theta})$ is decreasing, and even strictly decreasing on $[\delta_{\widetilde{\theta},h}, \eta_{\theta,h}]$, that is on the closure of the set where it is finite and non-zero. Thus,

$$\inf_{\widetilde{\theta}\in\mathcal{D}} g_\lambda(\widetilde{\theta}) \leq g_\lambda(\widetilde{\theta}_{\lambda-\varepsilon}) < g_{\lambda-\varepsilon}(\widetilde{\theta}_{\lambda-\varepsilon}) = \inf_{\widetilde{\theta}\in\mathcal{D}} g_{\lambda-\varepsilon}(\widetilde{\theta}) \,.$$

∎

We will finally prove the property we stated in Remark 25, in fact a property a little bit more general:

LEMMA 16. *If $\theta \in \{k\pi/4 \,|\, k \in \mathbb{N}\}$, then*

$$\mathcal{K}(\lambda) \,=\, \mathcal{I}_\theta(\lambda^+) \,.$$

Proof :
We fix a $\theta \in \{k\pi/4 \,|\, k \in \mathbb{N}\}$. The first property to notice is that $\theta \in \mathcal{D}$, so it is sufficient to prove that

$$\forall \lambda \geq 0\,,\ \forall \widetilde{\theta} \in [\theta - \pi/2, \theta + \pi/2]\,, \qquad \mathcal{I}_\theta(\lambda^+) \,\leq\, \frac{1}{\cos(\widetilde{\theta} - \theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda \cos(\widetilde{\theta} - \theta)^+) \,.$$

Let $\widetilde{\theta} \in]\theta - \pi/2, \theta + \pi/2[$. We use the same notations as in Lemma 12. We consider the non degenerate triangle (abc) such that $\widetilde{\theta}_c = \theta + \pi$ (so $\mathrm{cyl}_c(n)$ is a straight cylinder in the case $\theta = 0$), $\widetilde{\theta}_b = \max(\widetilde{\theta}, 2\theta - \widetilde{\theta})$, $\widetilde{\theta}_a = \min(\widetilde{\theta}, 2\theta - \widetilde{\theta})$, $l(ab) = 1$ and $l(ac) = l(bc) = (2\cos(\widetilde{\theta} - \theta))^{-1}$. Since the graph is invariant by a symmetry of axis $((0,0), (\cos\theta, \sin\theta))$ (respectively $((0,0), (1,1))$), we know that $\mathcal{I}_{\widetilde{\theta}_a} = \mathcal{I}_{\widetilde{\theta}_b}$ (respectively $\mathcal{I}_{\widetilde{\theta}_c} = \mathcal{I}_\theta$). Then Lemma 12 applied with $\alpha = 1/2$ states that for all $\lambda \geq 0$,

$$\mathcal{I}_\theta\left(\lambda^+\right) \,\leq\, \frac{1}{\cos(\widetilde{\theta} - \theta)} \mathcal{I}_{\widetilde{\theta}}(\lambda \cos(\widetilde{\theta} - \theta)^+) \,.$$

The inequality remains obviously valid for $\widetilde{\theta} \in \{\theta + \pi/2, \theta - \pi/2\}$, since we have seen in Remark 28 that the right hand side of the previous inequality equals $+\infty$ in this case. This ends the proof of Lemma 16.

∎

4.2.2. *Upper large deviations.* To handle the upper large deviations, we shall use the following result.

LEMMA 17. *Let A be a non-empty line-segment in $\mathbb{R}^2$, with euclidean length $l(A)$. Let $\theta \in [0, \pi[$ be such that $(\cos\theta, \sin\theta)$ is orthogonal to the hyperplane spanned by A and $(h(n))_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} h(n) = +\infty$. We suppose that F admits an exponential moment:*

$$\exists \gamma > 0 \text{ s.t.} \qquad \int e^{\gamma x} dF(x) \,<\, \infty \,.$$

Then for all $\lambda > \nu_\theta$ we have

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}\left[\frac{\phi_n}{nl(A)} \geq \lambda\right] \,=\, -\infty \,.$$

Lemma 17 is a consequence of Theorem 22, that will be proved in section 5. We admit this result for the moment. We decided to prove it after the lower large deviation principle, because its proof is not based on specificities of the dimension two, it is only an adaptation of the proofs presented in the Chapters 2 and 3 of the thesis.

4.2.3. *Upper bound.* Let $\mathcal{F}$ be a closed subset of $\mathbb{R}^+$. We want to prove that

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}\left[\frac{\phi_n}{nl(A)} \in \mathcal{F}\right] \le -\inf_{\lambda\in\mathcal{F}} \mathcal{K}(\lambda)\,.$$

If $\eta_{\theta,h}$ belongs to $\mathcal{F}$, then according to Corollary 2.1, we know that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\phi_n}{nl(A)} \in \mathcal{F}\right] = 1\,,$$

and so

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}\left[\frac{\phi_n}{nl(A)} \in \mathcal{F}\right] = 0 = -\inf_{\lambda\in\mathcal{F}} \mathcal{K}(\lambda)\,,$$

because $\mathcal{K}$ is non-negative, and $\mathcal{K}(\eta_{\theta,h}) = 0$. Let us suppose that $\eta_{\theta,h}$ does not belong to $\mathcal{F}$. The following proof is similar to the one of the upper bound in [**52**]. We define $f_1 = \sup(\mathcal{F}\cap[0,\eta_{\theta,h}])$ and $f_2 = (\inf \mathcal{F}\cap[\eta_{\theta,h},+\infty[)$. We suppose here that $\mathcal{F}\cap[0,\eta_{\theta,h}]$ and $\mathcal{F}\cap[\eta_{\theta,h},+\infty[$ are non empty, because it is the most complicated case (if one of these two sets is empty, part of the following study is sufficient). Since $\mathcal{F}$ is closed, we know that $f_1 < \eta_{\theta,h}$ and $f_2 > \eta_{\theta,h}$. Then

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{nl(A)} &\log\mathbb{P}\left[\frac{\phi_n}{nl(A)} \in \mathcal{F}\right]\\ &\le \limsup_{n\to\infty} \frac{1}{nl(A)} \log\left(\mathbb{P}\left[\frac{\phi_n}{nl(A)} \le f_1\right] + \mathbb{P}\left[\frac{\phi_n}{nl(A)} \ge f_2\right]\right)\,.\end{aligned}$$

On one hand, by (6.41), we know that

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}[\phi_n \le f_1 nl(A)] \le -\mathcal{K}(f_1)\,.$$

On the other hand, if we refer to Lemma 17, we know that

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}[\phi_n \ge f_2 nl(A)] = -\infty\,.$$

If $\mathcal{K}(f_1) = +\infty$, we have

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}\left[\frac{\phi_n}{nl(A)} \in \mathcal{F}\right] = -\infty = -\inf_{\mathcal{F}} \mathcal{K}\,,$$

because $\mathcal{K}$ is infinite on $[0, f_1]$ ($\mathcal{K}'$ is non-increasing) and on $[f_2,+\infty[$, so on $\mathcal{F}$. If $\mathcal{K}(f_1) < \infty$, we have

$$\limsup_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}\left[\frac{\phi_n}{nl(A)} \in \mathcal{F}\right] \le -\mathcal{K}(f_1) = -\inf_{\mathcal{F}} \mathcal{K}\,,$$

because $\mathcal{K}$ is non-increasing on $[0, f_1]$ and infinite on $[f_2,+\infty[$. So the upper bound is proved.

4.2.4. *Lower bound.* We have to prove that for all open subset $\mathcal{O}$ of $\mathbb{R}^+$, we have

$$\liminf_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}\left[\frac{\phi_n}{nl(A)} \in \mathcal{O}\right] \ge -\inf_{\lambda\in\mathcal{O}} \mathcal{K}(\lambda)\,.$$

Classically, it suffices to prove the local lower bound:

$$\forall a\in\mathbb{R}^+\,,\ \forall\varepsilon>0 \qquad \liminf_{n\to\infty} \frac{1}{nl(A)} \log \mathbb{P}\left[\frac{\phi_n}{nl(A)} \in [a-\varepsilon, a+\varepsilon]\right] \ge -\mathcal{K}(a)\,. \tag{6.56}$$

If $\mathcal{K}(a) = +\infty$, the result is obvious, so we suppose that $\mathcal{K}(a) < +\infty$. For all $\eta < \varepsilon$, we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{nl(A)} &\log\mathbb{P}\left[\frac{\phi_n}{nl(A)} \in [a-\varepsilon, a+\varepsilon]\right]\\ &\ge \liminf_{n\to\infty} \frac{1}{nl(A)} \log\left(\mathbb{P}\left[\frac{\phi_n}{nl(A)} \le a+\eta\right] - \mathbb{P}\left[\frac{\phi_n}{nl(A)} \le a-\varepsilon\right]\right)\,.\end{aligned} \tag{6.57}$$

From the strict decreasing of $\mathcal{K}$ (see Lemma 15), we deduce that for all $a \in \mathbb{R}^+$ such that $\mathcal{K}(a) < \infty$, for all positive η and ε, we have

$$\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}((a+\eta)\cos(\widetilde{\theta}-\theta)^-) < \inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}((a-\varepsilon)\cos(\widetilde{\theta}-\theta)^+)\,. \tag{6.58}$$

Indeed, for all positive η, we have

$$\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}((a+\eta)\cos(\widetilde{\theta}-\theta)^-) \le \mathcal{K}(a) < \mathcal{K}(a-\varepsilon)\,.$$

Then thanks to (6.39), (6.41) and (6.58), we know that the second term in the sum appearing in (6.57) is negligible compared to the first one, so we obtain that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{nl(A)} \log\mathbb{P}\left[\frac{\phi_n}{nl(A)} \in [a-\varepsilon, a+\varepsilon]\right] & \\ \ge -\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}((a+\eta)\cos(\widetilde{\theta}-\theta)^-) & \\ \ge -\inf_{\widetilde{\theta}\in\mathcal{D}} \lim_{\varepsilon'\to 0} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}((a+\eta)\cos(\widetilde{\theta}-\theta)-\varepsilon')\,. &\end{aligned}$$

Sending η to zero, we obtain that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{nl(A)} \log\mathbb{P}\left[\frac{\phi_n}{nl(A)} \in [a-\varepsilon, a+\varepsilon]\right] & \\ \ge -\liminf_{\eta\to 0}\inf_{\widetilde{\theta}\in\mathcal{D}} \lim_{\varepsilon'\to 0} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}((a+\eta)\cos(\widetilde{\theta}-\theta)-\varepsilon') & \\ \ge -\inf_{\widetilde{\theta}\in\mathcal{D}} \lim_{\eta\to 0}\lim_{\varepsilon'\to 0} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}((a+\eta)\cos(\widetilde{\theta}-\theta)-\varepsilon') & \\ \ge -\inf_{\widetilde{\theta}\in\mathcal{D}} \frac{1}{\cos(\widetilde{\theta}-\theta)}\mathcal{I}_{\widetilde{\theta}}(a\cos(\widetilde{\theta}-\theta)^+)\,, &\end{aligned}$$

and so the local lower bound is proved.

5. Upper large deviations

The proof of Theorem 22 follows the one of the upper large deviations for the maximal flow $\tau(nA, h(n))$ in tilted cylinders done in Chapter 3: we will consider the cylinder $\mathrm{cyl}(nA, h(n))$ on a macroscopic scale, and we will divide it into smaller cylinders on a mesoscopic scale. However, we will need a geometrical construction a little bit more complicated than in Chapter 3, because the mesoscopic cylinders we will consider are oriented towards the direction that minimizes the value of $\nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$ and not towards the direction θ itself.

5.1. Geometric construction. First of all, we recall that the hypothesis that the sequence $(\phi(nA, h(n))/(nl(A)))_{n\in\mathbb{N}}$ converges towards $\eta_{\theta,h}$ is equivalent to the hypothesis that

$$\inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,,$$

and $\eta_{\theta,h}$ is equal to those infima. We have already noticed that

$$\eta_{\theta,h} = \inf_{\mathrm{ad}(\underline{\mathcal{D}})} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}\,,$$

and since the function $\widetilde{\theta} \rightarrow \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$ is l.s.c. and $\mathrm{ad}(\underline{\mathcal{D}})$ is compact, we know that there exists $\widetilde{\theta}_0$ in $\mathrm{ad}(\underline{\mathcal{D}})$ such that

$$\eta_{\theta,h} \,=\, \frac{\nu_{\widetilde{\theta}_0}}{\cos(\widetilde{\theta}_0-\theta)}\,.$$

Let $\lambda > \eta_{\theta,h}$, and ε such that $\lambda \geq \eta_{\theta,h}+3\varepsilon$. We remark that $\widetilde{\theta}_0 \in]\theta-\pi/2,\theta+\pi/2[$. Then $\widetilde{\theta} \rightarrow \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$ is continuous at $\widetilde{\theta}_0$ so there exists $\delta>0$ such that

$$|\widetilde{\theta}-\widetilde{\theta}_0| \,\leq\, 2\delta \quad\Rightarrow\quad \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \,\leq\, \frac{\nu_{\widetilde{\theta}_0}}{\cos(\widetilde{\theta}_0-\theta)}+\varepsilon \,=\, \eta_{\theta,h}+\varepsilon\,.$$

The set $\underline{\mathcal{D}}$ is a symmetric segment centered at θ. We suppose for this paragraph that $\widetilde{\theta}_0 \neq \theta$ (and so that $\underline{\mathcal{D}} \smallsetminus \{\theta\} \neq \emptyset$). Then suppose for example that $\widetilde{\theta}_0 > \theta$. We know that $[\theta,\widetilde{\theta}_0[\subset \underline{\mathcal{D}}$. Up to taking a smaller δ, if we define $\widetilde{\theta}_1=\widetilde{\theta}_0-2\delta$, then we can have $\widetilde{\theta}_1+\delta=\widetilde{\theta}_0-\delta \in \underline{\mathcal{D}}$ and $\widetilde{\theta}_1>\theta$. On one hand, $\widetilde{\theta}_1-\theta$, δ and $\widetilde{\theta}_1-\theta+\delta$ are in $[0,\pi/2]$ so we have

$$\tan(\widetilde{\theta}_1-\theta+\delta) \,\geq\, \tan(\widetilde{\theta}_1-\theta)+\tan(\delta)\,.$$

On the other hand, we obtain the existence of a n_0 such that for all $n \geq n_0$, $\widetilde{\theta}_1+\delta \in \mathcal{D}_n$. Symmetrically, if $\widetilde{\theta}_0 < \theta$, we define $\widetilde{\theta}_1=\widetilde{\theta}_0+2\delta$ such that $\widetilde{\theta}_1<\theta$ and $\widetilde{\theta}_1-\delta \in \underline{\mathcal{D}}$. We obtain that $\widetilde{\theta}_1-\delta \in \mathcal{D}_n$ for all $n \geq n_0$ and that $\theta-\widetilde{\theta}_1$, δ and $\theta-\widetilde{\theta}_1+\delta$ are in $[0,\pi/2]$ so

$$\tan(\theta-\widetilde{\theta}_1+\delta) \,\geq\, \tan(\theta-\widetilde{\theta}_1)+\tan(\delta)\,.$$

Then, in both cases, we obtain that for all $n \geq n_0$,

$$2h(n) \,\geq\, nl(A)\tan\left(|\widetilde{\theta}_1-\theta|+\delta\right) \,\geq\, nl(A)\tan|\widetilde{\theta}_1-\theta|+nl(A)\tan\delta\,.$$

We define $\widetilde{h}(n)$ by

$$\widetilde{h}(n) \,=\, 2h(n)-nl(A)\tan|\widetilde{\theta}_1-\theta|\,.$$

We immediately notice that

$$\widetilde{h}(n) \,\geq\, nl(A)\tan\delta\,. \tag{6.59}$$

In fact if we consider the boundary conditions $\kappa_1=(0,\widetilde{\theta}_1)$ if $\widetilde{\theta}_1>\theta$ (respectively $\kappa_1=(1,\widetilde{\theta}_1)$ if $\widetilde{\theta}_1<\theta$) in $\mathrm{cyl}(nA,h(n))$, then $\widetilde{h}(n)$ is the distance between the point defined by the boundary condition κ_1 on the right side of $\mathrm{cyl}(nA,h(n))$ and the upper right corner (respectively the lower right corner) of $\mathrm{cyl}(nA,h(n))$, where the directions "right" and "upper" are given by the direction θ of the cylinder $\mathrm{cyl}(nA,h(n))$. See figure 5 for a picture that shows κ_1 and $\widetilde{h}(n)$.

In the case were $\widetilde{\theta}_0=\theta$, the equivalent definitions would be $\widetilde{\theta}_1=\widetilde{\theta}_0=\theta$, $\widetilde{h}(n)=h(n)$ and $\kappa_1=(1/2,\theta)$. Actually, we will not study this case here, because it has already been studied in Chapter 3. The cases $\widetilde{\theta}_0>\theta$ and $\widetilde{\theta}_0<\theta$ are symmetric, we will suppose from now on that $\widetilde{\theta}_0>\theta$ to simplify some notations but exactly the same proof works in the case $\widetilde{\theta}_0<\theta$.

We now consider the boundary condition κ_1 on $\mathrm{cyl}(nA,h(n))$ for $n \geq n_0$. As in the previous sections, we consider the two points x_n and y_n defined by κ_1 respectively on the left side and on the right side of $\mathrm{cyl}(nA,h(n))$. The point x_n corresponds to the lower left corner of $\mathrm{cyl}(nA,h(n))$. We know that a layer of thickness $\widetilde{h}(n)$ above $[x_n,y_n]$ remains inside $\mathrm{cyl}(nA,h(n))$ (see figure 5), and we divide it into thinner layers of thickness $p+\zeta$, were ζ is a constant greater that $2d$, and p is an integer that will define the size of the mesoscopic cylinders - thus we consider that p is a lot bigger than 1 but a lot smaller than n. The definitions that follow are illustrated by figures 5 and 6. We define as previously the vectors

$$\vec{v}(\theta) \,=\, (\cos\theta,\sin\theta)\,,\quad \vec{v}^{\perp}(\theta) \,=\, (-\sin\theta,\cos\theta)$$

and

$$\vec{v}(\widetilde{\theta}_1) \,=\, (\cos\widetilde{\theta}_1,\sin\widetilde{\theta}_1)\,,\quad \vec{v}^{\perp}(\widetilde{\theta}_1)=(-\sin\widetilde{\theta}_1,\cos\widetilde{\theta}_1)\,.$$

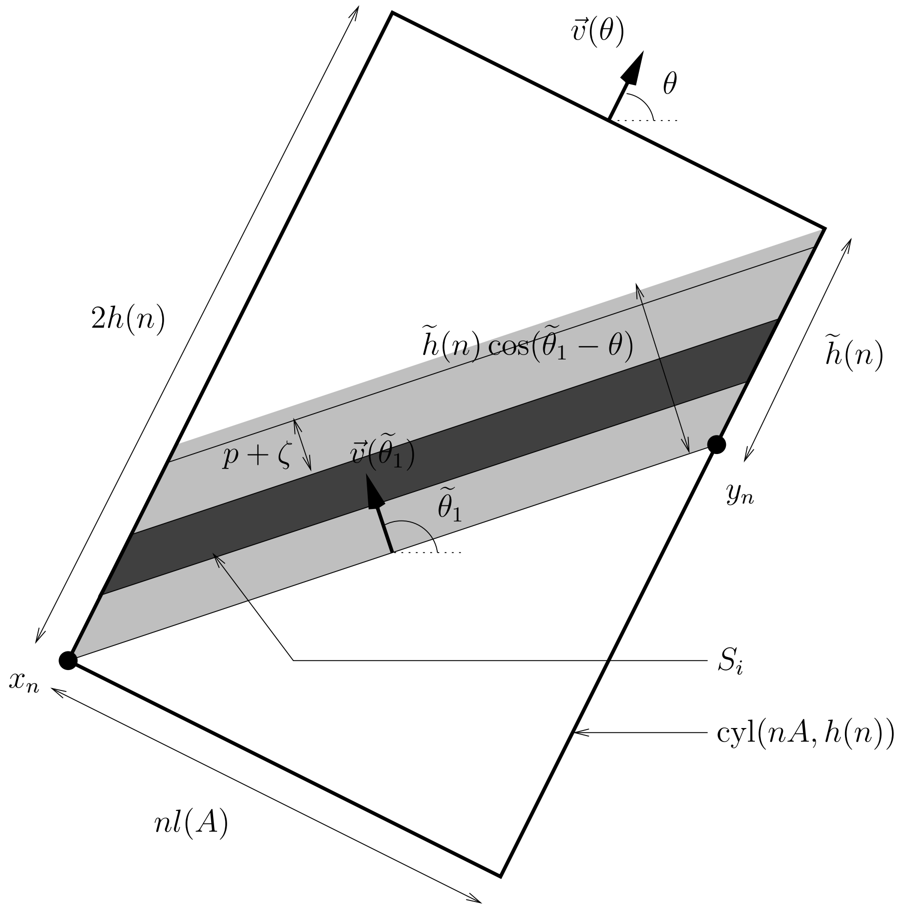

FIGURE 5. The cylinder $\mathrm{cyl}(nA, h(n))$ and the slabs S_i.

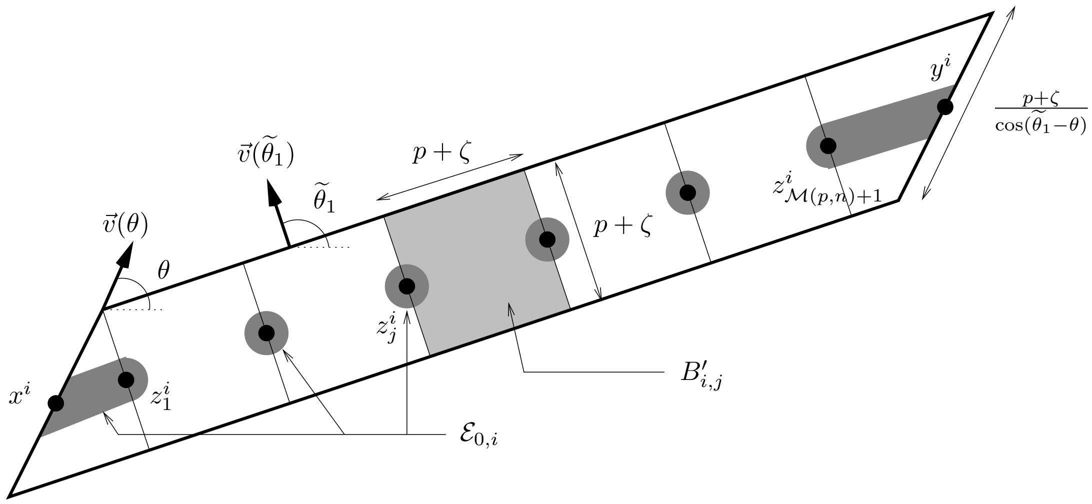

FIGURE 6. The slab S_i and the cylinders $B'_{i,j}$.

We define the layers S_i by

$$S_i \;=\; \{x + t\vec{v}(\widetilde{\theta}_1)\,|\,x \in \mathbb{R}(x_n - y_n) \text{ and } t \in [(i-1)(p+\zeta), i(p+\zeta)]\} \cap \mathrm{cyl}(nA, h(n))$$

for $i = 1, ..., M(p,n)$ with

$$M(p,n) \;=\; \left\lfloor \frac{\widetilde{h}(n)\cos(\widetilde{\theta}_1-\theta)}{p+\zeta} \right\rfloor .$$

The set S_i is a layer of thickness $p+\zeta$ in the direction defined by $\widetilde{\theta}_1$, and the definition of $M(p,n)$ implies that all the S_i intersect $\partial\,\mathrm{cyl}(nA, h(n))$ on the left side and right side of $\mathrm{cyl}(nA, h(n))$ (including the corners of the cylinder) but not on its top or on its bottom. We define

$$B \;=\; \{x + t\vec{v}(\widetilde{\theta}_1)\,|\, x\in[0, p\vec{v}^{\perp}(\widetilde{\theta}_1)] \text{ and } t\in[0,p]\}\,,$$

which is a cylinder of dimension $p\times p$ oriented towards the direction $\widetilde{\theta}_1$, and the bigger cylinder

$$B' \;=\; \{x + t\vec{v}(\widetilde{\theta}_1)\,|\, x\in[0, (p+\zeta)\vec{v}^{\perp}(\widetilde{\theta}_1)] \text{ and } t\in[0,p+\zeta]\}\,.$$

We will fill each S_i with disjoint translates of B', inside which we will define translates of B by a translation whose vector has integer coordinates. Since the slab S_i is tilted, it has not squared corners, so we have to be a little bit careful. We define x^i (respectively y^i), the middle of the left side (respectively the right side) of S_i (we forget the dependence on n and p to simplify the notations), by

$$x^i \;=\; x_n + (i-1/2)\frac{p+\zeta}{\cos(\widetilde{\theta}_1-\theta)}\vec{v}(\theta)$$

and

$$y^i \;=\; y_n + (i-1/2)\frac{p+\zeta}{\cos(\widetilde{\theta}_1-\theta)}\vec{v}(\theta)\,.$$

We then translate x_i in the direction given by $\vec{v}^{\perp}(\widetilde{\theta}_1)$ first at the point z_1^i at which the slab S_i has a thickness $p+\zeta$ in the direction $\widetilde{\theta}_1$, and then successively at z_j^i which is at distance $p+\zeta$ of z_{j-1}^i, i.e.,

$$z_j^i \;=\; x_n^i + \left[\frac{(p+\zeta)\tan|\widetilde{\theta}_1-\theta|}{2} + (j-1)(p+\zeta)\right]\vec{v}^{\perp}(\widetilde{\theta}_1)\,,$$

for $j = 1, ..., \mathcal{M}(p,n)+1$ with

$$\mathcal{M}(p,n) \;=\; \left\lfloor \frac{\frac{nl(A)}{\cos(\widetilde{\theta}_1-\theta)} - (p+\zeta)\tan|\widetilde{\theta}_1-\theta|}{p+\zeta} \right\rfloor .$$

For all $i = 1, ..., M(p,n)$ and $j = i, ..., \mathcal{M}(p,n)$, we define

$$B'_{i,j} \;=\; \mathrm{cyl}([z_j^i, z_{j+1}^i], (p+\zeta)/2)\,,$$

which are translates of B' with pairwise disjoint interiors, and we denote by $B_{i,j}$ a translate of B by a translation whose vector has integer coordinates such that $B_{i,j}\subset B'_{i,j}$ (so they also have pairwise disjoint interiors).

5.2. Probabilistic part of the proof. For each $i\in\{1, ..., M(p,n)\}$, $j\in\{1, ..., \mathcal{M}(p,n)\}$, we define

$$\tau_{i,j} \;=\; \tau(B_{i,j}, \vec{v}(\widetilde{\theta}_1))\,.$$

We want to compare $\phi(nA, h(n))$ with those $\tau_{i,j}$. Exactly as in Chapter 3, we have to add some edges to glue together cutsets in the different cylinders $B_{i,j}$ to obtain a cutset in $\mathrm{cyl}(nA, h(n))$. For $i = 1, ..., M(p,n)$, we define the set

$$\mathcal{E}_{0,i} \;=\; \mathcal{V}\left([x^i, z_1^i]\cup\bigcup_{i=2}^{\mathcal{M}(p,n)}\{z_j^i\}\cup[z^i_{\mathcal{M}(p,n)+1}, y^i], 3\zeta\right)\cap S_i\,,$$

and we denote by $E_{0,i}$ the set of the edges included in $\mathcal{E}_{0,i}$ (we have kept the same notation as in Chapter 3). Then if for all couple (i,j) in $\{1, ..., M(p,n)\}\times\{1, ..., \mathcal{M}(p,n)\}$ the set of edges

$\mathcal{F}_{i,j}$ separates the upper half cylinder from the lower half cylinder in $B_{i,j}$ for the direction $\widetilde{\theta}_1$, then $\cup_{j=1}^{\mathcal{M}(p,n)} \mathcal{F}_{i,j} \cup E_{0,i}$ separates the top from the bottom of $\mathrm{cyl}(nA, h(n))$. We obtain that

$$\phi(nA, h(n)) \,\leq\, \min_{i=1,\dots,M(p,n)} \sum_{j=1}^{\mathcal{M}(p,n)} \tau_{i,j} + V(E_{0,i})\,,$$

and thus by independence we have

$$\mathbb{P}(\phi(nA,h(n)) \geq \lambda n l(A)) \,\leq\, \prod_{i=1}^{M(p,n)} \left(\mathbb{P}\left[\sum_{j=1}^{\mathcal{M}(p,n)} \tau_{i,j} \geq \left(\frac{\nu_{\widetilde{\theta}_1}}{\cos(\widetilde{\theta}_1 - \theta)} + \varepsilon \right) n l(A) \right] + \mathbb{P}\left[V(E_{0,i}) \geq \varepsilon n l(A) \right] \right). \tag{6.60}$$

On one hand, we have

$$\begin{aligned} \mathbb{P}\Bigg[\sum_{j=1}^{\mathcal{M}(p,n)} \tau_{i,j} \geq \left(\frac{\nu_{\widetilde{\theta}_1}}{\cos(\widetilde{\theta}_1 - \theta)} + \varepsilon \right) n l(A) \Bigg] & \\ = \mathbb{P}\left[\frac{1}{\mathcal{M}(p,n)} \sum_{j=1}^{\mathcal{M}(p,n)} \frac{\tau_{i,j}}{p} \geq \frac{n l(A)}{p\mathcal{M}(p,n)} \left(\frac{\nu_{\widetilde{\theta}_1}}{\cos(\widetilde{\theta}_1 - \theta)} + \varepsilon \right) \right] & \\ \leq \mathbb{P}\left[\frac{1}{\mathcal{M}(p,n)} \sum_{j=1}^{\mathcal{M}(p,n)} \frac{\tau_{i,j}}{p} \geq \nu_{\widetilde{\theta}_1} + \cos(\widetilde{\theta}_1 - \theta)\varepsilon \right] . & \end{aligned}$$

The variables $(\tau_{i,j})_{j=1,\dots,\mathcal{M}(p,n)}$ are independent and identically distributed, with the same law as $\tau(B, \vec{v}(\widetilde{\theta}_1))$. Since $\mathbb{E}(\tau(B, \vec{v}(\widetilde{\theta}_1)))/p$ converges towards $\nu_{\widetilde{\theta}_1}$ when p goes to infinity, there exists a p_0 large enough so that for all $p \geq p_0$:

$$\frac{\mathbb{E}(\tau(B, \vec{v}(\widetilde{\theta}_1)))}{p} \,\leq\, \nu_{\widetilde{\theta}_1} + \frac{\cos(\widetilde{\theta}_1 - \theta)\varepsilon}{2}\,,$$

Since the law of the capacity of the edges admits an exponential moment, so does the variable $\tau(B, \vec{v}(\widetilde{\theta}_1))/p$, because we can compare it with the rescaled capacity of a flat cutset that contains $O(p)$ edges. We can then apply the Cramér theorem to obtain that for fixed $p \geq p_0$ and λ there exists a constant c (depending on the law of $\tau(B, \vec{v}(\widetilde{\theta}_1))$, λ and ε) such that

$$\limsup_{n\to\infty} \frac{1}{\mathcal{M}(p,N)} \log \mathbb{P}\left[\frac{1}{\mathcal{M}(p,n)} \sum_{j=1}^{\mathcal{M}(p,n)} \frac{\tau_{i,j}}{p} \geq \nu_{\widetilde{\theta}_1} + \cos(\widetilde{\theta}_1 - \theta)\varepsilon \right] \,\leq\, c \,<\, 0\,,$$

whence

$$\limsup_{n\to\infty} \frac{1}{n l(A)} \log \mathbb{P}\left[\sum_{j=1}^{\mathcal{M}(p,n)} \tau_{i,j} \geq \left(\frac{\nu_{\widetilde{\theta}_1}}{\cos(\widetilde{\theta}_1 - \theta)} + \varepsilon \right) n l(A) \right] \,\leq\, c' \,<\, 0\,, \tag{6.61}$$

for a constant c' depending on $l(A)$ and the same parameters as c. On the other hand, we know that there exists a constant C such that for all $i = 1, \dots, M(p,n)$

$$\mathrm{card}(E_{0,i}) \,\leq\, C\left(\frac{n}{p} + p \right)\,.$$

If $\gamma > 0$ is such that $\mathbb{E}(\exp(\gamma t(e))) < \infty$, by a simple Chebyshev inequality, we obtain:

$$\mathbb{P}\left(V(E_{0,i}) \geq \varepsilon n l(A) \right) \,\leq\, \mathbb{P}\left(\sum_{k=1}^{C(np^{-1}+p)} t(e_k) \geq \varepsilon n l(A) \right)$$

$$\begin{aligned}
&\leq \mathbb{E}\left(\exp(\gamma t(e))\right)^{C(np^{-1}+p)} \exp(-\varepsilon nl(A)) \\
&\leq \exp\left[-n\left(\varepsilon l(A) - C(p^{-1}+pn^{-1})\log \mathbb{E}(\exp(\gamma t(e)))\right)\right] .
\end{aligned}$$

Obviously there exists p_1 such that for all $p \geq p_1$, for all n large enough (how large depends on p), we have

$$\mathbb{P}\left(V(E_{0,i}) \geq \varepsilon nl(A)\right) \leq \exp\left(-nl(A)\frac{\varepsilon}{2}\right) . \tag{6.62}$$

Combining equations (6.60), (6.61) and (6.62) we obtain that, for every fixed $p \geq \max(p_0, p_1)$,

$$\liminf_{n\to\infty} \frac{-1}{nl(A)M(p,n)} \log \mathbb{P}\left[\phi(nA, h(n)) \geq \lambda nl(A)\right] > 0 . \tag{6.63}$$

The last thing we have to do to complete the proof of Theorem 22 is to analyze the behaviour of $M(p,n)$. Thanks to (6.59) we have

$$\begin{aligned}
M(p,n) &\geq h(n)\frac{\cos(\widetilde{\theta}_1-\theta)}{p+\zeta}\left(1-\frac{nl(A)\tan|\widetilde{\theta}_1-\theta|}{2h(n)}\right)-1 \\
&\geq h(n)\frac{\cos(\widetilde{\theta}_1-\theta)}{p+\zeta}\left(1-\frac{\tan|\widetilde{\theta}_1-\theta|}{\tan|\widetilde{\theta}_1-\theta|+\tan\delta}\right)-1 \\
&\geq K(p,\theta,\widetilde{\theta}_1,\delta)h(n)-1 ,
\end{aligned}$$

where $K(p,\theta,\widetilde{\theta}_1,\delta) > 0$ is a strictly positive constant depending on the given parameters p, θ, $\widetilde{\theta}_1$ and δ. We deduce from (6.63) that

$$\liminf_{n\to\infty} \frac{-1}{nl(A)h(n)} \log \mathbb{P}\left[\phi(nA, h(n)) \geq \lambda nl(A)\right] > 0 .$$

As we said previously, exactly the same proof works for $\widetilde{\theta}_0 < \theta$, only the precise definition of the cylinders $B_{i,j}$ has to be adapted. In the case $\widetilde{\theta}_0 = \theta$, we can fill the entire cylinder $\mathrm{cyl}(nA, h(n))$ with translates of B' and not only a subpart of height $h(n) - nl(A)\tan|\widetilde{\theta}_1 - \theta|$, this is the reason why the equivalent of $\widetilde{h}(n)$ in this case is simply $h(n)$; the proof in this particular case has already been written in Chapter 3, since $\phi(nA, h(n))/n$ converges almost surely towards ν_θ.

Part 4

Maximal flow through a domain of $\mathbb{R}^d$

CHAPTER 7

Law of large numbers for the maximal flow through a domain of $\mathbb{R}^d$

This chapter is a joint work with Raphaël Cerf.

We consider the standard first passage percolation model in the rescaled graph $\mathbb{Z}^d/n$ for $d \geq 2$, and a domain Ω of boundary Γ in $\mathbb{R}^d$. Let Γ^1 and Γ^2 be two disjoint parts of Γ, representing the area of Γ through which some water can enter and escape from Ω. We investigate the asymptotic behaviour of the flow ϕ_n through a discrete version Ω_n of Ω between the corresponding discrete sets Γ_n^1 and Γ_n^2. We prove that under some conditions on the regularity of the domain and on the law of the capacity of the edges, ϕ_n converges almost surely towards a positive constant ϕ_Ω, which is the solution of a continuous non-random max-flow min-cut problem. Moreover, we prove that the lower large deviations are of surface order, while the upper large deviations are of volume order.

Contents

1. Definitions and main results

1.1. Main results. We use many notations introduced in [**40**] and [**41**]. Let $d \geq 2$. We consider the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ having for vertices $\mathbb{Z}_n^d = \mathbb{Z}^d/n$ and for edges $\mathbb{E}_n^d$, the set of pairs of nearest neighbours for the standard L^1 norm. With each edge e in $\mathbb{E}_n^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}_n^d)$ is independent and identically distributed, with a common law Λ: this is the standard model of first passage percolation on the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$. We interpret $t(e)$ as the capacity of the edge e; it means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time.

We consider an open bounded subset Ω of $\mathbb{R}^d$ such that the boundary $\Gamma = \partial\Omega$ of Ω is piecewise of class $\mathcal{C}^1$ (in particular Γ has finite area: $\mathcal{H}^{d-1}(\Gamma) < \infty$). It means that Γ is included in the union of a finite number of hypersurfaces of class $\mathcal{C}^1$, i.e., in the union of a finite number of C^1 submanifold of $\mathbb{R}^d$ of codimension 1. Let Γ^1, Γ^2 be two disjoint subsets of Γ that are open in Γ We want to define the maximal flow from Γ^1 to Γ^2 through Ω for the capacities $(t(e), e \in \mathbb{E}_n^d)$. We consider a discrete version $(\Omega_n, \Gamma_n, \Gamma_n^1, \Gamma_n^2)$ of $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ defined by:

$$\begin{cases} \Omega_n = \{x \in \mathbb{Z}_n^d \,|\, d_\infty(x, \Omega) < 1/n\}\,, \\ \Gamma_n = \{x \in \Omega_n \,|\, \exists y \notin \Omega_n\,,\ \langle x, y\rangle \in \mathbb{E}_n^d\}\,, \\ \Gamma_n^i = \{x \in \Gamma_n \,|\, d_\infty(x, \Gamma^i) < 1/n\,,\ d_\infty(x, \Gamma^j) \geq 1/n\} \text{ for } i = 1,2 \text{ and } j \neq i\,, \end{cases}$$

where d_∞ is the L^∞-distance, the notation $\langle x, y\rangle$ corresponds to the edge of endpoints x and y (see figure 1).

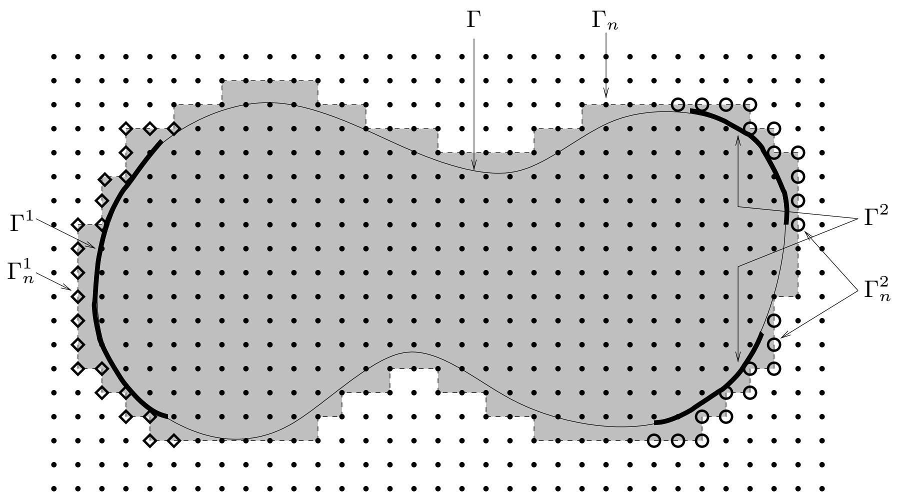

FIGURE 1. Domain Ω.

We want to study the maximal flow from Γ_n^1 to Γ_n^2 in Ω_n. Let us define properly the maximal flow $\phi(F_1 \rightarrow F_2 \textit{ in } C)$ from F_1 to F_2 in C, for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d$). We will say that an edge $e = \langle x, y\rangle$ belongs to a subset A of $\mathbb{R}^d$, which we denote by $e \in A$, if the segment joining x to y (eventually excluding these points) is included in A. We define $\widetilde{\mathbb{E}}_n^d$ as the set of all the oriented edges, i.e., an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}_n^d$ is an ordered pair of vertices which are nearest neighbours. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}_n^d$ by $\langle\langle x, y\rangle\rangle$, where x, $y \in \mathbb{Z}_n^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from x towards y. We consider the set $\mathcal{S}$ of all pairs of functions (g, o), with $g : \mathbb{E}_n^d \rightarrow \mathbb{R}^+$ and $o : \mathbb{E}_n^d \rightarrow \widetilde{\mathbb{E}}_n^d$ such that $o(\langle x, y\rangle) \in \{\langle\langle x, y\rangle\rangle, \langle\langle y, x\rangle\rangle\}$, satisfying:

- for each edge e in C we have
$$0 \leq g(e) \leq t(e)\,,$$
- for each vertex v in $C \smallsetminus (F_1 \cup F_2)$ we have
$$\sum_{e \in C\,:\, o(e) = \langle\langle v, \cdot\rangle\rangle} g(e) = \sum_{e \in C\,:\, o(e) = \langle\langle \cdot, v\rangle\rangle} g(e)\,,$$

where the notation $o(e) = \langle\langle v, .\rangle\rangle$ (respectively $o(e) = \langle\langle ., v\rangle\rangle$) means that there exists $y \in \mathbb{Z}_n^d$ such that $e = \langle v, y\rangle$ and $o(e) = \langle\langle v, y\rangle\rangle$ (respectively $o(e) = \langle\langle y, v\rangle\rangle$). A couple $(g, o) \in \mathcal{S}$ is a possible stream in C from F_1 to F_2: $g(e)$ is the amount of fluid that goes through the edge e, and $o(e)$ gives the direction in which the fluid goes through e. The two conditions on (g,o) express only the fact that the amount of fluid that can go through an edge is bounded by its capacity, and that there is no loss of fluid in the graph. With each possible stream we associate the corresponding flow

$$\mathrm{flow}(g,o) \,=\, \sum_{u\in F_2\,,\,v\notin C\,:\,\langle u,v\rangle\in\mathbb{E}_n^d} g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle u,v\rangle\rangle} - g(\langle u,v\rangle)\mathbb{1}_{o(\langle u,v\rangle)=\langle\langle v,u\rangle\rangle}\,.$$

This is the amount of fluid that crosses C from F_1 to F_2 if the fluid respects the stream (g,o). The maximal flow through C from F_1 to F_2 is the supremum of this quantity over all possible choices of streams

$$\phi(F_1\to F_2 \text{ in } C) \,=\, \sup\{\mathrm{flow}(g,o)\,|\,(g,o)\in\mathcal{S}\}\,.$$

We denote by

$$\phi_n \,=\, \phi(\Gamma_n^1\to\Gamma_n^2 \text{ in } \Omega_n)$$

the maximal flow from Γ_n^1 to Γ_n^2 in Ω_n. We will investigate the asymptotic behaviour of ϕ_n/n^{d-1} for large n. More precisely, we will show that $(\phi_n/n^{d-1})_{n\geq 1}$ converges towards a positive constant ϕ_Ω (depending on Ω, Γ^1, Γ^2, Λ and d) when n goes to infinity, that the lower deviations are of surface order, and that the upper deviations are of volume order. The description of ϕ_Ω will be given in section 1.2. Here we state the precise theorems:

THEOREM 23. *If the law Λ of the capacity of an edge admits an exponential moment:*

$$\exists\theta>0 \qquad \int_{\mathbb{R}^+} e^{\theta x}d\Lambda(x) \,<\, +\infty\,,$$

and if $\Lambda(0)<1-p_c(d)$, then there exists a finite constant ϕ_Ω such that for all $\lambda\in[0,\phi_\Omega[$,

$$\limsup_{n\to\infty}\frac{1}{n^{d-1}}\log\mathbb{P}[\phi_n\leq\lambda n^{d-1}] \,<\, 0\,.$$

THEOREM 24. *We suppose that $d(\Gamma^1,\Gamma^2)>0$. If the law Λ of the capacity of an edge admits an exponential moment:*

$$\exists\theta>0 \qquad \int_{\mathbb{R}^+} e^{\theta x}d\Lambda(x) \,<\, +\infty\,,$$

and if $\Lambda(0)<1-p_c(d)$, then there exists a finite constant $\widetilde{\phi_\Omega}$ such that for all $\lambda\in]\widetilde{\phi_\Omega},+\infty[$,

$$\limsup_{n\to\infty}\frac{1}{n^{d}}\log\mathbb{P}[\phi_n\geq\lambda n^{d-1}] \,<\, 0\,.$$

REMARK 29. In the theorem 24 we need to impose that $d(\Gamma^1,\Gamma^2)>0$ because otherwise we cannot be sure that $\widetilde{\phi_\Omega}<\infty$, as we will see at the beginning of section 3. Moreover, if $d(\Gamma^1,\Gamma^2)=0$, there exists a set of edges of constant cardinality (not depending on n) that forms paths from Γ_n^1 to Γ_n^2 through Ω_n for all n along the common boundary of Γ^1 and Γ^2, and so it is sufficient for these edges to have a huge capacity to obtain that ϕ_n is abnormally big too. Thus, we cannot hope to obtain upper large deviations of volume order.

THEOREM 25. *In addition to the hypothesis that Γ is piecewise of class $\mathcal{C}^1$, we suppose that Ω is a Lipschitz domain, i.e., its boundary Γ can be locally represented by the graph of a Lipschitz function defined on some open ball of $\mathbb{R}^{d-1}$, and that Γ is included in the union of a finite number of oriented hypersurfaces $\mathcal{S}_1,...,\mathcal{S}_p$ of class $\mathcal{C}^1$ which are transverse to each other. In addition to the hypothesis that Γ^1 and Γ^2 are open in Γ, we suppose that their relative boundaries $\partial_\Gamma\Gamma^1$ and $\partial_\Gamma\Gamma^2$ in Γ have null $\mathcal{H}^{d-1}$ measure, and that $d(\Gamma^1,\Gamma^2)>0$. Then*

$$\phi_\Omega \,=\, \widetilde{\phi_\Omega}\,.$$

Moreover, if $\Lambda(0) < 1 - p_c(d)$ *and* Λ *admits a moment of order* 1*, i.e.,*

$$\int_{[0,+\infty[} x\, d\Lambda(x) \,<\, \infty\,,$$

then this constant is strictly positive.

As a corollary of Theorem 23, Theorem 24 and Theorem 25, we immediately obtain the following law of large numbers:

THEOREM 26. *We suppose that* Ω *is a Lipschitz domain and that* Γ *is included in the union of a finite number of oriented hypersurfaces* $\mathcal{S}_1,...,\mathcal{S}_r$ *of class* $\mathcal{C}^1$ *which are transverse to each other. We also suppose that* Γ^1 *and* Γ^2 *are open in* Γ*, that their relative boundaries* $\partial_\Gamma\Gamma^1$ *and* $\partial_\Gamma\Gamma^2$ *in* Γ *have null* $\mathcal{H}^{d-1}$ *measure, and that* $d(\Gamma^1,\Gamma^2)>0$*. We suppose that the law* Λ *of the capacity of an edge admits an exponential moment:*

$$\exists\theta>0 \qquad \int_{\mathbb{R}^+} e^{\theta x} d\Lambda(x) \,<\, +\infty\,,$$

and that $\Lambda(0) < 1-p_c(d)$*. Then there exists a positive and finite constant* $\phi_\Omega > 0$ *such that*

$$\lim_{n\to\infty} \frac{\phi_n}{n^{d-1}} \,=\, \phi_\Omega \quad \textit{a.s.}$$

REMARK 30. The large deviations we obtain are of the relevant order. Indeed, if all the edges in Ω_n have a capacity which is abnormally big, then the maximal flow ϕ_n will be abnormally big too. The probability for these edges to have an abnormally large capacity is of order $\exp -Cn^d$ for a constant C, because the number of edges in Ω_n is $C'n^d$ for a constant C'. On the opposite, if all the edges in a flat layer that separates Γ_n^1 from Γ_n^2 in Ω_n have abnormally small capacity, then ϕ_n will be abnormally small. Since the cardinality of such a set of edges is $D'n^{d-1}$ for a constant D', the probability of this event is of order $\exp -Dn^{d-1}$ for a constant D.

REMARK 31. The condition $\Lambda(0) < 1-p_c(d)$ is relevant. Indeed, Zhang proved in [**58**] that in the particular case where $d=3$ and Ω is a straight cube of bottom Γ^1 and top Γ^2, if Λ admits an exponential moment and $\Lambda(0) = 1-p_c(d)$, then $\lim_{n\to\infty}\phi_n/n^{d-1} = 0$ a.s. The heuristic is the following: if $\Lambda(0)\geq 1-p_c(d)$, then the edges of capacity strictly positive do not percolate, and therefore they cannot convey a strictly positive amount of fluid through Ω when n goes to infinity. However, Kesten stated the first results about maximal flows in this model in [**41**] under a stronger hypothesis on $\Lambda(\{0\})$. It is only in 2007 that Zhang succeeded in relaxing the constraint on Λ in his remarkable article [**59**].

1.2. Computation of ϕ_Ω and $\widetilde{\phi_\Omega}$.

1.2.1. *Geometric notations.* We start with some geometric definitions. For a subset X of $\mathbb{R}^d$, we denote by $\mathcal{H}^s(X)$ the s-dimensional Hausdorff measure of X (we will use $s=d-1$ and $s=d-2$). The r-neighbourhood $\mathcal{V}_i(X,r)$ of X for the distance d_i, that can be the Euclidean distance if $i=2$ or the L^∞-distance if $i=\infty$, is defined by

$$\mathcal{V}_i(X,r) \,=\, \{y\in\mathbb{R}^d \,|\, d_i(y,X)<r\}\,.$$

If X is a subset of $\mathbb{R}^d$ included in an hyperplane of $\mathbb{R}^d$ and of codimension 1 (for example a non degenerate hyperrectangle), we denote by $\mathrm{hyp}(X)$ the hyperplane spanned by X, and we denote by $\mathrm{cyl}(X,h)$ the cylinder of basis X and of height $2h$ defined by

$$\mathrm{cyl}(X,h) \,=\, \{x+tv \,|\, x\in X\,,\, t\in[-h,h]\}\,,$$

where v is one of the two unit vectors orthogonal to $\mathrm{hyp}(X)$ (see figure 2).

For $x\in\mathbb{R}^d$, $r\geq 0$ and a unit vector v, we denote by $B(x,r)$ the closed ball centered at x of radius r, by $\mathrm{disc}(x,r,v)$ the closed disc centered at x of radius r and normal vector v, and by

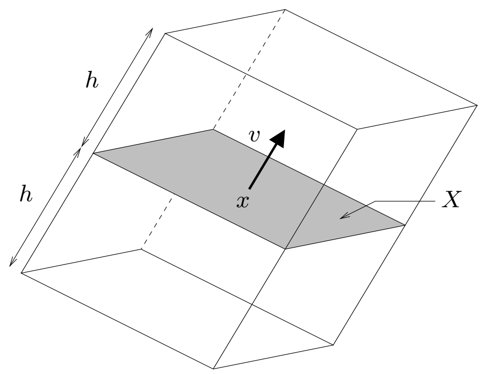

FIGURE 2. Cylinder $\mathrm{cyl}(X, h)$.

$B^+(x, r, v)$ (respectively $B^-(x, r, v)$) the upper (respectively lower) half part of $B(x, r)$ where the direction is determined by v (see figure 3), i.e.,

$$B^+(x, r, v) = \{y \in B(x, r) \,|\, (y - x) \cdot v \geq 0\},$$
$$B^-(x, r, v) = \{y \in B(x, r) \,|\, (y - x) \cdot v \leq 0\}.$$

We denote by α_d the volume of a unit ball in $\mathbb{R}^d$, and α_{d-1} the $\mathcal{H}^{d-1}$ measure of a unit disc.

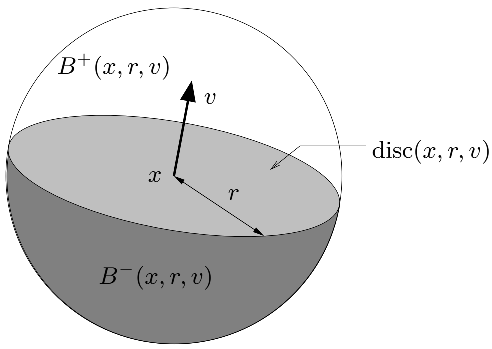

FIGURE 3. Ball $B(x, r)$.

1.2.2. *Flow in a cylinder.* Here are some particular definitions of the flow through a box. It is important to know them, because all our work consists in comparing the maximal flow ϕ_n in Ω_n with the maximal flows in small cylinders. Let A be a non degenerate hyperrectangle, i.e., a box of dimension $d-1$ in $\mathbb{R}^d$. All hyperrectangles will be supposed to be closed in $\mathbb{R}^d$. We denote by v one of the two unit vectors orthogonal to $\mathrm{hyp}(A)$. For h a positive real number, we consider the cylinder $\mathrm{cyl}(A, h)$. The set $\mathrm{cyl}(A, h) \smallsetminus \mathrm{hyp}(A)$ has two connected components, which we denote by $\mathcal{C}_1(A, h)$ and $\mathcal{C}_2(A, h)$. For $i = 1, 2$, let A_i^h be the set of the points in $\mathcal{C}_i(A, h) \cap \mathbb{Z}_n^d$ which have a nearest neighbour in $\mathbb{Z}_n^d \smallsetminus \mathrm{cyl}(A, h)$:

$$A_i^h = \{x \in \mathcal{C}_i(A, h) \cap \mathbb{Z}_n^d \,|\, \exists y \in \mathbb{Z}_n^d \smallsetminus \mathrm{cyl}(A, h),\ \langle x, y\rangle \in \mathbb{E}_n^d\}.$$

Let $T(A, h)$ (respectively $B(A, h)$) be the top (respectively the bottom) of $\mathrm{cyl}(A, h)$, i.e.,

$$T(A, h) = \{x \in \mathrm{cyl}(A, h) \,|\, \exists y \notin \mathrm{cyl}(A, h),\ \langle x, y\rangle \in \mathbb{E}_n^d \text{ and } \langle x, y\rangle \text{ intersects } A + hv\}$$

and

$$B(A, h) = \{x \in \mathrm{cyl}(A, h) \,|\, \exists y \notin \mathrm{cyl}(A, h),\ \langle x, y\rangle \in \mathbb{E}_n^d \text{ and } \langle x, y\rangle \text{ intersects } A - hv\}.$$

For a given realisation $(t(e), e \in \mathbb{E}_n^d)$ we define the variable $\tau(A, h) = \tau(\mathrm{cyl}(A, h), v)$ by

$$\tau(A, h) = \tau(\mathrm{cyl}(A, h), v) = \phi(A_1^h \to A_2^h \text{ in } \mathrm{cyl}(A, h)),$$

and the variable $\phi(A,h) = \phi(\mathrm{cyl}(A,h), v)$ by

$$\phi(A,h) \,=\, \phi(\mathrm{cyl}(A,h), v) \,=\, \phi(B(A,h) \to T(A,h) \text{ in } \mathrm{cyl}(A,h))\,,$$

where $\phi(F_1 \to F_2 \; in \; C)$ is the maximal flow from F_1 to F_2 in C, for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d$) defined previously.

1.2.3. *Max-flow min-cut theorem.* The maximal flow $\phi(F_1 \to F_2 \text{ in } C)$ can be expressed differently thanks to the max-flow min-cut theorem (see [**12**]). We need some definitions to state this result. A path on the graph $\mathbb{Z}_n^d$ from v_0 to v_m is a sequence $(v_0, e_1, v_1, ..., e_m, v_m)$ of vertices $v_0, ..., v_m$ alternating with edges $e_1, ..., e_m$ such that v_{i-1} and v_i are neighbours in the graph, joined by the edge e_i, for i in $\{1, ..., m\}$. A set E of edges in C is said to cut F_1 from F_2 in C if there is no path from F_1 to F_2 in $C \setminus E$. We call E an (F_1, F_2)-cut if E cuts F_1 from F_2 in C and if no proper subset of E does. With each set E of edges we associate its capacity which is the variable

$$V(E) \,=\, \sum_{e\in E} t(e)\,.$$

The max-flow min-cut theorem states that

$$\phi(F_1 \to F_2 \; in \; C) \,=\, \min\{\, V(E) \,|\, E \text{ is a } (F_1, F_2)\text{-cut}\,\}\,.$$

1.2.4. *Definition of ν.* Only in this section, we consider the standard first passage percolation model on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ instead of the rescaled graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$. We present here some important theorems that have been proved about maximal flows.

The asymptotic behaviour of the variable $\tau(nA, h(n))$ for large n, for A a non degenerate hyperrectangle of orthogonal unit vector v, and h a height function with values in $\mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, has already been studied. By a subadditive argument, we know that as soon as the capacities of the edges are in L^1, there exists a constant $\nu(v)$ (depending on Λ, d and v but not on h and on A itself) such that

$$\lim_{n\to\infty} \frac{\mathbb{E}[\tau(nA, h(n))]}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(v)\,.$$

Moreover, under some added hypotheses on A and v, or on F, we know that

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(v) \qquad \text{a.s.}$$

(see the introduction of the thesis for more details).

Kesten and Zhang have studied the maximal flow between the top and the bottom of straight cylinders. Let us denote by $D(\mathbf{k}, m)$ the cylinder

$$D(\mathbf{k}, m) \,=\, \prod_{i=1}^{d-1} [0, k_i] \times [0, m]\,,$$

where $\mathbf{k} = (k_1, ..., k_{d-1}) \in \mathbb{R}^{d-1}$. We denote by $\phi(\mathbf{k}, m)$ the maximal flow in $D(\mathbf{k}, m)$ from its top $\prod_{i=1}^{d-1}[0, k_i] \times \{m\}$ to its bottom $\prod_{i=1}^{d-1}[0, k_i] \times \{0\}$. Kesten proved in [**41**] the following result:

THEOREM 27 (Kesten). *Let $d = 3$. We suppose that $\Lambda(0) < p_0$ for some fixed $p_0 \geq 1/27$, and that*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x}\, d\Lambda(x) \,<\, \infty\,.$$

If $m = m(\mathbf{k})$ goes to infinity with $k_1 \geq k_2$ in such a way that

$$\exists \delta > 0 \qquad \lim_{k_1 \geq k_2 \to \infty} k^{-1+\delta} \log m(\mathbf{k}) \,=\, 0\,,$$

then

$$\lim_{k_1 \geq k_2 \to \infty} \frac{\phi(\mathbf{k}, m)}{k_1 k_2} \,=\, \nu((0,0,1)) \qquad \textit{a.s. and in } L^1\,.$$

Moreover, if $\Lambda(0) > 1 - p_c(d)$*, where* $p_c(d)$ *is the critical parameter for the standard percolation model by edges on* $\mathbb{Z}^d$*, and if*

$$\int_{[0,+\infty[} x^6 \, d\Lambda(x) \, < \, \infty \, ,$$

there exists a constant $C = C(F) < \infty$ *such that for all* $m = m(\mathbf{k})$ *that goes to infinity with* $k_1 \geq k_2$ *and satisfies*

$$\liminf_{k_1 \geq k_2 \to \infty} \frac{m(\mathbf{k})}{k_1 k_2} \, > \, C \, ,$$

for all $k_1 \geq k_2$ *sufficiently large, we have*

$$\phi(\mathbf{k}, m) \, = \, 0 \qquad \textit{a.s.}$$

Zhang improved this result in [**59**] where he proved the following theorem:

THEOREM 28 (Zhang). *Let* $d \geq 2$*. We suppose that*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, d\Lambda(x) \, < \, \infty \, .$$

Then for all $m = m(\mathbf{k})$ *that goes to infinity when all the* k_i*,* $i = 1, ..., d-1$ *go to infinity in such a way that*

$$\exists \delta \in]0,1] \qquad \log m(\mathbf{k}) \, \leq \, \max_{i=1,\dots,d-1} (k_i^{1-\delta}) \, ,$$

we have

$$\lim_{k_1,\dots,k_{d-1} \to \infty} \frac{\phi(\mathbf{k}, m)}{\prod_{i=1}^{d-1} k_i} \, = \, \nu((0, ..., 0, 1)) \qquad \textit{a.s. and in } L^1 \, .$$

Moreover, this limit is positive if and only if $\Lambda(0) < 1 - p_c(d)$*.*

To show this theorem, Zhang obtains first a control on the number of edges in a minimal cutset. We will present and use this result in section 2.2.

Garet studied in [**30**] the maximal flow $\sigma(A)$ between a convex bounded set A and infinity in the case $d = 2$. By an extension of the max flow - min cut theorem to non finite graphs, Garet proves in [**30**] that this maximal flow is equal to the minimal capacity of a set of edges that cuts all paths from A to infinity. Let ∂A be the boundary of A, and $\partial^* A$ the set of the points $x \in \partial A$ at which A admits a unique exterior normal unit vector $v_A(x)$ in a measure theoretic sense (see [**19**], section 13, for a precise definition). If A is a convex set, the set $\partial^* A$ is also equal to the set of the points $x \in \partial A$ at which A admits a unique exterior normal vector in the classical sense, and this vector is $v_A(x)$. Garet proved the following theorem:

THEOREM 29 (Garet). *Let* $d = 2$*. We suppose that* $\Lambda(0) < 1 - p_c(2) = 1/2$ *and that*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, d\Lambda(x) \, < \, \infty \, .$$

Then for all convex bounded set A *containing* 0 *in its interior, we have*

$$\lim_{n \to \infty} \frac{\sigma(nA)}{n} \, = \, \int_{\partial^* A} \nu(v_A(x)) d\mathcal{H}^1(x) \, = \, \mathcal{I}(A) \, > \, 0 \qquad \textit{a.s.}$$

Moreover, for all $\varepsilon > 0$*, there exist constants* C_1*,* $C_2 > 0$ *depending on* ε *and* Λ *such that*

$$\forall n \geq 0 \qquad \mathbb{P} \left[\frac{\sigma(nA)}{n\mathcal{I}(A)} \notin]1-\varepsilon, 1+\varepsilon[\right] \, \leq \, C_1 \exp(-C_2 n) \, .$$

Nevertheless, the maximal flow from the top to the bottom of a tilted cylinder for $d \geq 3$ was not studied yet. In fact, the lack of symmetry of the graph induced by the slope of the box is a major issue to extend the result concerning straight cylinders to tilted cylinders. The theorem of Garet was not extended to dimension $d \geq 3$ either.

We recall some geometric properties of $\nu : v \in S^{d-1} \mapsto \nu(v)$, under the only condition on F that $\mathbb{E}(t(e)) < \infty$. They have been proved in the section 4.5 of the Chapter 5 of this thesis. There exists a unit vector v_0 such that $\nu(v_0) = 0$ if and only if for all unit vector v, $\nu(v) = 0$, and it happens if and only if $\Lambda(\{0\}) \geq 1 - p_c(d)$. Moreover, ν satisfies the weak triangle inequality, i.e., if (ABC) is a non degenerate triangle in $\mathbb{R}^d$ and v_A, v_B and v_C are the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane spanned by A, B, C, then

$$\mathcal{H}^1([AB])\nu(v_C) \leq \mathcal{H}^1([AC])\nu(v_B) + \mathcal{H}^1([BC])\nu(v_A)\,.$$

This implies that the homogeneous extension ν_0 of ν to $\mathbb{R}^d$, defined by $\nu_0(0) = 0$ and for all w in $\mathbb{R}^d$,

$$\nu_0(w) = |w|_2 \nu(w/|w|_2)\,,$$

is a convex function; in particular, since ν_0 is finite, it is continuous on $\mathbb{R}^d$. We denote by $\nu_{\min}$ (respectively $\nu_{\max}$) the infimum (respectively supremum) of ν on S^{d-1}.

We consider the rescaled graph $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$ again, and for the rest of the chapter.

1.2.5. *Continuous min-cut.* We give here a definition of ϕ_Ω and $\widetilde{\phi_\Omega}$ in term of the map ν. For a subset F of $\mathbb{R}^d$, we define the perimeter of F in Ω by

$$\mathcal{P}(F,\Omega) = \sup\left\{\int_F \operatorname{div} f(x) d\mathcal{L}^d(x),\ f \in \mathcal{C}_c^\infty(\Omega, B(0,1))\right\}\,,$$

where $\mathcal{C}_c^\infty(\Omega, B(0,1))$ is the set of the functions of class $\mathcal{C}^\infty$ from $\mathbb{R}^d$ to $B(0,1)$, the ball centered at 0 and of radius 1 in $\mathbb{R}^d$, having a compact support included in Ω, and div is the usual divergence operator. The perimeter $\mathcal{P}(F)$ of F is defined as $\mathcal{P}(F,\mathbb{R}^d)$. We denote by ∂F the boundary of F, and by $\partial^* F$ the reduced boundary of F. At any point x of $\partial^* F$, the set F admits a unit exterior normal vector $v_F(x)$ at x in a measure theoretic sense (for definitions see for example [**19**] section 13). For all $F \subset \mathbb{R}^d$ of finite perimeter in Ω, we define

$$\begin{aligned}\mathcal{I}_\Omega(F) = &\int_{\partial^* F \cap \Omega} \nu(v_F(x)) d\mathcal{H}^{d-1}(x) + \int_{\Gamma^2 \cap \partial^*(F\cap\Omega)} \nu(v_F(x)) d\mathcal{H}^{d-1}(x)\\ &+ \int_{\Gamma^1 \cap \partial^*(\Omega \smallsetminus F)} \nu(v_\Omega(x)) d\mathcal{H}^{d-1}(x)\,.\end{aligned}$$

If $\mathcal{P}(F,\Omega) = +\infty$, we define $\mathcal{I}_\Omega(F) = +\infty$. Finally, we define

$$\begin{aligned}\phi_\Omega &= \inf\{\mathcal{I}_\Omega(F)\,|\, F \subset \mathbb{R}^d\}\\ &= \inf\{\mathcal{I}_\Omega(F)\,|\, F \subset \Omega\}\,.\end{aligned}$$

When a hypersurface $\mathcal{S}$ is piecewise of class $\mathcal{C}^1$, we say that $\mathcal{S}$ is transverse to Γ if for all $x \in \mathcal{S} \cap \Gamma$, the normal unit vectors to $\mathcal{S}$ and Γ at x are not collinear; if the normal vector to $\mathcal{S}$ (respectively to Γ) at x is not well defined, this property must be satisfied by all the vectors which are limits of normal unit vectors to $\mathcal{S}$ (respectively Γ) at $y \in \mathcal{S}$ (respectively $y \in \Gamma$) when we send y to x - there is at most a finite number of such limits. We say that a subset P of $\mathbb{R}^d$ is polyhedral if its boundary ∂P is included in the union of a finite number of hyperplanes. For each point x of such a set P which is on the interior of one face of ∂P, we denote by $v_P(x)$ the exterior unit vector orthogonal to P at x. For $A \subset \mathbb{R}^d$, we denote by $\overset{\circ}{A}$ the interior of A. We define $\widetilde{\phi_\Omega}$ by

$$\begin{aligned}\widetilde{\phi_\Omega} &= \inf\left\{\int_{\partial P\cap\Omega} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) \,\middle|\, \begin{array}{c} P \subset \mathbb{R}^d\,,\ \overline{\Gamma^1} \subset \overset{\circ}{P}\,,\ \overline{\Gamma^2} \subset \overset{\circ}{\widehat{\mathbb{R}^d \smallsetminus P}} \\ P \text{ is polyhedral}\,,\ \partial P \text{ is transverse to } \Gamma \end{array}\right\}\\ &= \inf\left\{\mathcal{I}_\Omega(P) \,\middle|\, \begin{array}{c} P \subset \mathbb{R}^d\,,\ \overline{\Gamma^1} \subset \overset{\circ}{P}\,,\ \overline{\Gamma^2} \subset \overset{\circ}{\widehat{\mathbb{R}^d \smallsetminus P}} \\ P \text{ is polyhedral}\,,\ \partial P \text{ is transverse to } \Gamma \end{array}\right\}.\end{aligned}$$

See figure 4 to have an example of such a polyhedral set P. Theorem 25 is in fact a result of

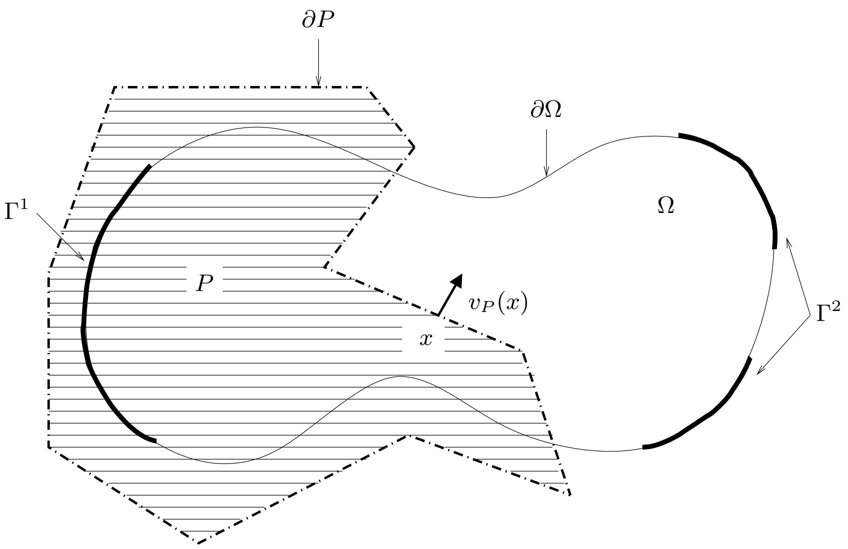

FIGURE 4. Polyhedral set P as in the definition of $\widetilde{\phi_\Omega}$.

polyhedral approximation of sets having finite perimeter.

The definitions of the constants ϕ_Ω and $\widetilde{\phi_\Omega}$ are not very intuitive. We propose to define the notion of continuous cutset to have a better understanding of these constants. We say that $\mathcal{S} \subset \mathbb{R}^d$ cuts Γ^1 from Γ^2 in $\overline{\Omega}$ if every continuous path from Γ^1 to Γ^2 in $\overline{\Omega}$ intersects $\mathcal{S}$. Actually, if P is a polyhedral set of $\mathbb{R}^d$ such that

$$\overline{\Gamma^1} \subset \overset{\circ}{P} \quad \text{and} \quad \overline{\Gamma^2} \subset \overset{\circ}{\widehat{\mathbb{R}^d \setminus P}},$$

then $\partial P \cap \overline{\Omega}$ is a continuous cutset from Γ^1 to Γ^2 in $\overline{\Omega}$. Moreover, for any set $F \subset \mathbb{R}^d$ of finite perimeter in Ω, the set

$$\overline{\Omega} \cap \partial\left((F \cup \Gamma^1) \setminus \Gamma^2\right)$$

is also a continuous cutset separating Γ^1 from Γ^2 in $\overline{\Omega}$ (figure 5 shows the localisation of this continuous cutset). Since $\nu(v)$ is the average amount of fluid that can cross a hypersurface of

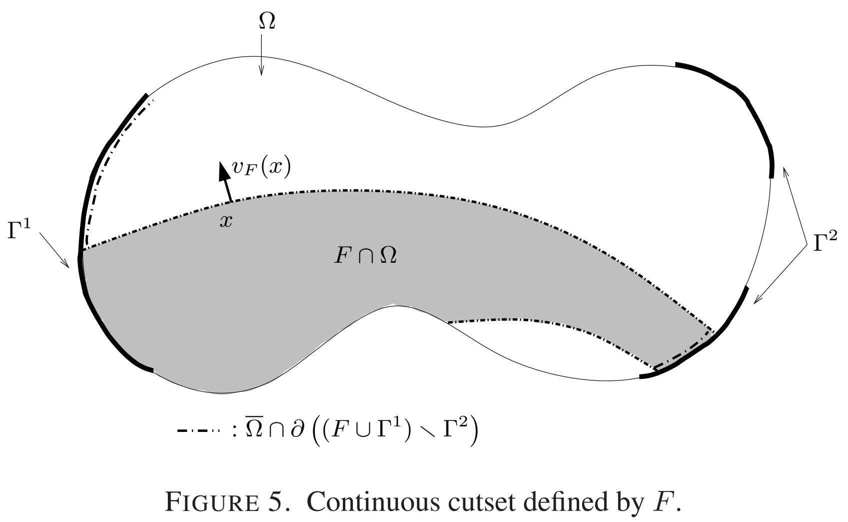

FIGURE 5. Continuous cutset defined by F.

area one in the direction v per unit of time, it can be interpreted as the capacity of a unitary hypersurface. Thus $\mathcal{I}_\Omega(F)$ can be interpreted as the capacity of the continuous cutset defined by F. The constants ϕ_Ω and $\widetilde{\phi_\Omega}$ are solutions of min cuts problems, because they are equal to the infimum of the capacity of a continuous cutset that satisfies some specific properties. We can define two other constants, that are solutions of possibly more intuitive min cuts problems. If $\mathcal{S}$ is a hypersurface which is piecewise of class $\mathcal{C}^1$, we denote by $v_{\mathcal{S}}(x)$ one of the two normal unit vectors to A at x for every point x at which $\mathcal{S}$ is regular. The $\mathcal{H}^{d-1}$ measure of the points at which $\mathcal{S}$ is not regular is null. We define

$$\hat{\phi}_\Omega = \inf \left\{ \int_{\mathcal{S}\cap\overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \,\middle|\, \begin{array}{c} \mathcal{S} \text{ hypersurface piecewise of class } \mathcal{C}^1 \\ \mathcal{S} \text{ cuts } \Gamma^1 \text{ from } \Gamma^2 \text{ in } \overline{\Omega} \end{array} \right\}$$

and

$$\check{\phi}_\Omega = \inf \left\{ \int_{\mathcal{S}\cap\overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \,\middle|\, \begin{array}{c} \mathcal{S} \text{ polyhedral hypersurface} \\ \mathcal{S} \text{ cuts } \Gamma^1 \text{ from } \Gamma^2 \text{ in } \overline{\Omega} \end{array} \right\}.$$

We remark that by definition,

$$\hat{\phi}_\Omega \leq \check{\phi}_\Omega \leq \widetilde{\phi_\Omega}.$$

We claim that $\phi_\Omega \leq \hat{\phi}_\Omega$. Let $\mathcal{S}$ be a hypersurface which is piecewise of class $\mathcal{C}^1$, which cuts Γ^1 from Γ^2 in $\overline{\Omega}$, and such that

$$\int_{\mathcal{S}\cap\overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \leq \hat{\phi}_\Omega + \eta$$

for some positive η. Let F be the set of the points of $\overline{\Omega} \smallsetminus \mathcal{S}$ that can be joined to a point of Γ^1 by a continuous path. Then

$$(\partial F \cap \Omega) \cup (\Gamma^1 \cap \partial(\Omega \smallsetminus F)) \cup (\Gamma^2 \cap \partial(F \cap \Omega)) \subset \mathcal{S} \cap \overline{\Omega}.$$

Thus F is of finite perimeter in Ω, and $\mathcal{I}_\Omega(F)$ satisfies

$$\mathcal{I}_\Omega(F) \leq \int_{\mathcal{S}\cap\overline{\Omega}} \nu(v_{\mathcal{S}}(x)) d\mathcal{H}^{d-1}(x) \leq \hat{\phi}_\Omega + \eta.$$

Thus we have proved that

$$\phi_\Omega \leq \hat{\phi}_\Omega \leq \check{\phi}_\Omega \leq \widetilde{\phi}_\Omega.$$

If all the hypotheses of theorem 25 are satisfied, then all these constants are equal.

We remark that the capacity $\mathcal{I}_\Omega$ of a continuous cutset is exactly the same as the one defined by Garet in [**30**] in dimension two, except that we consider a maximal flow through a bounded domain, so our capacity is adapted to the problems of boundaries that arise.

2. Lower large deviations

2.1. Sketch of the proof. We are studying the lower large deviations of ϕ_n/n^{d-1}: they are controlled by what happens around a minimal cutset. First, we will use the estimate of the number of edges in a minimal cutset made by Zhang in [**59**] to restrict the problem to cutsets having a number of edges at most cn^{d-1} for a constant c; we can then conclude that the minimal cutset is "near" the boundary of a subset F of Ω belonging to a compact space. By making an adequate covering of this space, we need only to deal with a finite number of sets and their neighbourhoods. We will then cover the boundary of such a set F by balls of very small radius, such that ∂F is "almost flat" in each ball; we will also show that if ϕ_n is smaller than $\phi_\Omega(1-\varepsilon)n^{d-1}$ for some positive ε, then some local event happens in each ball of the covering of ∂F (this event will be denoted by $G(B, v_F(x))$ for the ball B centered at $x \in \partial F$). After that, we will construct a link between this local event in a ball and the fact that the maximal flow through a cylinder (included in the ball) is abnormally small. The lower large deviations for the maximal flow through a cylinder are already known (see [**52**] or the Chapter 5 of the thesis). Finally, we calibrate the constants to get Theorem 23.

This proof is largely inspired by the methods used to study the Wulff crystal in Ising model in dimension $d \geq 3$ (see for example [**19**]).

2.2. Number of edges in a minimal cutset and compactness. We consider a (Γ_n^1, Γ_n^2)-cut $\mathcal{E}_n$ in Ω_n of minimal capacity, i.e., $\phi_n = V(\mathcal{E}_n)$, and of minimal number of edges (if there are more than one such cutset, we select one of them by a deterministic algorithm). According to Theorem 1 in [**59**], adapted to our case as said in Remark 2 in [**59**], we know that:

THEOREM 30 (Zhang). *If the law of the capacity of the edges admits an exponential moment, and if $\Lambda(0) < 1 - p_c(d)$, then there exist constants $\beta_0 = \beta_0(\Lambda, d)$, $C_i = C_i(\Lambda, d)$ for $i = 1, 2$ and $N = N(\Lambda, d, \Omega, \Gamma, \Gamma^1, \Gamma^2)$ such that for all $\beta \geq \beta_0$, for all $n \geq N$, we have*

$$\mathbb{P}[\mathrm{card}(\mathcal{E}_n) \geq \beta n^{d-1}] \,\leq\, C_1 \exp(-C_2 \beta n^{d-1}) \,.$$

We will always consider such large $n \geq N$ during the section 2. Thus with high probability the (Γ_n^1, Γ_n^2)-cut $\mathcal{E}_n$ has not "too much" edges. We want now to change a little bit our point of view in order to work with a subset of $\mathbb{R}^d$ rather than the cutset $\mathcal{E}_n$. We define for each edge e the variable $t'(e) = \mathbb{1}_{\{e \notin \mathcal{E}_n\}}$, and the set $\widetilde{E}_n \subset \mathbb{Z}_n^d$ by

$$\widetilde{E}_n = \{x \in \Omega_n \,|\, x \text{ is in an open cluster connected to } \Gamma_n^1 \text{ for the percolation process } (t'(e))_{e \in \Omega_n}\}.$$

Then the edge boundary $\partial^e \widetilde{E}_n$ of $\widetilde{E}_n$, defined by

$$\partial^e \widetilde{E}_n \,=\, \{e = \langle x, y \rangle \in \mathbb{Z}_n^d \cap \Omega_n \,|\, x \in \widetilde{E}_n \text{ and } y \notin \widetilde{E}_n\} \,,$$

is exactly equal to $\mathcal{E}_n$. We consider now the "non discrete version" E_n of $\widetilde{E}_n$ defined by

$$E_n \,=\, \{x \in \Omega \,|\, d_\infty(x, \widetilde{E}_n) \leq 1/(2n)\} \,=\, \left(\widetilde{E}_n + [-1/(2n), 1/(2n)]^d\right) \cap \Omega \,.$$

For all $F \subset \mathbb{R}^d$, we recall that the perimeter of F in Ω is defined by

$$\mathcal{P}(F, \Omega) \,=\, \sup \left\{ \int_F \mathrm{div}\, f(x) d\mathcal{L}^d(x),\ f \in \mathcal{C}_c^\infty(\Omega, B(0,1)) \right\} \,.$$

We know that if $\mathrm{card}(\mathcal{E}_n) \leq \beta n^{d-1}$, then $\mathcal{P}(E_n, \Omega) \leq \mathcal{P}(E_n) \leq \beta$.

We define

$$\mathcal{C}_\beta \,=\, \{F \subset \Omega \,|\, \mathcal{P}(F, \Omega) \leq \beta\} \,,$$

endowed with the topology L^1 associated to the distance $d(F, F') = \mathcal{L}^d(F \triangle F')$, where $F \triangle F'$ is the symmetric difference between these two sets. For this topology the set $\mathcal{C}_\beta$ is compact. With every F in $\mathcal{C}_\beta$ we associate a positive ε_F, that we will choose later. The collection of sets $\mathcal{V}(F, \varepsilon_F)$, $F \in \mathcal{C}_\beta$, where $\mathcal{V}(F, \varepsilon_F)$ is the neighbourhood of F of size ε_F for the distance defined previously, covers $\mathcal{C}_\beta$ so we can extract a finite covering: $\mathcal{C}_\beta \subset \cup_{i=1\dots N} \mathcal{V}(F_i, \varepsilon_{F_i})$. We then obtain that for a fixed $\beta \geq \beta_0$, for all λ we have

$$\begin{aligned} \mathbb{P}[\phi_n \leq \lambda n^{d-1}] \,&\leq\, e^{-\beta n^{d-1}} + \mathbb{P}[V(\mathcal{E}_n) \leq \lambda n^{d-1} \text{ and } \mathcal{P}(E_n, \Omega) \leq \beta] \\ &\leq\, e^{-\beta n^{d-1}} + \sum_{i=1}^N \mathbb{P}[V(\mathcal{E}_n) \leq \lambda n^{d-1} \text{ and } \mathcal{L}^d(E_n \triangle F_i) \leq \varepsilon_i] \,. \end{aligned}$$

It remains to study

$$\mathbb{P}[V(\mathcal{E}_n) \leq \lambda n^{d-1} \text{ and } \mathcal{L}^d(E_n \triangle F) \leq \varepsilon_F]$$

for a generic F in $\mathcal{C}_\beta$ and the corresponding ε_F.

2.3. Covering of ∂F by balls.

2.3.1. *Geometric tools.* We recall an important result about the Minkowski content of a subset of $\mathbb{R}^d$ (see for example Appendix A in [**18**]). Whenever E is a closed $(d-1)$-rectifiable subset of $\mathbb{R}^d$ (i.e., there exists a Lipschitz function mapping some bounded subset of $\mathbb{R}^{d-1}$ onto E), the

Minkowski content of E, defined by

$$\lim_{r\to 0} \frac{1}{2r}\mathcal{L}^d(\mathcal{V}_2(E,r))\,,$$

exists and is equal to $\mathcal{H}^{d-1}(E)$.

We will also use the Vitali covering theorem for $\mathcal{H}^{d-1}$ (see theorem 1.10 in [**28**]). A collection of sets $\mathcal{U}$ is called a Vitali class for a Borel set E of $\mathbb{R}^d$ if for each $x\in E$ and $\delta>0$, there exists a set $U\in\mathcal{U}$ containing x such that $0<\operatorname{diam} U<\delta$, where $\operatorname{diam} U$ is the diameter of the set U. We now recall the Vitali covering theorem for $\mathcal{H}^{d-1}$ (see for instance [**28**], Theorem 1.10):

THEOREM 31. *Let E be a $\mathcal{H}^{d-1}$ measurable subset of $\mathbb{R}^d$ and $\mathcal{U}$ be a Vitali class of closed sets for E. Then we may select a (countable) disjoint sequence $(U_i)_{i\in I}$ from $\mathcal{U}$ such that*

$$\textit{either } \sum_{i\in I}(\operatorname{diam} U_i)^{d-1} \,=\, +\infty \textit{ or } \mathcal{H}^{d-1}(E\smallsetminus \cup_{i\in I}U_i) \,=\, 0\,.$$

If $\mathcal{H}^{d-1}(E)<\infty$, then given $\varepsilon>0$, we may also require that

$$\mathcal{H}^{d-1}(E) \,\leq\, \frac{\alpha_{d-1}}{2^{d-1}}\sum_{i\in I}(\operatorname{diam} U_i)^{d-1}\,.$$

We recall next the Besicovitch differentiation theorem in $\mathbb{R}^d$ (see for example [**6**]):

THEOREM 32. *Let $\mathfrak{M}$ be a finite positive Radon measure on $\mathbb{R}^d$. For any Borel function $f\in L^1(\mathfrak{M})$, the quotient*

$$\frac{1}{\mathfrak{M}(B(x,r))}\int_{B(x,r)} f(y)d\mathfrak{M}(y)$$

converges $\mathfrak{M}$-almost surely towards $f(x)$ as r goes to 0.

We state a result of covering that we will use in our study of the lower deviations of ϕ_n:

LEMMA 18. *Let F be a subset of Ω of finite perimeter. For every positive constants δ and η, there exists a finite family of closed disjoint balls $(B_i)_{i\in I\cup J\cup K} = (B(x_i,r_i),v_i)_{i\in I\cup J\cup K}$ such that (the vector v_i defines B_i^-)*

$$\begin{aligned}
&\forall i\in I\,,\ x_i\in\partial^*F\cap\Omega\,,\ r_i\in]0,1[\,,\ B_i\subset\Omega\,,\ \mathcal{L}^d((F\cap B_i)\triangle B_i^-)\,\leq\,\delta\alpha_d r_i^d\,,\\
&\forall i\in J\,,\ x_i\in\Gamma^1\cap\partial^*(\Omega\smallsetminus F)\,,\ r_i\in]0,1[\,,\ \partial\Omega\cap B_i\subset\Gamma^1\,,\ \mathcal{L}^d((B_i\cap\Omega)\triangle B_i^-)\,\leq\,\delta\alpha_d r_i^d\,,\\
&\forall i\in K\,,\ x_i\in\Gamma^2\cap\partial^*F\,,\ r_i\in]0,1[\,,\ \partial\Omega\cap B_i\subset\Gamma^2\,,\ \mathcal{L}^d((F\cap B_i)\triangle B_i^-)\,\leq\,\delta\alpha_d r_i^d\,,
\end{aligned}$$

and finally

$$\left|\mathcal{I}_\Omega(F)-\sum_{i\in I\cup K}\alpha_{d-1}r_i^{d-1}\nu(v_F(x_i))-\sum_{i\in J}\alpha_{d-1}r_i^{d-1}\nu(v_\Omega(x_i))\right|\,\leq\,\eta\,.$$

We will prove Lemma 18 with the help of Theorems 31 and 32, following the proof of Lemma 14.6 in [**19**]. For E a set of finite perimeter, we denote by $||\nabla_{\chi_E}||$ the measure defined by

$$\forall A \text{ Borel set in } \mathbb{R}^d \qquad ||\nabla_{\chi_E}||(A)\,=\,\mathcal{H}^{d-1}(A\cap\partial^*E)\,.$$

We consider a subset F of Ω of finite perimeter. We recall that the function $\nu: S^{d-1}\to\mathbb{R}^+$ is continuous. The map $x\in\partial^*F\cap\Omega\mapsto v_F(x)$ is $||\nabla_{\chi_F}||$-measurable, so we can apply the Besicovitch differentiation theorem in $\mathbb{R}^d$ to the maps $x\in\partial^*F\cap\Omega\mapsto\nu(v_F(x))$ and $x\in\partial^*F\cap\Omega\mapsto 1$ to obtain that for $\mathcal{H}^{d-1}$-almost all $x\in\partial^*F\cap\Omega$

$$\lim_{r\to 0}\frac{1}{\alpha_{d-1}r^{d-1}}\mathcal{H}^{d-1}(B(x,r)\cap\partial^*F\cap\Omega)\,=\,1\,,$$

$$\lim_{r\to 0}\frac{1}{\alpha_{d-1}r^{d-1}}\int_{B(x,r)\cap\partial^*F\cap\Omega}\nu(v_F(y))d\mathcal{H}^{d-1}(y)\,=\,\nu(v_F(x))\,.$$

We denote by $\mathcal{R}_1$ the set of the points of $\partial^* F \cap \Omega$ where the two preceding identities hold simultaneously, thus $\mathcal{H}^{d-1}((\partial^* F \cap \Omega) \smallsetminus \mathcal{R}_1) = 0$. Similarly, let $\mathcal{R}_2$ be the set of the points x belonging to $\Gamma^2 \cap \partial^* F$ such that

$$\lim_{r\to 0} \frac{1}{\alpha_{d-1} r^{d-1}} \mathcal{H}^{d-1}(B(x,r) \cap \Gamma^2 \cap \partial^* F) \,=\, 1\,,$$
$$\lim_{r\to 0} \frac{1}{\alpha_{d-1} r^{d-1}} \int_{B(x,r)\cap\Gamma^2\cap\partial^* F} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) \,=\, \nu(v_F(x))\,.$$

We also know that $\mathcal{H}^{d-1}((\Gamma^2 \cap \partial^* F) \smallsetminus \mathcal{R}_2) = 0$. Since the map $x \in \Gamma^1 \cap \partial^*(\Omega \smallsetminus F) \mapsto v_\Omega(x)$ is $||\nabla_{\chi\Omega}||$-measurable, the same arguments imply that the set $\mathcal{R}_3$ of the points x of $\Gamma^1 \cap \partial^*(\Omega \smallsetminus F)$ such that

$$\lim_{r\to 0} \frac{1}{\alpha_{d-1} r^{d-1}} \mathcal{H}^{d-1}(B(x,r) \cap \Gamma^1 \cap \partial^*(\Omega \smallsetminus F)) \,=\, 1\,,$$
$$\lim_{r\to 0} \frac{1}{\alpha_{d-1} r^{d-1}} \int_{B(x,r)\cap\Gamma^1\cap\partial^*(\Omega\smallsetminus F)} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) \,=\, \nu(v_\Omega(x))\,,$$

satisfies $\mathcal{H}^{d-1}(\Gamma^1 \cap \partial^*(\Omega \smallsetminus F) \smallsetminus \mathcal{R}_3) = 0$. Moreover, from the theory of sets of finite perimeter (see for example section 13 in [**19**]), we know that

$$\begin{cases} \forall x \in \partial^* F\,, & \lim_{r\to 0} r^{-d} \mathcal{L}^d(F \triangle B^-(x,r,v_F(x))) \,=\, 0\,, \\ \forall x \in \partial^*(\Omega \smallsetminus F)\,, & \lim_{r\to 0} r^{-d} \mathcal{L}^d(\Omega \triangle B^-(x,r,v_\Omega(x))) \,=\, 0\,. \end{cases}$$

We fix two parameters $\eta > 0$ and $\delta > 0$. For all $x \in \mathcal{R}_1$, there exists a positive $r(x,\eta,\delta)$ such that for all $r < r(x,\eta,\delta)$ we have

$$|\mathcal{H}^{d-1}(B(x,r) \cap \partial^* F \cap \Omega) - \alpha_{d-1} r^{d-1}| \,\leq\, \eta \alpha_{d-1} r^{d-1}\,,$$
$$\left| \frac{1}{\alpha_{d-1} r^{d-1}} \int_{B(x,r)\cap\partial^* F\cap\Omega} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \nu(v_F(x)) \right| \,\leq\, \eta\,,$$
$$\mathcal{L}^d((F \cap B(x,r)) \triangle B^-(x,r,v_F(x))) \,\leq\, \delta \alpha_d r^d \quad \text{and} \quad B(x,r) \,\subset\, \Omega\,.$$

For all x in $\mathcal{R}_2$, there exists a positive $r(x,\eta,\delta)$ such that for all $r < r(x,\eta,\delta)$ we have

$$|\mathcal{H}^{d-1}(B(x,r) \cap \Gamma^2 \cap \partial^* F) - \alpha_{d-1} r^{d-1}| \,\leq\, \eta \alpha_{d-1} r^{d-1}\,,$$
$$\left| \frac{1}{\alpha_{d-1} r^{d-1}} \int_{B(x,r)\cap\Gamma^2\cap\partial^* F} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \nu(v_F(x)) \right| \,\leq\, \eta\,,$$
$$\mathcal{L}^d((F \cap B(x,r)) \triangle B^-(x,r,v_F(x))) \,\leq\, \delta \alpha_d r^d \quad \text{and} \quad B(x,r) \cap \Gamma \,\subset\, \Gamma^2\,.$$

For all x in $\mathcal{R}_3$, there exists a positive $r(x,\eta,\delta)$ such that for all $r < r(x,\eta,\delta)$ we have

$$|\mathcal{H}^{d-1}(B(x,r) \cap \Gamma^1 \cap \partial^*(\Omega \smallsetminus F)) - \alpha_{d-1} r^{d-1}| \,\leq\, \eta \alpha_{d-1} r^{d-1}\,,$$
$$\left| \frac{1}{\alpha_{d-1} r^{d-1}} \int_{B(x,r)\cap\Gamma^1\cap\partial^*(\Omega\smallsetminus F)} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) - \nu(v_\Omega(x)) \right| \,\leq\, \eta\,,$$
$$\mathcal{L}^d((\Omega \cap B(x,r)) \triangle B^-(x,r,v_F(x))) \,\leq\, \delta \alpha_d r^d \quad \text{and} \quad B(x,r) \cap \Gamma \,\subset\, \Gamma^1\,.$$

The family of balls

$$(B(x,r), x \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3, r < r(x,\eta,\delta))$$

is a Vitali relation for $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$. By the Vitali covering theorem for $\mathcal{H}^{d-1}$, we may select from this collection of balls a finite or countable collection of disjoint balls $B(x_i, r_i), i \in I_1$ such that either

$$\mathcal{H}^{d-1}\left((\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3) \smallsetminus \bigcup_{i\in I_1} B(x_i, r_i) \right) \,=\, 0$$

or

$$\sum_{i\in I_1} r_i^{d-1} \,=\, \infty\,.$$

We know that Ω and F have finite perimeter, and that

$$(\partial^* F \cap \Omega) \cup (\Gamma^2 \cap \partial^* F) \cup (\Gamma^1 \cap \partial^*(\Omega \smallsetminus F)) \subset \Gamma \cup \partial^* F \,,$$

so

$$\begin{aligned}(1-\eta) \sum_{i \in I_1} \alpha_{d-1} r_i^{d-1} &\leq \mathcal{H}^{d-1} \left((\partial^* F \cap \Omega) \cup (\Gamma^2 \cap \partial^* F) \cup (\Gamma^1 \cap \partial^*(\Omega \smallsetminus F)) \right) \\ &\leq \mathcal{H}^{d-1}(\Gamma \cup \partial^* F) < \infty \,,\end{aligned}$$

thus the first case occurs in the Vitali covering theorem, so we may select a finite subset I_2 of I_1 such that

$$\mathcal{H}^{d-1} \left((\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3) \smallsetminus \bigcup_{i \in I_2} B(x_i, r_i) \right) \leq \eta \mathcal{H}^{d-1}(\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3) \,.$$

We claim that the collection of balls $(B(x_i, r_i), i \in I_2)$ enjoys the desired properties. We define the sets

$$\begin{aligned} I &= \{ i \in I_2 \,|\, x_i \in \partial^* F \cap \Omega \} \,, \\ J &= \{ i \in I_2 \,|\, x_i \in \Gamma^1 \cap \partial^*(\Omega \smallsetminus F) \} \,, \\ K &= \{ i \in I_2 \,|\, x_i \in \Gamma^2 \cap \partial^* F \} \,, \end{aligned}$$

and $v_i = v_F(x_i)$ for $i \in I \cup K$ and $v_i = v_\Omega(x_i)$ for $i \in J$. Finally, we only have to check that

$$\left| \mathcal{I}_\Omega(F) - \sum_{i \in I \cup K} \alpha_{d-1} r_i^{d-1} \nu(v_F(x_i)) - \sum_{i \in J} \alpha_{d-1} r_i^{d-1} \nu(v_\Omega(x_i)) \right| \leq \eta \,.$$

We recall that $\nu_{\max}$ is the supremum of ν over S^{d-1}; we have

$$\begin{aligned} &\left| \mathcal{I}_\Omega(F) - \sum_{i \in I \cup K} \alpha_{d-1} r_i^{d-1} \nu(v_F(x_i)) - \sum_{i \in J} \alpha_{d-1} r_i^{d-1} \nu(v_\Omega(x_i)) \right| \\ &\qquad \leq \left| \int_{\mathcal{R}_1} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \sum_{i \in I} \alpha_{d-1} r_i^{d-1} \nu(v_F(x_i)) \right| \\ &\qquad\quad + \left| \int_{\mathcal{R}_2} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \sum_{i \in K} \alpha_{d-1} r_i^{d-1} \nu(v_F(x_i)) \right| \\ &\qquad\quad + \left| \int_{\mathcal{R}_3} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) - \sum_{i \in J} \alpha_{d-1} r_i^{d-1} \nu(v_\Omega(x_i)) \right| \\ &\qquad \leq \int_{\mathcal{R}_1 \smallsetminus \cup_{i \in I} B(x_i, r_i)} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) \\ &\qquad\quad + \sum_{i \in I} \left| \int_{\mathcal{R}_1 \cap B(x_i, r_i)} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \alpha_{d-1} r_i^{d-1} \nu(v_F(x)) \right| \\ &\qquad\quad + \int_{\mathcal{R}_2 \smallsetminus \cup_{i \in K} B(x_i, r_i)} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) \\ &\qquad\quad + \sum_{i \in K} \left| \int_{\mathcal{R}_2 \cap B(x_i, r_i)} \nu(v_F(y)) d\mathcal{H}^{d-1}(y) - \alpha_{d-1} r_i^{d-1} \nu(v_F(x)) \right| \\ &\qquad\quad + \int_{\mathcal{R}_3 \smallsetminus \cup_{i \in J} B(x_i, r_i)} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) \\ &\qquad\quad + \sum_{i \in J} \left| \int_{\mathcal{R}_3 \cap B(x_i, r_i)} \nu(v_\Omega(y)) d\mathcal{H}^{d-1}(y) - \alpha_{d-1} r_i^{d-1} \nu(v_\Omega(x)) \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \eta\mathcal{H}^{d-1}(\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3)\nu_{\max}+\eta\sum_{i\in I\cup J\cup K}\alpha_{d-1}r_i^{d-1}\\
&\leq \eta\mathcal{H}^{d-1}(\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3)\nu_{\max}\,+\,2\eta\mathcal{H}^{d-1}(\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3)\\
&\leq \eta(\nu_{\max}+2)(\mathcal{P}(F,\Omega)+\mathcal{P}(\Omega))\,.
\end{aligned}
$$

Since $(\nu_{\max}+2)(\mathcal{P}(F,\Omega)+\mathcal{P}(\Omega))$ does not depend on η, we have the required estimate.

2.3.2. *Definition of a local event.* We consider a set F in $\mathcal{C}_\beta$, and a positive ε_F that we have to choose adequately. Thanks to Lemma 18, we know that for every positive fixed δ and η, there exists a finite family of closed disjoint balls $(B_i)_{i\in I\cup J\cup K}=(B(x_i,r_i),v_i)_{i\in I\cup J\cup K}$ such that (the vector v_i defines B_i^-)

$$
\begin{aligned}
&\forall i\in I\,,\ x_i\in\partial^*F\cap\Omega\,,\ r_i\in]0,1[\,,\ B_i\subset\Omega\,,\ \mathcal{L}^d((F\cap B_i)\triangle B_i^-)\,\leq\,\delta\alpha_d r_i^d\,,\\
&\forall i\in J\,,\ x_i\in\Gamma^1\cap\partial^*(\Omega\smallsetminus F)\,,\ r_i\in]0,1[\,,\ \partial\Omega\cap B_i\subset\Gamma^1\,,\ \mathcal{L}^d((B_i\cap\Omega)\triangle B_i^-)\,\leq\,\delta\alpha_d r_i^d\,,\\
&\forall i\in K\,,\ x_i\in\Gamma^2\cap\partial^*F\,,\ r_i\in]0,1[\,,\ \partial\Omega\cap B_i\subset\Gamma^2\,,\ \mathcal{L}^d((F\cap B_i)\triangle B_i^-)\,\leq\,\delta\alpha_d r_i^d\,,
\end{aligned}
$$

and finally

$$
\left|\mathcal{I}_\Omega(F)-\sum_{i\in I\cup K}\alpha_{d-1}r_i^{d-1}\nu(v_F(x_i))-\sum_{i\in J}\alpha_{d-1}r_i^{d-1}\nu(v_\Omega(x_i))\right|\,\leq\,\eta\,.
$$

It is obvious that $\phi_\Omega<\infty$ because

$$
\phi_\Omega\,\leq\,\mathcal{I}_\Omega(\Omega)\,=\,\int_{\Gamma^2\cap\partial^*\Omega}\nu(v_\Omega(x))d\mathcal{H}^{d-1}(x)\,\leq\,\nu_{\max}\mathcal{H}^{d-1}(\Gamma^2)\,<\,\infty\,.
$$

We suppose for the rest of the section 2 that $\phi_\Omega>0$ otherwise we do not have to study any lower large deviations. We consider $\lambda<\phi_\Omega$. There exists a positive s (we can choose it smaller than 1) such that $\lambda\leq\phi_\Omega(1-2s)\leq\mathcal{I}_\Omega(F)(1-2s)$. We choose

$$
\eta\,=\,\frac{s\mathcal{I}_\Omega(F)}{4}\,,
$$

and then we obtain that

$$
\begin{aligned}
&\left|\mathcal{I}_\Omega(F)-\sum_{i\in I\cup K}\alpha_{d-1}r_i^{d-1}\nu(v_F(x_i))-\sum_{i\in J}\alpha_{d-1}r_i^{d-1}\nu(v_\Omega(x_i))\right|\\
&\qquad\leq\,\left(\sum_{i\in I\cup K}\alpha_{d-1}r_i^{d-1}\nu(v_F(x_i))+\sum_{i\in J}\alpha_{d-1}r_i^{d-1}\nu(v_\Omega(x_i))\right)\frac{s}{2}\,,
\end{aligned}
$$

and that

$$
\lambda\,\leq\,\left(\sum_{i\in I\cup K}\alpha_{d-1}r_i^{d-1}\nu(v_F(x_i))+\sum_{i\in J}\alpha_{d-1}r_i^{d-1}\nu(v_\Omega(x_i))\right)(1-s)\,.
$$

Since the $(B_i)_{i\in I\cup J\cup K}$ are disjoint, we also know that

$$
\phi_n\,\geq\,\sum_{i\in I\cup J\cup K}V(\mathcal{E}_n\cap B_i)\,.
$$

Then

$$
\begin{aligned}
&\mathbb{P}[V(\mathcal{E}_n)\leq\lambda n^{d-1}\text{ and }\mathcal{L}^d(E_n\triangle F)\leq\varepsilon_F]\\
&\qquad\leq\,\mathbb{P}\left[\begin{array}{rl}\sum_{i\in I\cup J\cup K}V(\mathcal{E}_n\cap B_i)\leq(1-s) & n^{d-1}\big(\sum_{i\in I\cup K}\alpha_{d-1}r_i^{d-1}\nu(v_F(x_i))\\ & \qquad+\sum_{i\in J}\alpha_{d-1}r_i^{d-1}\nu(v_\Omega(x_i))\big)\\ \text{and }\mathcal{L}^d(E_n\triangle F) & \leq\,\varepsilon_F\end{array}\right]\,.
\end{aligned}
$$

From now on we choose ε_F to be

$$
\varepsilon_F\,=\,\min_{i\in I\cup J\cup K}\alpha_d r_i^d\delta\,,
$$

for a fixed δ that we will choose later. For all $i \in I$, we then have

$$\mathcal{L}^d((E_n \cap B_i)\triangle B_i^-) \leq \mathcal{L}^d((F \cap B_i)\triangle B_i^-) + \mathcal{L}^d(E_n \triangle F) \leq 2\delta\alpha_d r_i^d .$$

We want to evaluate $\mathrm{card}(((E_n \cap B_i)\triangle B_i^-) \cap \mathbb{Z}_n^d)$. It is equivalent to evaluate

$$n^d \mathcal{L}^d(((E_n \cap B_i)\triangle B_i^-) \cap \mathbb{Z}_n^d + [-1/2n, 1/2n]^d) .$$

By definition, for all $x \in E_n \cap \mathbb{Z}_n^d = \widetilde{E}_n$, $x + [-1/2n, 1/2n]^d \subset E_n$, so

$$\begin{aligned}
&((E_n \cap B_i)\triangle B_i^-) \cap \mathbb{Z}_n^d + [-1/2n, 1/2n]^d \\
&\qquad \subset ((E_n \cap B_i)\triangle B_i^-) \cup (\mathcal{V}_\infty(B_i, 1/n) \smallsetminus B_i) \cup (\mathcal{V}_\infty(B_i^-, 1/n) \smallsetminus B_i^-) \\
&\qquad \subset ((E_n \cap B_i)\triangle B_i^-) \cup (\mathcal{V}_2(B_i, 2d/n) \smallsetminus B_i) \cup (\mathcal{V}_2(B_i^-, 2d/n) \smallsetminus B_i^-) ,
\end{aligned}$$

Since ∂B_i and ∂B_i^- are very regular, the result about the Minkowski content implies that

$$\lim_{n\to\infty} \frac{n}{2d} \mathcal{L}^d(\mathcal{V}_2(B_i, 2d/n) \smallsetminus B_i) = \mathcal{H}^{d-1}(\partial B_i)$$

and

$$\lim_{n\to\infty} \frac{n}{2d} \mathcal{L}^d(\mathcal{V}_2(B_i^-, 2d/n) \smallsetminus B_i^-) = \mathcal{H}^{d-1}(\partial B_i^-) .$$

For n large enough, we then obtain that

$$\mathcal{L}^d(((E_n \cap B_i)\triangle B_i^-) \cap \mathbb{Z}_n^d + [-1/2n, 1/2n]^d) \leq 2\delta\alpha_d r_i^d + \frac{4d(\mathcal{H}^{d-1}(\partial B_i) + \mathcal{H}^{d-1}(\partial B_i^-))}{n} ,$$

and then for all n large enough

$$\begin{aligned}
\mathrm{card}(((E_n \cap B_i)\triangle B_i^-) \cap \mathbb{Z}_n^d) &\leq 2\delta\alpha_d r_i^d n^d + 4d(\mathcal{H}^{d-1}(\partial B_i) + \mathcal{H}^{d-1}(\partial B_i^-))n^{d-1} \\
&\leq 4\delta\alpha_d r_i^d n^d .
\end{aligned}$$

For $i \in K$, exactly the same arguments imply that

$$\mathrm{card}(((E_n \cap B_i)\triangle B_i^-) \cap \mathbb{Z}_n^d) \leq 4\delta\alpha_d r_i^d n^d$$

for n large enough.

We study now what happens in the balls B_i for $i \in J$. We recall that $\widetilde{E}_n = E_n \cap \mathbb{Z}_n^d$. We define $\widetilde{E}'_n = \widetilde{E}_n \cup \Omega_n^c$ (where $\Omega_n^c = \mathbb{Z}_n^d \smallsetminus \Omega_n$) and $E'_n = \widetilde{E}'_n + [-1/(2n), 1/(2n)]^{d-1}$. Then $E'_n \cap \Omega = E_n$. In a ball B_i, we have $\partial^e \widetilde{E}'_n \cap B_i = \mathcal{E}_n \cap B_i$. Indeed, we know that $\Gamma \cap B_i \subset \Gamma^1$. The sets Γ^1 and Γ^2 are open in Γ and disjoint, so $\Gamma^1 \cap \overline{\Gamma^2} = \varnothing$, where $\overline{\Gamma^2}$ is the adherence of Γ^2, and then $B_i \cap \overline{\Gamma^2} = \varnothing$. Since B_i is closed, we obtain that $d(B_i, \overline{\Gamma^2}) > 0$, and thus for n large enough, $\Gamma_n \cap B_i \subset \Gamma_n^1$. Moreover, we know that $\Gamma_n^1 \subset \widetilde{E}_n \subset \widetilde{E}'_n$. We obtain that $\partial^e \widetilde{E}'_n \cap \Omega_n^c \cap B_i = \varnothing$, i.e., all the edges of $\partial^e \widetilde{E}'_n$ in B_i have both endpoints in Ω_n (see figure 6). Now we have

$$\begin{aligned}
\mathcal{L}^d((E'_n \cap B_i)\triangle B_i^+) &\leq \mathcal{L}^d((E'_n \cap B_i)\triangle(\Omega^c \cap B_i)) + \mathcal{L}^d((\Omega^c \cap B_i)\triangle B_i^+) \\
&\leq \mathcal{L}^d(E'_n \cap B_i \cap \Omega) + \mathcal{L}^d((\Omega^c \smallsetminus E'_n) \cap B_i) + \mathcal{L}^d((\Omega \cap B_i)\triangle B_i^-) \\
&\leq \mathcal{L}^d(E_n \triangle F) + \mathcal{L}^d(\mathcal{V}_\infty(\Gamma, 1/n) \cap B_i) + \delta\alpha_d r_i^d \\
&\leq \varepsilon_F + \mathcal{L}^d(\mathcal{V}_\infty(\Gamma, 1/n) \cap B_i) + \delta\alpha_d r_i^d \\
&\leq 3\delta\alpha_d r_i^d ,
\end{aligned}$$

for n large enough, where the last inequality results from the properties of the Minkowski content. As previously, we obtain that for n large enough,

$$\mathrm{card}(((E'_n \cap B_i)\triangle B_i^+) \cap \mathbb{Z}_n^d) \leq 4\delta\alpha_d r_i^d n^d .$$

We conclude that for n large enough,

$$\mathbb{P}[V(\mathcal{E}_n) \leq \lambda n^{d-1} \text{ and } \mathcal{L}^d(E_n \triangle F) \leq \varepsilon_F]$$

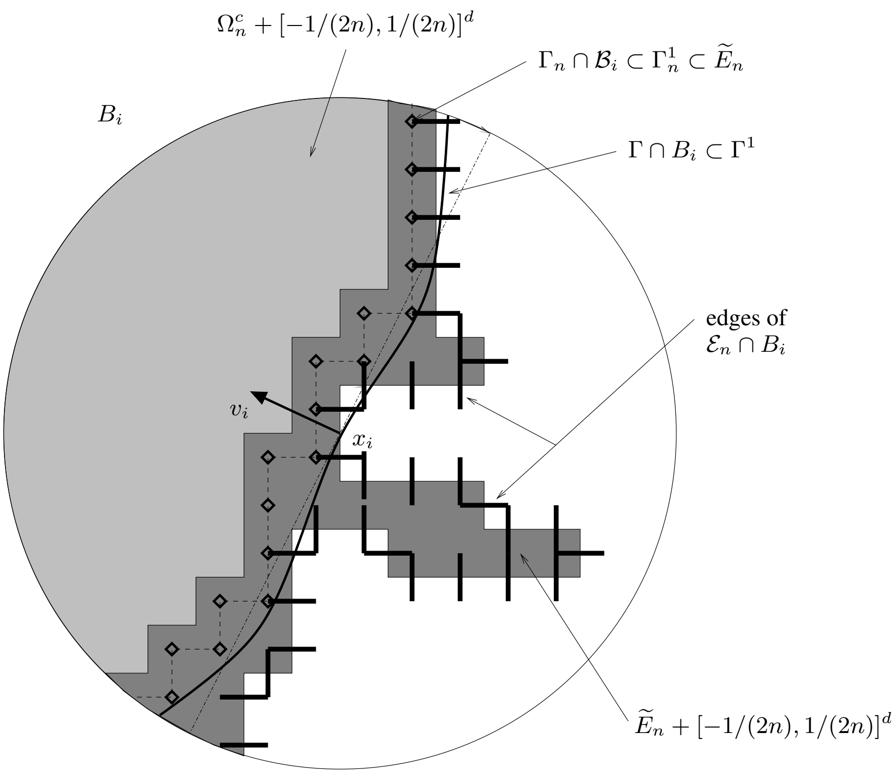

FIGURE 6. A ball B_i for $i \in J$.

$$
\begin{aligned}
&\leq \sum_{i\in I} \mathbb{P}\left[\begin{array}{c} V(\partial^e \widetilde{E}_n \cap B_i) \leq (1-s)\alpha_{d-1} r_i^{d-1} \nu(v_F(x_i)) \text{ and} \\ \operatorname{card}((\widetilde{E}_n \cap B_i) \triangle (B_i^- \cap \mathbb{Z}_n^d)) \leq 4\delta\alpha_d r_i^d n^d \end{array}\right] \\
&\quad + \sum_{i\in J} \mathbb{P}\left[\begin{array}{c} V(\partial^e \widetilde{E}'_n \cap B_i) \leq (1-s)\alpha_{d-1} r_i^{d-1} \nu(v_F(x_i)) \text{ and} \\ \operatorname{card}((\widetilde{E}'_n \cap B_i) \triangle (B_i^+ \cap \mathbb{Z}_n^d)) \leq 4\delta\alpha_d r_i^d n^d \end{array}\right] \\
&\quad + \sum_{i\in K} \mathbb{P}\left[\begin{array}{c} V(\partial^e \widetilde{E}_n \cap B_i) \leq (1-s)\alpha_{d-1} r_i^{d-1} \nu(v_F(x_i)) \text{ and} \\ \operatorname{card}((\widetilde{E}_n \cap B_i) \triangle (B_i^- \cap \mathbb{Z}_n^d)) \leq 4\delta\alpha_d r_i^d n^d \end{array}\right] \\
&\leq \sum_{i \in I\cup J\cup K} \mathbb{P}[G(x_i, r_i, v_i)]\,,
\end{aligned}
$$

where $G(x, r, v)$ is the event that there exists a set $U \subset B \cap \mathbb{Z}_n^d$ such that:

$$
\begin{cases} \operatorname{card}(U \triangle B^-) \leq 4\delta\alpha_d r^d n^d\,, \\ V(\partial^e U \cap B) \leq (\alpha_{d-1} r^{d-1} \nu(v(x)))(1-s) n^{d-1}\,. \end{cases}
$$

Notice that this event depends only on the edges in $B = B(x, r)$. This event seems to be complicated, but indeed when $G(x, r, v)$ happens, it means in a sense that the flow between the lower half part of $B(x, r)$ (for the direction v) and the upper half part of B is abnormally small. We will examine the consequence of the event $G(x, r, v)$ over the maximal flow in $B(x, r)$ in the next section.

2.4. Surgery in a ball to define an almost flat cutset. We consider a fixed ball $B = B(x, r)$ and a unit vector v (corresponding to one generic ball of the previous covering). We want to interpret the event $G(x, r, v)$ in term of the maximal flow through a cylinder whose basis is a disc, included in the ball B, and oriented along the direction v. We define

$$\gamma_{\max} \,=\, \rho r \,,$$

where ρ is a constant depending on δ and B which we can imagine very small, it will be chosen later. The constant $\gamma_{\max}$ is in fact the height of the cylinder we are constructing, namely

$$\mathcal{C} \,=\, \mathrm{cyl}(\mathrm{disc}(x,r',v),\gamma_{\max})\,.$$

We want $\mathcal{C}$ to be included in B, so we choose

$$r' \,=\, r\cos(\arcsin\rho)\,.$$

We would like to analyse the implication of the event $G(x,r,v)$ on the flow $\phi_{\mathcal{C}}$ between the top and the bottom of $\mathcal{C}$ for the direction v. As we said previously, the event $G(x,r,v)$ means that the maximal flow between a set U that "looks like" B^- (for the direction given by v) and the set U^c that "looks like" B^+ is a bit too small. Here "looks like" means that B^- and U are closed in volume, but the set U might have some thin strands (of small volume, but that can be long) that go deeply into B^+ and symmetrically the set U^c might have some thin strands that go deeply into B^- (see figure 7). What we have to do to control $\phi_{\mathcal{C}}$ is to cut these strands: by adding edges to

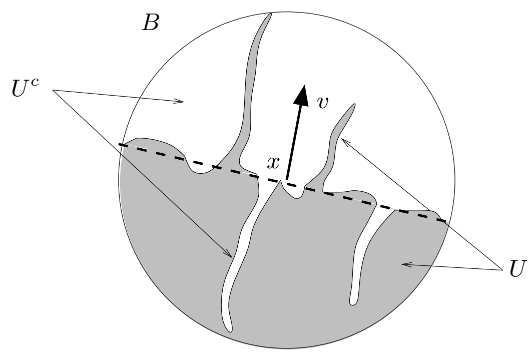

FIGURE 7. Event $G(x,r,v)$.

$\partial^e U$ at a fixed height in $\mathcal{C}$ to close the strands, we obtain a cutset in $\mathcal{C}$. The point is that we have to control the capacity of these edges we have added to $\partial^e U$. This is the reason why we choose the height at which we add edges to be sure we add not too many edges, and then we control their capacity thanks to a property of independence.

We suppose that the event $G(x,r,v)$ happens, and we denote by U a fixed set satisfying the properties described in the definition of $G(x,r,v)$. For each γ in $\{1/n,...,(\lfloor n\gamma_{\max}\rfloor-1)/n\}$, we define

$$\left\{\begin{array}{l} D(\gamma) \,=\, \mathrm{cyl}(\mathrm{disc}(x,r',v),\gamma)\,,\\ \partial^+ D(\gamma) \,=\, \{y\in D(\gamma)\,|\,\exists z\in\mathbb{Z}_n^d\,,\ (z-x)\cdot v>\gamma \text{ and } |z-y|=1\}\,,\\ \partial^- D(\gamma) \,=\, \{y\in D(\gamma)\,|\,\exists z\in\mathbb{Z}_n^d\,,\ (z-x)\cdot v<-\gamma \text{ and } |z-y|=1\}\,.\end{array}\right.$$

These sets are represented in figure 8. The sets $\partial^+ D(\gamma)\cup\partial^- D(\gamma)$ are pairwise disjoint for different γ, and we know that

$$\sum_{\gamma=1/n,...,(\lfloor n\gamma_{\max}\rfloor-1)/n} \mathrm{card}((\partial^+ D(\gamma)\cap U)\cup(\partial^- D(\gamma)\cap U^c)) \,\leq\, 4\delta\alpha_d r^d n^d\,,$$

so there exists a γ_0 in $\{1/n,...,(\lfloor n\gamma_{\max}\rfloor-1)/n\}$ such that

$$\mathrm{card}((\partial^+ D(\gamma_0)\cap U)\cup(\partial^- D(\gamma_0)\cap U^c)) \,\leq\, \frac{4\delta\alpha_d r^d n^d}{\lfloor n\gamma_{\max}\rfloor-1} \,\leq\, \frac{5\delta\alpha_d r^d n^{d-1}}{\gamma_{\max}}$$

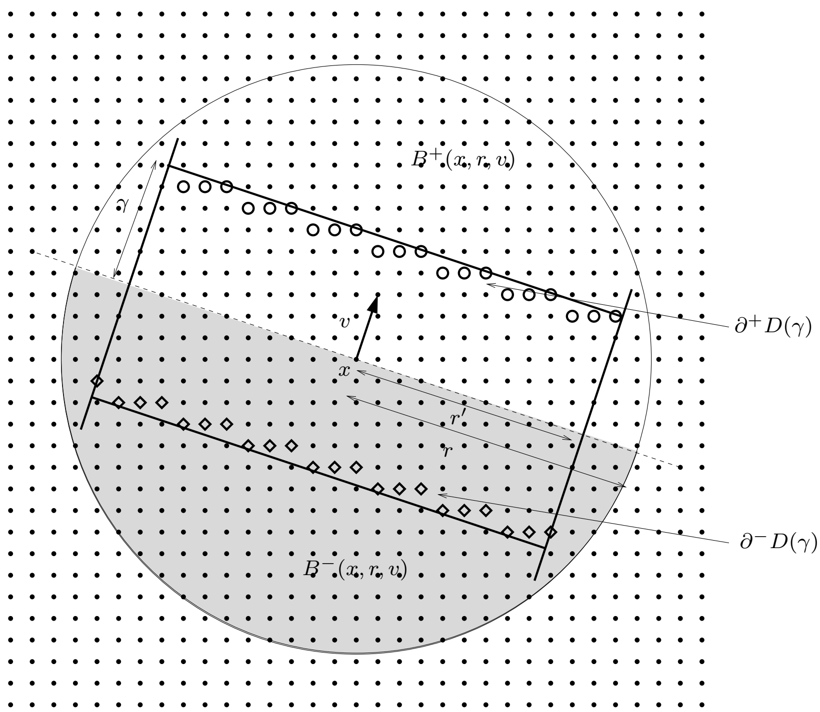

FIGURE 8. Representation of $D(\gamma)$.

for n sufficiently large. We define the event $G^*(x,r,v,\gamma)$ (depending only on the edges in $D(\gamma)$)) to be the existence of a set $X \subset D(\gamma) \cap \mathbb{Z}_n^d$ with the following properties:

$$\begin{cases} \operatorname{card}((\partial^+ D(\gamma) \cap X) \cup (\partial^- D(\gamma) \cap X^c)) \,\leq\, 5\delta\alpha_d r^d n^{d-1} \gamma_{\max}^{-1} \,=\, 5\delta\alpha_d \rho^{-1} r^{d-1} n^{d-1}\,, \\ V(\partial^e X \cap D(\gamma)) \,\leq\, \alpha_{d-1} r^{d-1} \nu(v)(1-s) n^{d-1}\,. \end{cases}$$

We have proved that if $G(x,r,v)$ occurs, there exists a γ in $\{1/n, ..., (\lfloor n\gamma_{\max} \rfloor - 1)/n\}$ such that $G^*(x,r,v,\gamma)$ happens. On $G^*(x,r,v,\gamma)$, we select a set of edges X that satisfies the properties described in the definition of $G^*(B, v(x), \gamma)$ with a deterministic procedure, and we define

$$\begin{cases} X^+ \,=\, \{\langle x,y \rangle \,|\, x \in \partial^+ D(\gamma) \cap X\,,\ y \notin D(\gamma)\}\,, \\ X^- \,=\, \{\langle x,y \rangle \,|\, x \in \partial^- D(\gamma) \smallsetminus X\,,\ y \notin D(\gamma)\}\,. \end{cases}$$

The set of edges $(\partial^e X \cap D(\gamma)) \cup X^+ \cup X^-$ cuts the top from the bottom of $\mathcal{C} = D(\gamma_{\max})$, so on $G^*(x,r,v,\gamma)$, we have

$$\phi_{\mathcal{C}} \,\leq\, V(\partial^e X \cap D(\gamma)) + V(X^+ \cup X^-)\,.$$

(Recall that $\partial^e X \cap D(\gamma)$ is the set of the edges of $\partial^e X$ which are included in $D(\gamma)$). Moreover

$$\begin{aligned} \operatorname{card}(X^+ \cup X^-) \,&\leq\, 2d \operatorname{card}((\partial^+ D(\gamma) \cap X) \cup (\partial^- D(\gamma) \smallsetminus X)) \\ &\leq\, 2d \frac{5\delta\alpha_d r^d n^{d-1}}{\gamma_{\max}} \,=\, C r^{d-1} \delta \rho^{-1} n^{d-1}\,, \end{aligned}$$

where $C = 10d\alpha_d$ is a constant depending on the dimension. We obtain that

$$\begin{aligned} \mathbb{P}[G(x,r,v)] \,&\leq \sum_{\gamma = 1/n, \dots, (\lfloor n\gamma_{\max} \rfloor - 1)/n} \mathbb{P}[G^*(x,r,v,\gamma)] \\ &\leq \sum_{\gamma} \mathbb{P}[G^*(x,r,v,\gamma) \cap \{V(X^+ \cup X^-) \leq \alpha_{d-1} r^{d-1} \nu(v) n^{d-1} s/4\}] \\ &\qquad + \mathbb{P}[G^*(x,r,v,\gamma) \cap \{V(X^+ \cup X^-) \geq \alpha_{d-1} r^{d-1} \nu(v) n^{d-1} s/4\}]\,. \end{aligned}$$

On one hand, we have proved that

$$\begin{aligned}\mathbb{P}[G^*(x,r,v,\gamma)\cap\{V(X^+\cup X^-)\leq \alpha_{d-1}r^{d-1}\nu(v)n^{d-1}s/4\}]\\ \leq \mathbb{P}[\phi_{\mathcal{C}}\leq \alpha_{d-1}r^{d-1}\nu(v)(1-3s/4)n^{d-1}]\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathbb{P}[G^*(x,r,v,\gamma)\cap\{V(X^+\cup X^-)\geq \alpha_{d-1}r^{d-1}\nu(v)n^{d-1}s/4\}]\\
&\quad\leq \mathbb{E}\left(\mathbb{P}(G^*(x,r,v,\gamma)\cap\{V(X^+\cup X^-)\geq \alpha_{d-1}r^{d-1}\nu(v)n^{d-1}s/4\}\,|\,(t(e))_{e\in D(\gamma)})\right)\\
&\quad\leq \mathbb{E}\Bigg(\mathbb{P}(G^*(x,r,v,\gamma)\cap\bigcup_{F\subset\mathbb{E}_n^d}(\{X^+\cup X^-=F\}\\
&\qquad\qquad\cap\{V(F)\geq \alpha_{d-1}r^{d-1}\nu(v)n^{d-1}s/4\})\,|\,(t(e))_{e\in D(\gamma)})\Bigg)\\
&\quad\leq \mathbb{E}\Bigg(\mathbb{1}_{G^*(x,r,v,\gamma)}\sum_{F\subset\mathbb{E}_n^d}\mathbb{1}_{\{X^+\cup X^-=F\}}\mathbb{P}(V(F)\geq \alpha_{d-1}r^{d-1}\nu(v)n^{d-1}s/4)\Bigg)\\
&\quad\leq \mathbb{P}\left[\sum_{i=1}^{Cr^{d-1}\delta\rho^{-1}n^{d-1}}t(e_i)\geq \alpha_{d-1}r^{d-1}\nu(v)n^{d-1}s/4\right]\,,\end{aligned}$$

where the last inequality comes from the fact that for all F such that $\mathbb{P}[X^+\cup X^-=F]>0$, $\mathrm{card}(F)\leq Cr^{d-1}\delta\rho^{-1}n^{d-1}$. Here we have used the following essential property of $X^+\cup X^-$: the position of the edges of $X^+\cup X^-$ is $\sigma(t(e),e\in D(\gamma))$-measurable, but their capacities are independent of $(t(e))_{e\in D(\gamma)}$. Finally, we obtain that

$$\begin{aligned}\mathbb{P}[G^*(x,r,v,\gamma)]\;\leq\;&\gamma_{\max}n\mathbb{P}[\phi_{\mathcal{C}}\leq(\alpha_{d-1}r^{d-1}\nu(v))(1-3s/4)n^{d-1}]\\
&+\gamma_{\max}n\mathbb{P}\left[\sum_{i=1}^{Cr^{d-1}\delta\rho^{-1}n^{d-1}}t(e_i)\geq(\alpha_{d-1}r^{d-1}\nu(v))n^{d-1}s/4\right]\,.\end{aligned}$$

We want to consider cylinders whose basis are hyperrectangles instead of discs, and the variable τ instead of ϕ in these cylinders, because we only know the lower large deviations of the flow in this case (see **[52]** or the Chapter 5 of the thesis). There exists a constant $c=c(d)$ such that, for any positive κ, there exists a finite family $(A_i)_{i\in I}$ of disjoint closed hyperrectangles included in $\mathrm{disc}(x,r',v)$ such that

$$\begin{cases}\sum_{i\in I}\mathcal{H}^{d-1}(A_i)\;\geq\;\alpha_{d-1}r'^{d-1}-\kappa\,,\\ \sum_{i\in I}\mathcal{H}^{d-2}(\partial A_i)\leq cr'^{d-2}\,,\end{cases}$$

(see figure 9). Thanks to the max-flow min-cut theorem, we know that for each i, the maximal flow $\tau_{\mathrm{cyl}(A_i,\gamma_{max})}$ is equal to the smallest capacity of a set of edges in $\mathrm{cyl}(A_i,\gamma_{\max})$ that cuts the lower half part from the upper half part of the boundary of the cylinder along the direction given by v. We denote by $\mathcal{E}_i$ such a cutset in $\mathrm{cyl}(A_i,\gamma_{\max})$. It is a set of edges that is pinned at the boundary of A_i (which is the common boundary of the two halves of the boundary of the cylinder $\mathrm{cyl}(A_i,\gamma_{\max})$ between which the flow $\tau_{\mathrm{cyl}(A_i,\gamma_{\max})}$ goes). Thus the different sets $\mathcal{E}_i$ in each cylinder $\mathrm{cyl}(A_i,\gamma_{\max})$ can be glued together along $\cup_{i\in I}\partial A_i$ to create a cutset in $\mathcal{C}$ if we provide some "glue", i.e., if we add some edges in a small neighbourhood of $\cup_{i\in I}\partial A_i$. For each $i\in I$, we define the set $\mathcal{P}_i(n)\subset\mathbb{R}^d$ by

$$\mathcal{P}_i(n)\;=\;\mathrm{cyl}(\mathcal{V}(\partial A_i,\zeta/n)\cap\mathrm{hyp}(A_i),\gamma_{\max})\,,$$

where ζ is a fixed constant bigger than $2d$, and we denote by $P_i(n)$ the set of the edges included in $\mathcal{P}_i(n)$. Then $\cup_{i\in I}E_i\cup P_i(n)$ cuts the top from the bottom of $\mathcal{C}$. Thanks to the max-flow min-cut

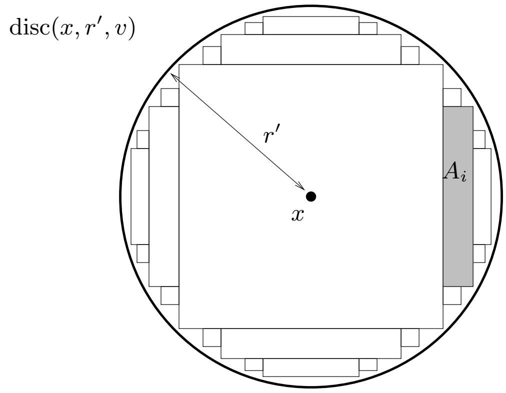

FIGURE 9. Disc $\mathrm{disc}(x, r', v)$.

theorem again, we thus obtain that

$$\phi_{\mathcal{C}} \;\leq\; \sum_{i\in I} \tau_{\mathrm{cyl}(A_i,\gamma_{\max})} + V(\cup_{i\in I} P_i(n))\,.$$

We can evaluate the number of edges in $\cup_{i\in I} P_i(n)$ as follows:

$$\mathrm{card}(\cup_{i\in I} P_i(n)) \;\leq\; c' r'^{d-2} \gamma_{\max} n^{d-1} \;\leq\; c' \rho r^{d-1} n^{d-1}\,,$$

where c' is a constant depending on ζ and d. Therefore

$$\begin{aligned}
&\mathbb{P}[\phi_{\mathcal{C}} \leq \alpha_{d-1} r^{d-1} \nu(v)(1-3s/4) n^{d-1}] \\
&\quad\leq\; \mathbb{P}\left[\sum_{i\in I} \tau_{\mathrm{cyl}(A_i,\gamma_{\max})} \leq \alpha_{d-1} r^{d-1} \nu(v)(1-s/2) n^{d-1}\right] \\
&\qquad + \mathbb{P}\left[\sum_{i=1}^{c'\rho r^{d-1} n^{d-1}} t(e_i) \geq \alpha_{d-1} r^{d-1} \nu(v) \frac{s}{4} n^{d-1}\right] \\
&\quad\leq\; \mathbb{P}\left[\sum_{i\in I} \tau_{\mathrm{cyl}(A_i,\gamma_{\max})} \leq (1-s/4) n^{d-1} \sum_{i\in I} \mathcal{H}^{d-1}(A_i)\nu(v)\right] \\
&\qquad + \mathbb{P}\left[\sum_{i=1}^{c'\rho r^{d-1} n^{d-1}} t(e_i) \geq \alpha_{d-1} r^{d-1} \nu(v) \frac{s}{4} n^{d-1}\right]\,,
\end{aligned}$$

as soon as the constants satisfy the condition

$$(\kappa + \alpha_{d-1}(r^{d-1} - r'^{d-1}))(1-s/2) \;\leq\; \sum_{i\in I} \mathcal{H}^{d-1}(A_i) \nu_{\min} s/4\,. \tag{7.1}$$

Then

$$\begin{aligned}
\mathbb{P}[G^*(x,r,v,\gamma)] \;\leq\; &\rho r n \sum_{i\in I} \mathbb{P}[\tau_{\mathrm{cyl}(A_i,\gamma_{\max})} \leq \mathcal{H}^{d-1}(A_i)\nu(v)(1-s/4) n^{d-1}] \\
&+ \rho r n \mathbb{P}\left[\sum_{i=1}^{C r^{d-1}\delta\rho^{-1} n^{d-1}} t(e_i) \geq \alpha_{d-1} r^{d-1} \nu(v) n^{d-1} s/4\right] \\
&+ \rho r n \mathbb{P}\left[\sum_{i=1}^{c'\rho r^{d-1} n^{d-1}} t(e_i) \geq \alpha_{d-1} r^{d-1} \nu(v) n^{d-1} s/4\right]. \\
\leq\; &\rho r n \sum_{i\in I} \mathbb{P}[\tau_{\mathrm{cyl}(A_i,\gamma_{\max})} \leq \mathcal{H}^{d-1}(A_i)\nu(v)(1-s/4) n^{d-1}]
\end{aligned}$$

$$+ 2\rho r n \mathbb{P}\left[\sum_{i=1}^{C'(\delta\rho^{-1}+\rho)r^{d-1}n^{d-1}} t(e_i) \geq \alpha_{d-1} r^{d-1} \nu(v) n^{d-1} s/2\right],$$

where C' is a constant depending on ζ and d.

2.5. Calibration of the constants. From now on we suppose that the law Λ of the capacity of the edges admits an exponential moment. Then as soon as the constants satisfy the condition

$$C'(\rho+\delta\rho^{-1})r^{d-1}\mathbb{E}(t(e)) \,<\, (\alpha_{d-1}r^{d-1}\nu_{\min})\frac{s}{2}\,, \tag{7.2}$$

the Cramér Theorem in $\mathbb{R}$ allows us to affirm that there exists positive constants $\mathcal{D}$ and $\mathcal{D}'$ (depending on Λ, δ, ρ, ζ, s and d) such that

$$\mathbb{P}\left[\sum_{i=1}^{C'(\delta\rho^{-1}+\rho)r^{d-1}n^{d-1}} t(e_i) \geq (\alpha_{d-1} r^{d-1} \nu(v) n^{d-1} s/2\right] \,\leq\, \mathcal{D}' e^{-\mathcal{D}n^{d-1}}\,.$$

If we also suppose that $\Lambda(0) \,<\, 1 - p_c(d)$, we know from Theorem 1 in [**52**] that there exist a positive constant $\mathcal{D}''$ (depending only on s, d, Λ and v) and a constant $\mathcal{D}'''$ (possibly depending on Λ, d, A_i, $\gamma_{\max} = \rho r$ and s) such that

$$\mathbb{P}[\tau_{\mathrm{cyl}(A_i,\gamma_{\max})} \leq \mathcal{H}^{d-1}(A_i)\nu(v)(1-s/4)n^{d-1}] \,\leq\, \mathcal{D}''' e^{-\mathcal{D}'' n^{d-1}}\,.$$

We have thus proved that if we can choose, for a fixed F, the constants δ, ρ and κ such that for every ball B in the collection of balls $(B_i)_{i\in I\cup J\cup K}$ the conditions (7.1) and (7.2) are satisfied, then there exists positive constants $\widetilde{\mathcal{D}}$ and $\hat{\mathcal{D}}$ (depending on d, Λ, Ω, Γ^1, Γ^2 and λ) such that

$$\mathbb{P}[\phi_n \leq \lambda n^{d-1}] \,\leq\, \hat{\mathcal{D}} e^{-\widetilde{\mathcal{D}} n^{d-1}}\,,$$

and this yields Theorem 23.

We just have to calibrate the constants. In condition (7.2) appears the factor $(\rho + \delta\rho^{-1})$: to make it small, we choose $\rho = \sqrt{\delta}$. Then the condition (7.2) is equivalent to

$$\sqrt{\delta} \,<\, \frac{\alpha_{d-1}\nu_{\min} s}{2C'\mathbb{E}(t(e))}\,,$$

for a constant C' that depends on ζ and d, and thus it is satisfied if we choose δ small enough (clearly since $\Lambda(0) \,<\, 1 - p_c(d)$ we know that $\mathbb{E}(t(e)) \,>\, 0$ and $\nu_{\min} \,>\, 0$). To see that the condition (7.1) can also be satisfied, we just choose $\kappa \leq \alpha_{d-1}(r^{d-1} - r'^{d-1})/2$ (so κ depends on δ) and we remark that

$$1 - (\cos\arcsin\sqrt{\delta})^{d-1} \,=\, (d-1)\delta/2 + o(\delta)\,,$$

so for δ small enough, condition (7.1) is satisfied as soon as

$$\delta \,\leq\, \frac{2\nu_{\min}}{12(d-1)(1-s/2)}\,,$$

which can obviously be satisfied (remember that $s \,<\, 1$ and $\nu_{\min} \,>\, 0$). This ends the proof of Theorem 23.

3. Upper large deviations

3.1. Sketch of the proof. We first prove that $\widetilde{\phi_\Omega}$ is finite, i.e., that there exists a polyhedral set $P \subset \mathbb{R}^d$ such that ∂P is transverse to Γ and

$$\overline{\Gamma^1} \,\subset\, \overset{\circ}{P}\,,\ \overline{\Gamma^2} \,\subset\, \widehat{\mathbb{R}^d \smallsetminus \overset{\circ}{P}}\,.$$

Then, we consider such a polyhedral set P whose capacity $\mathcal{I}_\Omega(P)$ is close to $\widetilde{\phi_\Omega}$. We construct a set Ω' that contains a small neighbourhood of Ω, thus Ω' contains Ω_n for all large n, and such that $\mathcal{H}^{d-1}(\partial P \cap (\Omega' \smallsetminus \Omega))$ is very small. We need the property that ∂P is transverse to Γ to obtain this control on $\mathcal{H}^{d-1}(\partial P \cap (\Omega' \smallsetminus \Omega))$. We want to construct a (Γ^1_n, Γ^2_n)-cut in Ω_n that is close to $\partial P \cap \Omega'$. We cover $\partial P \cap \Omega'$ with cylinders of arbitrarily small height; this is the reason why we need to consider a polyhedral set P. A part of $\partial P \cap \Omega'$ of very small area is missing in this covering. We construct then a (Γ^1_n, Γ^2_n)-cut in Ω_n with the help of cutsets in the cylinders constructed on $\partial P \cap \Omega'$. To achieve this, we have to add edges to cover the part of $\partial P \cap \Omega'$ missing in the covering by the cylinders, and to glue together the cutsets in the different cylinders. Thanks to our study of the upper large deviations for the maximal flow through cylinders (see Part 1 of the thesis), we obtain that the probability that the flow ϕ_n is greater than $\mathcal{I}_\Omega(P) n^{d-1}$ goes to zero. We want to prove that this probability decays exponentially fast in n^d. For that purpose, we have to consider a collection of cardinality of order n of possible sets of edges we can add to construct the cutset in Ω_n, and to choose the set that has the minimal capacity.

The notations (especially the constants) introduced in section 3 are independent of those introduced in section 2.

3.2. The constant $\widetilde{\phi_\Omega}$ is finite. To prove that $\widetilde{\phi_\Omega} < \infty$, it is sufficient to exhibit a set P satisfying all the conditions given in the definition of $\widetilde{\phi_\Omega}$. Indeed, if such a set P exists, then

$$\widetilde{\phi_\Omega} \le \nu_{\max} \mathcal{H}^{d-1}(\partial P \cap \Omega) < \infty$$

since a polyhedral set has finite perimeter in Ω. We will construct such a set P. The idea of the proof is the following. We will cover $\overline{\Gamma^1}$ with small hypercubes which are transverse to Γ^1 and at positive distance of $\overline{\Gamma^2}$. Then, by compactness, we will extract a finite covering. We will denote by P the union of the hypercubes of this finite covering. Then P satisfies the desired properties. The method we will use to construct these small hypercubes transverse to Γ^1 is one of the techniques used to prove theorem 25, so it will be recalled in section 4.1.

We prove a geometric lemma:

LEMMA 19. *Let Γ be an hypersurface (that is a C^1 submanifold of $\mathbb{R}^d$ of codimension 1) and let K be a compact subset of Γ. There exists a positive $M = M(\Gamma, K)$ such that:*

$$\forall \varepsilon > 0 \quad \exists r > 0 \quad \forall x, y \in K \qquad |x-y|_2 \le r \quad \Rightarrow \quad d_2(y, \tan(\Gamma, x)) \le M \varepsilon |x-y|_2 .$$

($\tan(\Gamma, x)$ is the tangent hyperplane of Γ at x).

Proof :
By a standard compactness argument, it is enough to prove the following local property:

$$\forall x \in \Gamma \quad \exists M(x) > 0 \quad \forall \varepsilon > 0 \quad \exists r(x,\varepsilon) > 0 \quad \forall y, z \in \Gamma \cap B(x, r(x,\varepsilon))$$
$$d_2(y, \tan(\Gamma, z)) \le M(x) \varepsilon |y-z|_2 .$$

Indeed, if this property holds, we cover K by the open balls $\overset{o}{B}(x, r(x,\varepsilon)/2)$, $x \in K$, we extract a finite subcovering $\overset{o}{B}(x_i, r(x_i,\varepsilon)/2)$, $1 \le i \le k$, and we set

$$M = \max\{ M(x_i) : 1 \le i \le k \}, \quad r = \min\{ r(x_i,\varepsilon)/2 : 1 \le i \le k \} .$$

Let now y, z belong to K with $|y-z|_2 \le r$. Let i be such that y belongs to $B(x_i, r(x_i,\varepsilon)/2)$. Since $r \le r(x_i,\varepsilon)/2$, then both y, z belong to the ball $B(x_i, r(x_i,\varepsilon))$ and it follows that

$$d_2(y, \tan(\Gamma, z)) \le M(x_i) \varepsilon |y-z|_2 \le M \varepsilon |y-z|_2 .$$

We turn now to the proof of the above local property. Since Γ is an hypersurface, for any x in Γ there exists a neighbourhood V of x in $\mathbb{R}^d$, a diffeomorphism $f : V \mapsto \mathbb{R}^d$ of class C^1 and a $(d-1)$ dimensional vector space Z of $\mathbb{R}^d$ such that $Z \cap f(V) = f(\Gamma \cap V)$ (see for instance [**29**], 3.1.19). Let A be a compact neighbourhood of x included in V. Since f is a diffeomorphism, the maps $y \in A \mapsto df(y) \in \mathrm{End}(\mathbb{R}^d)$, $u \in f(A) \mapsto df^{-1}(u) \in \mathrm{End}(\mathbb{R}^d)$ are continuous. Therefore they are bounded:

$$\exists M>0 \quad \forall y \in A \quad ||df(y)|| \leq M\,, \quad \forall u \in f(A) \quad ||df^{-1}(u)|| \leq M$$

(here $||df(x)|| = \sup\{\,|df(x)(y)|_2 : |y|_2 \leq 1\,\}$ is the standard operator norm in End($\mathbb{R}^d$)). Since $f(A)$ is compact, the differential map df^{-1} is uniformly continuous on $f(A)$:

$$\forall \varepsilon>0 \quad \exists \delta>0 \quad \forall u,v \in f(A) \qquad |u-v|_2 \leq \delta \quad \Rightarrow \quad ||df^{-1}(u)-df^{-1}(v)|| \leq \varepsilon\,.$$

Let ε be positive and let δ be associated to ε as above. Let ρ be positive and small enough so that $\rho<\delta/2$ and $B(f(x),\rho) \subset f(A)$ (since f is a C^1 diffeomorphism, $f(A)$ is a neighbourhood of $f(x)$). Let r be such that $0<r<\rho/M$ and $B(x,r)\subset A$. We claim that M associated to x and r associated to ε, x answer the problem. Let y,z belong to $\Gamma\cap B(x,r)$. Since $[y,z]\subset B(x,r)\subset A$, and $||df(\zeta)|| \leq M$ on A, then

$$|f(y)-f(x)|_2 \leq M|y-x|_2 \leq Mr < \rho\,, \quad |f(z)-f(x)|_2<\rho\,,$$
$$|f(y)-f(z)|_2<\delta\,, \quad |f(y)-f(z)|_2 < M|y-z|_2\,.$$

We apply next a classical lemma of differential calculus (see [**45**], I, 4, Corollary 2) to the map f^{-1} and the interval $[f(z),f(y)]$ (which is included in $B(f(x),\rho)\subset f(A)$) and the point $f(z)$:

$$|y-z-df^{-1}(f(z))(f(y)-f(z))|_2 \leq$$
$$|f(y)-f(z)|_2 \sup\{\,||df^{-1}(\zeta)-df^{-1}(f(z))|| : \zeta\in[f(z),f(y)]\,\}\,.$$

The right–hand member is less than $M|y-z|_2\,\varepsilon$. Since $z+df^{-1}(f(z))(f(y)-f(z))$ belongs to $\tan(\Gamma,z)$, we are done.

■

We come back to our case. The boundary Γ of Ω is piecewise of class $\mathcal{C}^1$, i.e., it is included in a finite union of $\mathcal{C}^1$ hypersurfaces, which we denote by $(S_1,...,S_p)$. The hypersurfaces $S_1,\dots,S_p$ being $\mathcal{C}^1$ and the set Γ compact, the maps $x\in\Gamma\mapsto v_{S_k}(x)$, $1\leq k\leq p$ (where $v_{S_k}(x)$ is the unit normal vector to S_k at x) are uniformly continuous:

$$\forall\delta>0 \quad \exists\eta>0 \quad \forall k\in\{\,1,\dots,p\,\} \quad \forall x,y\in S_k\cap\Gamma \quad |x-y|_2\leq\eta \Rightarrow \Big|v_{S_k}(x)-v_{S_k}(y)\Big|_2<\delta\,.$$

Let η^* be associated to $\delta=1$ by this property. Let $k\in\{\,1,\dots,p\,\}$. The set $S_k\cap\Gamma$ is a compact subset of the hypersurface S_k. Applying the previous lemma, we get:

$$\exists M_k\ \forall\delta_0>0\ \exists\,\eta_k>0\ \forall x,y\in S_k\cap\Gamma \quad |x-y|_2\leq\eta_k \Rightarrow d_2\big(y,\tan(S_k,x)\big)\leq M_k\delta_0|x-y|_2\,.$$

Let $M_0 = \max_{1\leq k\leq p} M_k$ and let δ_0 in $]0,1/2[$ be such that $M_0\delta_0<1/2$. For each k in $\{\,1,\dots,p\,\}$, let η_k be associated to δ_0 as in the above property and let

$$\eta_0 \,=\, \min\Big(\min_{1\leq k\leq p}\eta_k,\,\eta^*,\,\frac{1}{8d}\,\mathrm{dist}(\Gamma^1,\Gamma^2)\Big)\,.$$

We build a family of cubes $Q(x,r)$, indexed by $x\in\Gamma$ and $r\in]0,r_\Gamma[$ such that $Q(x,r)$ is a cube centered at x of side length r which is transverse to Γ. For $x\in\mathbb{R}^d$ and $k\in\{\,1,\dots,p\,\}$, let $p_k(x)$ be a point of $S_k\cap\Gamma$ such that

$$|x-p_k(x)|_2 \,=\, \inf\,\Big\{\,|x-y|_2 : y\in S_k\cap\Gamma\,\Big\}\,.$$

Such a point exists since $S_k\cap\Gamma$ is compact. We define then for $k\in\{\,1,\dots,p\,\}$

$$\forall x\in\mathbb{R}^d \qquad v_k(x)\,=\,v_{S_k}(p_k(x))\,.$$

We define also

$$d_r \,=\, \inf_{v_1,\dots,v_p\in S^{d-1}}\ \max_{b\in\mathcal{B}_d}\ \min_{\substack{1\leq k\leq r\\ e\in b}}\ \Big(|e-v_i|_2,|-e-v_i|_2\Big)$$

where $\mathcal{B}_d$ is the collection of the orthonormal basis of $\mathbb{R}^d$ and S^{d-1} is the unit sphere of $\mathbb{R}^d$. Let η be associated to $d_r/4$ as in the above continuity property. We set

$$r_\Gamma = \frac{\eta}{2d}\,.$$

Let $x \in \Gamma$. By the definition of d_r, there exists an orthonormal basis b_x of $\mathbb{R}^d$ such that

$$\forall e \in b_x \quad \forall k \in \{1,\dots,p\} \quad \min\Big(|e - v_k(x)|_2, |-e - v_k(x)|_2\Big) > \frac{d_r}{2}\,.$$

Let $Q(x,r)$ be the cube centered at x of sidelength r whose sides are parallel to the vectors of b_x. We claim that $Q(x,r)$ is transverse to Γ for $r < r_\Gamma$. Indeed, let $y \in Q(x,r) \cap \Gamma$. Suppose that $y \in S_k$ for some $k \in \{1,\dots,p\}$, so that $v_k(y) = v_{S_k}(y)$ and $|x - p_k(x)|_2 < dr_\Gamma$. In particular, we have $|y - p_k(x)|_2 < 2dr_\Gamma < \eta$ and $|v_{S_k}(y) - v_k(x)|_2 < d_r/4$. For $e \in b_x$,

$$\frac{d_r}{2} \le |e - v_k(x)|_2 \le |e - v_{S_k}(y)|_2 + |v_{S_k}(y) - v_k(x)|_2$$

whence

$$|e - v_{S_k}(y)|_2 \ge \frac{d_r}{2} - \frac{d_r}{4} = \frac{d_r}{4}\,.$$

This is also true for $-e$, therefore the faces of the cube $Q(x,r)$ are transverse to S_k.

Now we consider the collection

$$(\mathring{Q}(x,r), x \in \overline{\Gamma^1}, r < r_\Gamma)\,.$$

It covers $\overline{\Gamma^1}$. By compactness of $\overline{\Gamma^1}$, we can extract a finite covering $(\mathring{Q}(x_i,r_i), i \in I)$ from this collection. We define

$$P = \cup_{i\in I} Q(x_i,r_i)\,,$$

We claim that P satisfies all the hypothesis in the definition of $\widetilde{\phi_\Omega}$. Indeed, P is obviously polyhedral and transverse to Γ. Moreover, we know that

$$\overline{\Gamma^1} \subset \mathring{P}\,,$$

and since $d(P, \overline{\Gamma^2}) > 0$ we also obtain that

$$\overline{\Gamma^2} \subset \overset{\circ}{\widehat{\mathbb{R}^d \setminus P}}\,.$$

Actually, we could have considered a family of hypercubes transverse to Γ of the form

$$(\mathring{Q}(x,r), x \in \overline{\Gamma^1}, r < r(x,\Gamma)\,,$$

where $r(x,\Gamma)$ also depends on x. However, we will need a r_Γ independent of x in section 4.1, and it is not much more difficult to obtain than a $r(x,\Gamma)$ depending on x, so we defined and used this r_Γ in the proof.

3.3. Definition of the set Ω'.

Let λ be in $]\widetilde{\phi_\Omega}, +\infty[$. We are studying

$$\mathbb{P}[\phi_n \ge \lambda n^{d-1}]\,.$$

The definitions we will give here correspond to a strictly positive $\widetilde{\phi_\Omega}$. Indeed, we could adapt the definitions of s, P and δ to the case $\widetilde{\phi_\Omega} = 0$. However, as we will prove in section 4.2, $\Lambda(0) < 1 - p_c(d)$ implies that $\widetilde{\phi_\Omega} > 0$ with no more assumptions than in Theorem 24, so we will not do this adaptation.

There exists a positive s such that $\lambda > \widetilde{\phi_\Omega}(1+s)^2$. By definition of $\widetilde{\phi_\Omega}$, for every positive s, there exists a polyhedral subset P of $\mathbb{R}^d$, such that ∂P is transverse to Γ,

$$\overline{\Gamma^1} \subset \mathring{P}\,,\ \overline{\Gamma^2} \subset \overset{\circ}{\widehat{\mathbb{R}^d \setminus P}}$$

and

$$\mathcal{I}_\Omega(P) \,\leq\, \widetilde{\phi_\Omega}(1+s)\,.$$

Then $\lambda > \mathcal{I}_\Omega(P)(1+s)$ and

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \,\leq\, \mathbb{P}[\phi_n \geq \mathcal{I}_\Omega(P)(1+s)n^{d-1}]\,.$$

Since ∂P is transverse to Γ, we know that there exists $\delta_0 > 0$ (depending on λ, P and Γ) such that for all $\delta \leq \delta_0$,

$$\mathcal{H}^{d-1}(\partial P \cap (\mathcal{V}_2(\Omega,\delta) \smallsetminus \Omega)) \,\leq\, \frac{s\mathcal{I}_\Omega(P)}{2\nu_{\max}}\,.$$

Thus, for any set Ω' satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega,\delta_0)$, we have

$$\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) \,\leq\, \mathcal{I}_\Omega(P)(1+s/2)\,,$$

then $\lambda > (1+s/2)(\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x))$ and

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \,\leq\, \mathbb{P}\left[\phi_n \geq \left(\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x)\right)(1+s/2)n^{d-1}\right]\,.$$

We will construct a particular set Ω' satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega,\delta_0)$.

In the previous section, we have associated to each couple (x,r) in $\Gamma \times]0, r_\Gamma[$ a hypercube $Q(x,r)$ centered at x, of sidelength r, and which is transverse to Γ. Using exactly the same method, we can build a family of hypercubes

$$(Q'(x,r), x \in \Gamma, r < r_{(\Gamma,P)})$$

such that $Q'(x,r)$ is centered at x, of sidelength r, and it is transverse to Γ and ∂P. The family

$$(\overset{\circ}{Q'}(x,r), x \in \Gamma, r < \min(r_{(\Gamma,P)}, \delta_0/(2d)))$$

is a covering of the compact set Γ, thus we can extract a finite covering from this collection, we denote it by $(\overset{\circ}{Q'}(x_i,r_i), i \in J)$. We define

$$\Omega' \,=\, \Omega \cup \bigcup_{i \in J} \overset{\circ}{Q'}(x_i,r_i)\,.$$

Since $r_i \leq \delta_0/(2d)$ for all $i \in J$, we have $\Omega' \subset \mathcal{V}_2(\Omega,\delta_0)$. Moreover, ∂P is transverse to the boundary Γ' of Ω'. Finally, if we define

$$\delta_1 \,=\, \min_{i \in J} r_i/2\,,$$

we know that $\mathcal{V}_2(\Omega,\delta_1) \subset \Omega'$, and thus for all $n \geq 2d/\delta_1$, we have $\Omega_n \subset \Omega'$.

3.4. Existence of a family of (Γ_n^1, Γ_n^2)-cuts. In this section we prove that we can construct a family of disjoint (Γ_n^1, Γ_n^2)-cuts in Ω_n. Let ζ be a fixed constant larger than $2d$. We consider a parameter $h < h_0 = d(\partial P, \Gamma^1 \cup \Gamma^2)$. For $k \in \{0, ..., \lfloor hn/\zeta \rfloor\}$ we define

$$P(k) \,=\, \{x \in \mathbb{R}^d \,|\, d(x,P) \leq k\zeta/n\}\,,$$

and for $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$ we define

$$\begin{aligned} \mathcal{U}(k) \,&=\, \overset{\circ}{\widehat{(\mathbb{R}^d \smallsetminus P_{k+1})}} \smallsetminus \overset{\circ}{P}_k \\ &=\, \{x \in \mathbb{R}^d \,|\, k\zeta/n \leq d(x,P) < (k+1)\zeta/n\}\,, \end{aligned}$$

and $\mathcal{M}'(k) = \mathcal{U}(k) \cap \Omega'$ (see figure 10). We will prove the following lemma:

LEMMA 20. *There exists N large enough such that for all $n \geq N$, every path on the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ from Γ_n^1 to Γ_n^2 in Ω_n contains at least one edge which is included in the set $\mathcal{M}'(k)$ for $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$.*

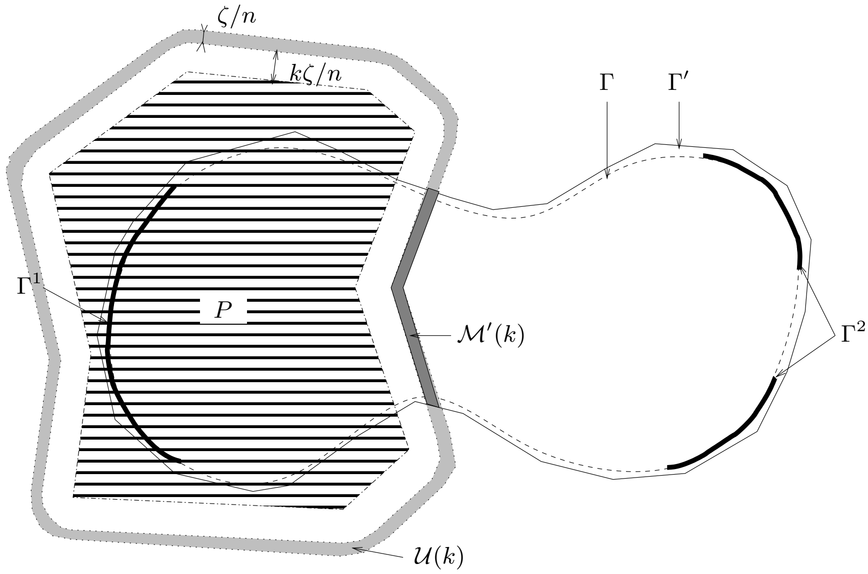

FIGURE 10. The sets $P, \mathcal{U}(k)$ and $\mathcal{M}'(k)$.

This lemma states precisely that for all $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$, $\mathcal{M}'(k)$ contains a (Γ_n^1, Γ_n^2)-cut in Ω_n.

Proof :

Let $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$. Let γ be a discrete path from Γ_n^1 to Γ_n^2 in Ω_n. In particular, γ is continuous, so we can parametrise it : $\gamma = (\gamma_t)_{0 \leq t \leq 1}$. There exists N large enough such that for all $n \geq N$, we have

$$\Omega_n \subset \Omega' , \quad \Gamma_n^1 \subset \mathcal{V}_2(\Gamma^1, 2d/n) \subset \overset{\circ}{P}_k , \quad \text{and} \quad \Gamma_n^2 \subset \mathcal{V}_2(\Gamma^2, 2d/n) \subset \overset{\circ}{\widehat{\mathbb{R}^d \smallsetminus P_{k+1}}} .$$

Since γ is continuous, we know that there exists $t_1, t_2 \in]0, 1[$ such that

$$t_1 = \sup\{t \in [0,1] \,|\, \gamma_t \in \overset{\circ}{P}_k\} ,$$

$$t_2 = \inf\{t \geq t_1 \,|\, \gamma_t \in \overset{\circ}{\widehat{\mathbb{R}^d \smallsetminus P_{k+1}}}\} .$$

Since

$$\overset{\circ}{P}_k \cup \mathcal{U}(k) \cup \overset{\circ}{\widehat{\mathbb{R}^d \smallsetminus P_{k+1}}}$$

is a partition of $\mathbb{R}^d$, we know that $(\gamma_t)_{t_1 \leq t < t_2}$, which is a continuous path, is included in $\mathcal{U}(k)$. The length of $(\gamma_t)_{t_1 \leq t < t_2}$ is larger than $d(\gamma_{t_1}, \gamma_{t_2})$. The segment $[\gamma_{t_1}, \gamma_{t_2}]$ intersects

$$\{x \in \mathbb{R}^d \,|\, d(x, P) = (k + 1/2)\zeta/n\}$$

at a point z, and we know that

$$\mathcal{V}_2(z, \zeta/(2n)) \subset \overset{\circ}{\widehat{V(k)}} .$$

Thus $d(\gamma_{t_1}, \gamma_{t_2}) \geq \zeta/n$, and then the length of $(\gamma_t)_{t_1 \leq t < t_2}$ is larger than ζ/n. Finally, γ is composed of edges of length $1/n$, and $\zeta \geq 2d$, so $(\gamma_t)_{t_1 \leq t < t_2}$, and thus γ, contains at least one edge which is included in $\mathcal{U}(k)$. Noticing that for all $n \geq N$,

$$\gamma \subset \Omega_n \subset \Omega' ,$$

we obtain that this edge belongs to $\mathcal{U}(k) \cap \Omega' = \mathcal{M}'(k)$.

■

3.5. Convering of $\partial P \cap \Omega'$ by cylinders. From now on we only consider $n \geq N$. According to lemma 20, we know that each set $\mathcal{M}'(k)$ for $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$ contains a (Γ_n^1, Γ_n^2)-cut in Ω_n, thus if we denote by $M'(k)$ the set of the edges included in $\mathcal{M}'(k)$, we obtain

$$\phi_n \leq \min\{V(M'(k)), \, k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}\}.$$

However, we do not have estimates on $V(M'(k))$ that allow us to control ϕ_n using only the previous inequality. The estimates we can use are the one of the upper large deviations for the maximal flow from the top to the bottom of a cylinder (see Chapter 3 of the thesis). In this section, we will transform our family of cuts $(M'(k))$ by replacing a huge part of the edges in each $\mathcal{M}'(k)$ by the edges of minimal cutsets in cylinders.

We denote by $H_i, i = 1, ..., \mathcal{N}$ the intersection of the faces of ∂P with Ω'. For each $i = 1, ..., \mathcal{N}$, we denote by v_i the exterior normal unit vector to P along H_i. We will cover $\partial P \cap \Omega'$ by cylinders, except a surface of $\mathcal{H}^{d-1}$ measure controlled by a parameter ε. To explain the construction of a cutset we will do with a huge number of cylinders, we present first the simpler construction of a cutset using one cylinder. Let R be a hyperrectangle that is included in H_j for a $j \in \{1, ..., \mathcal{N}\}$, and let B be the cylinder defined by

$$B = \{x + tv_j \,|\, x \in R, \, t \in [0, h]\},$$

where $h \leq h_0$ is the same parameter as previously. The cylinder B is built on $\partial P \cap \Omega'$, in $\mathbb{R}^d \smallsetminus \overset{\circ}{P}$. We recall that $h_0 = d(\partial P, \Gamma^1 \cup \Gamma^2) > 0$, so we know that $d(B, \Gamma^1 \cup \Gamma^2) > 0$. We denote by E_a the set of the edges included in

$$\mathcal{E}_a = \{x + tv_j \,|\, x \in R, \, d(x, \partial R) < \zeta/n, \, t \in [0, h]\}.$$

The set $\mathcal{E}_a$ is a neighbourhood in B of the "vertical" faces of B, i.e., the faces of B that are collinear to v_j. We denote by E_b a set of edges in B that cuts the top $R + hv_j$ from the bottom R of B. Let $M'(k)$ be the set of the edges included in $\mathcal{M}'(k)$, for a $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$. Let B' be the thinner cylinder

$$B' = \{x + tv_j \,|\, x \in R, \, d(x, \partial R) \geq \zeta/n, \, t \in [0, h]\}.$$

Thus for all $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$, the set of edges

$$(M'(k) \cap (\mathbb{R}^d \smallsetminus B')) \cup E_a \cup E_b$$

cuts Γ_n^1 from Γ_n^2 in Ω_n. Indeed, the set of edges $M'(k)$ is already a cut between Γ_n^1 and Γ_n^2 in Ω_n. We remove from it the edges that are inside B' which is in the interior of B, and we add to it a cutset E_b from the top to the bottom of B, and the set of edges E_a that glue together E_b and $M'(k) \cap (\mathbb{R}^d \smallsetminus B')$. This property is illustrated in the figure 11.

REMARK 32. In this figure, we have represented E_b as a surface (so a path in dimension 2) that separates the top from the bottom of the cylinder to illustrate the fact that E_b cuts all discrete paths from the bottom to the top of B. Actually, we can mention that it is possible to define an object which could be the dual of an edge in dimension $d \geq 2$ (as a generalization of the dual of a planar graph). This object is a plaquette, i.e., a hypersquare of sidelength $1/n$ that is orthogonal to the edge and cuts it in its middle, and whose sides are parallel to the hyperplanes of the axis. Then the dual of a cutset is a hypersurface of plaquettes, thus the figure 11 is somehow intuitive.

We do exactly the same construction, but with a large number of cylinders, that will almost cover $\partial P \cap \Omega'$. We consider a fixed $\varepsilon > 0$. There exists a l sufficiently small (depending on F, P and ε) such that there exists a finite collection $(R_{i,j}, i = 1, ..., \mathcal{N}, j = 1, ..., N_i)$ of hypersquares of side l of disjoint interiors satisfying $R_{i,j} \subset H_i$ for all $i \in \{1, ..., \mathcal{N}\}$ and $j \in \{1, ..., N_i\}$, and

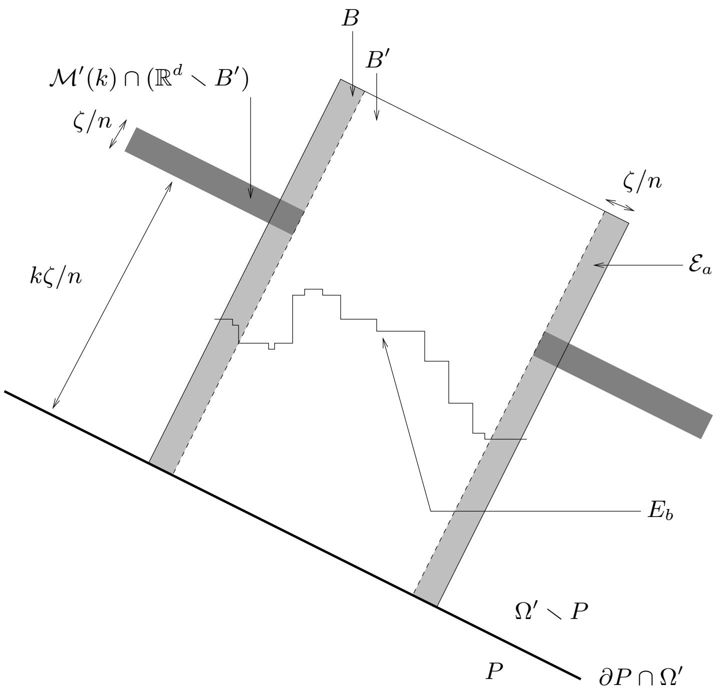

FIGURE 11. Construction of a (Γ_n^1, Γ_n^2)-cut in Ω_n using a cutset in a cylinder.

for all $i \in \{1, ..., \mathcal{N}\}$,

$$\{x \in H_i \,|\, d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-1}\} \subset \bigcup_{j=1}^{N_i} R_{i,j} \subset$$
$$\subset \{x \in H_i \,|\, d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-1} 2^{-1}\}\,.$$

We immediately obtain that

$$\mathcal{H}^{d-1}\left((\partial P \cap \Omega') \smallsetminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} R_{i,j} \right) \leq \varepsilon\,.$$

We remark that

$$\int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) \geq \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\,,$$

so that

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}\left[\phi_n \geq (1 + s/2) n^{d-1} \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i) \right]\,.$$

Let $h < h_0$. For all $i \in \{1, ..., \mathcal{N}\}$ and $j \in \{1, ..., N_i\}$, we define

$$B_{i,j} = \{x + t v_i \,|\, x \in R_{i,j}\,,\, t \in [0, h]\}\,.$$

Since all the $B_{i,j}$ are at strictly positive distance of ∂H_i, there exists a positive h_1 such that for all $h < h_1$, the cylinders $B_{i,j}$ have pairwise disjoint interiors. We thus consider $h < \min(h_0, h_1)$ (see figure 12 for example). At this point, we could define a neighbourhood of the vertical faces of each cylinder $B_{i,j}$, and do the same construction as in the previous example with one cylinder. Actually, we need to choose a little bit more carefully the sets of edges we define along the vertical faces of the cylinders. We will not consider only each cylinder $B_{i,j}$, but also thinner versions of these cylinders of the type

$$B_{i,j}(k) = \{x + t v_j \,|\, x \in R_{i,j}\,,\, d(x, \partial R_{i,j}) > k\zeta/n\,,\, t \in [0, h]\}$$

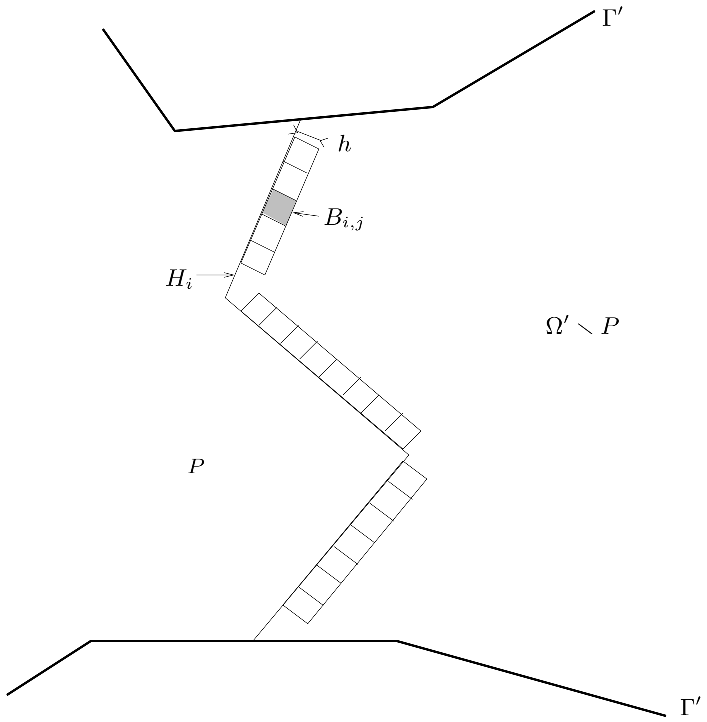

FIGURE 12. Covering of $\partial P \cap \Omega'$ by cylinders.

for different values of k. We will then consider the edges included in a neighbourhood of the vertical faces of each $B_{i,j}(k)$ (see the set $\mathcal{W}_{i,j}(k)$ above), and choose k to minimize the capacity of the union over i and j of these edges. The reason why we need this optimization is also the reason why we built a family $(M'(k))$ of cutsets and not only one cutset from Γ_n^1 to Γ_n^2 in Ω_n, we will try to explain it in remark 33.

Here are the precise definitions of the sets of edges. We still consider the same constants ζ bigger than $2d$ and $h < \min(h_0, h_1)$. We define another positive constant η that we will choose later (depending on P, s and Ω). For i in $\{1, ..., \mathcal{N}\}$ and j in $\{1, ..., N_i\}$ we recall the definition of $B_{i,j}$:

$$B_{i,j} = \{x + tv_i \,|\, x \in R_{i,j}\,,\ t \in [0,h]\}\,,$$

and we define the following subsets of $\mathbb{R}^d$:

$$B'_{i,j} = \{x + tv_i \,|\, x \in R_{i,j}\,,\ d(x, \partial R_{i,j}) > \eta\,,\ t \in [0,h]\}\,,$$

$$\forall k \in \{0, ..., \lfloor \eta n/\zeta - 1 \rfloor\}\,,\ \mathcal{W}_{i,j}(k) = \{x \in B_{i,j} \,|\, k\zeta/n \leq d_2(x, \partial R_{i,j} + \mathbb{R}v_i) < (k+1)\zeta/n\}\,,$$

$$\forall k \in \{0, ..., \lfloor hn\kappa/\zeta - 1 \rfloor\}\,,\ \mathcal{M}(k) = \mathcal{M}'(k) \setminus \left(\bigcup_{i,j} B'_{i,j} \right)\,,$$

(see figures 13 and 14). We denote by $W_{i,j}(k)$ the set of the edges included in $\mathcal{W}_{i,j}(k)$ and we define $W(k) = \cup_{i,j} W_{i,j}(k)$. We also denote by $M(k)$ the edges included in $\mathcal{M}(k)$. Exactly as in the construction of a cutset with one cylinder, we obtain a cutset that is built with cutsets in each cylinders $B_{i,j}$. Indeed, if we denote by $E_{i,j}$ a set of edges that is a cutset from the top to the bottom of $B_{i,j}$ (oriented towards the direction given by v_i), then for each $k_1 \in \{0, ..., \lfloor \eta n/\zeta - 1 \rfloor\}$ and $k_2 \in \{0, ..., \lfloor hn/\zeta - 1 \rfloor\}$, the set of edges:

$$\bigcup_{\substack{i=1,...,\mathcal{N} \\ j=1,...,N_i}} E_{i,j} \cup W(k_1) \cup M(k_2)$$

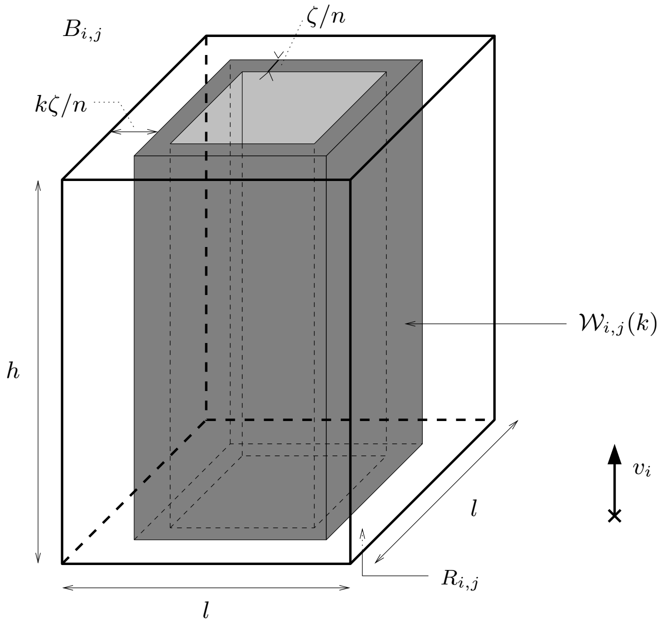

FIGURE 13. The set $\mathcal{W}_{i,j}(k)$.

contains a cutset from Γ_n^1 to Γ_n^2 in Ω_n. We deduce that

$$\phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + \min_{k_1} V(W(k_1)) + \min_{k_2} V(M(k_2)) . \tag{7.3}$$

3.6. Control of the cardinality of the sets of edges W and M. For the sake of clarity, we do not recall the sets in which the parameters take its values, we always assume that they are the following: $i \in \{1, ..., \mathcal{N}\}$, $j \in \{1, ..., N_i\}$, $k_1 \in \{0, ..., , \lfloor \eta n/\zeta - 1 \rfloor\}$ and $k_2 \in \{0, ..., \lfloor hn/\zeta - 1 \rfloor\}$. We have to evaluate the number of edges in the sets $W(k_1)$ and $M(k_2)$ to control the terms $\min_{k_1} V(W(k_1))$ and $\min_{k_2} V(M(k_2))$ in (7.3). There exist constants $c_1(d, \Omega)$, $c_2(P, d, \Omega)$ such that

$$\operatorname{card} W(k_1) \leq c_1 \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega')}{l^{d-1}} \zeta l^{d-2} h n^{d-1} \leq c_2 l^{-1} h n^{d-1} .$$

The cardinality of $M(k_2)$ is a little bit more complicated to control. We will divide $M(k)$ (respectively $\mathcal{M}(k)$) into three parts: $M(k) \subset M_1(k) \cup M_2(k) \cup M_3(k)$ (respectively $\mathcal{M}(k) \subset \mathcal{M}_1(k) \cup \mathcal{M}_2(k) \subset \mathcal{M}_3(k)$), that are represented in figure 14.

We define $R'_{i,j} = \{x \in R_{i,j} \,|\, d(x, \partial R_{i,j}) > \eta\}$ which is the basis of $B'_{i,j}$. The set $\mathcal{M}_1(k)$ is a translation of the sets $H_i \smallsetminus (\cup_{j=1}^{N_i} R'_{i,j})$ along the direction given by v_i enlarged with a thickness $\zeta/(n\kappa)$:

$$\mathcal{M}_1(k) \subset \bigcup_{i=1}^{\mathcal{N}} \{x + t v_i \,|\, x \in H_i \smallsetminus (\cup_{j=1}^{N_i} R'_{i,j}) \,,\; t \in [k\zeta/n, (k+1)\zeta/n[\} .$$

Here we have an inclusion and not an equality because $\mathcal{M}_1(k)$ can be a truncated version of this set (truncated at the junction between the translates of two different faces). Since we know that

$$\mathcal{H}^{d-1} \left((\partial P \cap \Omega') \smallsetminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} R_{i,j} \right) \leq \varepsilon ,$$

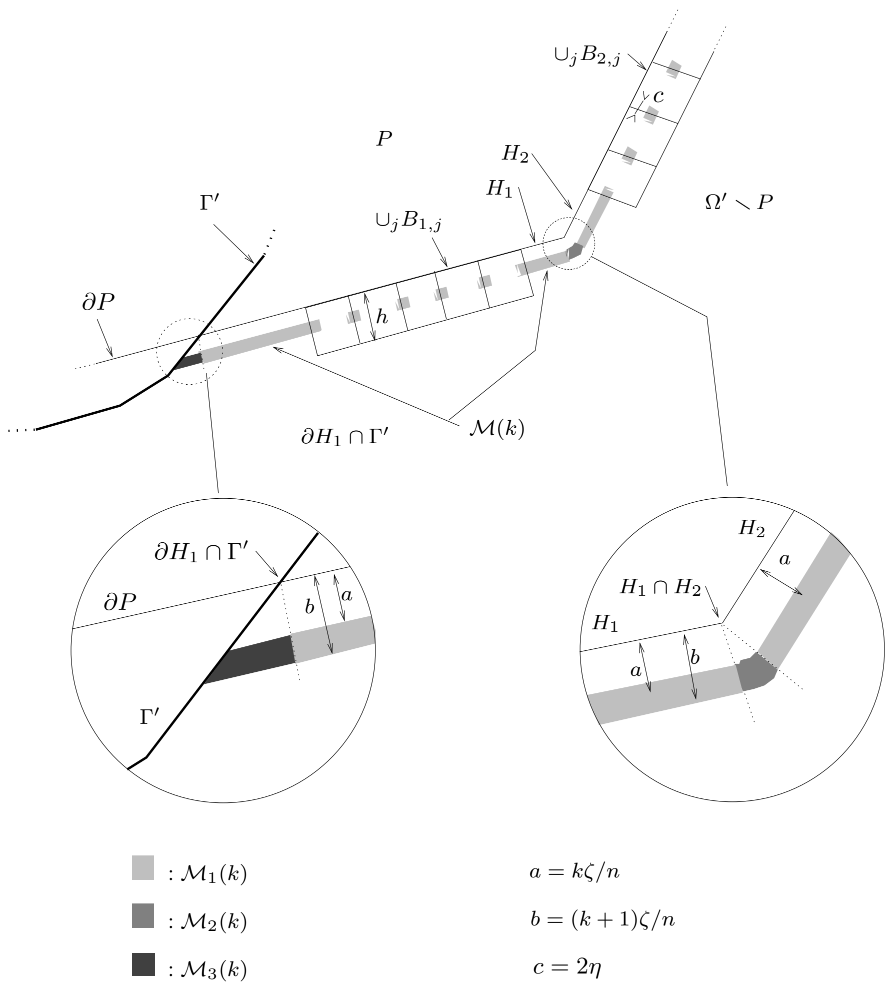

FIGURE 14. The set $\mathcal{M}(k)$.

and

$$\mathcal{H}^{d-1}\left(\bigcup_{i=1}^{\mathcal{N}}\bigcup_{j=1}^{N_i}(R_{i,j}\setminus R'_{i,j})\right) \leq \frac{\mathcal{H}^{d-1}(\partial P\cap\Omega')}{l^{d-1}}l^{d-2}\eta = \mathcal{H}^{d-1}(\partial P\cap\Omega')l^{-1}\eta\,,$$

we have the following bound on the cardinality of $M_1(k)$:

$$\mathrm{card}(M_1(k)) \leq c_3(\varepsilon+l^{-1}\eta)n^{d-1}\,,$$

for a constant $c_3(d,P,\Omega,\Omega')$.

The part $M_2(k)$ corresponds to the edges included in the ”bends” of the neighbourhood of ∂P located around the boundary of the faces of ∂P in Ω', denoted by $\mathcal{M}_2(k)$, i.e.:

$$\mathcal{M}_2(k) \subset \bigcup_{i,j}(\mathcal{V}_2(H_i\cap H_j,(k+1)\zeta/n)\setminus\mathcal{V}_2(H_i\cap H_j,k\zeta/n))\,,$$

and there exists a constant $c_4(d,P,\Omega')$ such that

$$\mathrm{card}\,M_2(k) \leq c_4|k\zeta/n|^{d-2}n^{d-1} \leq c_4h^{d-2}n^{d-1}\,.$$

The last part $\mathcal{M}_3(k)$ corresponds to the part of $\mathcal{M}(k)$ that is near the boundary Γ' of Ω'. Indeed, Γ' is not orthogonal to ∂P, thus for some k, the set $\mathcal{M}(k)$ may contain edges that are not included in

$$\bigcup_{i=1}^{\mathcal{N}} \{x + tv_i \,|\, x \in H_i \smallsetminus (\cup_{j=1}^{N_i} R'_{i,j})\,,\ t \in [k\zeta/n, (k+1)\zeta/n[\}\,,$$

neither in

$$\bigcup_{i,j} \left(\mathcal{V}_2(H_i \cap H_j, (k+1)\zeta/n) \smallsetminus \mathcal{V}_2(H_i \cap H_j, k\zeta/n)\right)\,,$$

(see figure 14). However, $\mathcal{M}(k) \subset \mathcal{U}(k)$, the problem is to evaluate the difference of cardinality between the different $M(k)$ due to the intersection of $\mathcal{U}(k)$ with Ω'. We have constructed Ω' such that Γ' is transverse to ∂P precisely to obtain this control. The sets Γ' and ∂P are polyhedral surfaces which are transverse. We denote by $(\mathcal{H}_i, i \in I)$ (resp. $(\mathcal{H}'_j, j \in J)$) the hyperplanes that contain ∂P (resp. Γ'), and by v_i (resp. v'_j) the exterior normal unit vector to P along $\mathcal{H}_i$ (resp. Ω' along $\mathcal{H}'_j$). The set $\Gamma' \cap \partial P$ is included in the union of a finite number of intersections $\mathcal{H}_i \cap \mathcal{H}'_j$ of transverse hyperplanes. To each such intersection $\mathcal{H}_i \cap \mathcal{H}'_j$, we can associate the angles between v_i and v'_j, and between v_i and $-v'_j$, in the plane of dimension 2 spanned by v_i and v'_j. Each such angle is strictly positive because $\mathcal{H}_i$ is transverse to $\mathcal{H}'_j$, and so the minimum θ_0 over the finite number of defined angles is strictly positive. This θ_0 and the measure $\mathcal{H}^{d-2}(\partial P \cap \Gamma')$ give to us a control on the volume of $\mathcal{M}_3(k)$, and thus on $\mathrm{card}(M_3(k))$, as soon as these sets belong to a neighbourhood of $\partial P \cap \Gamma'$ (see figure 15). Thus, there exist $h_2(\Omega', P) > 0$ and a constant

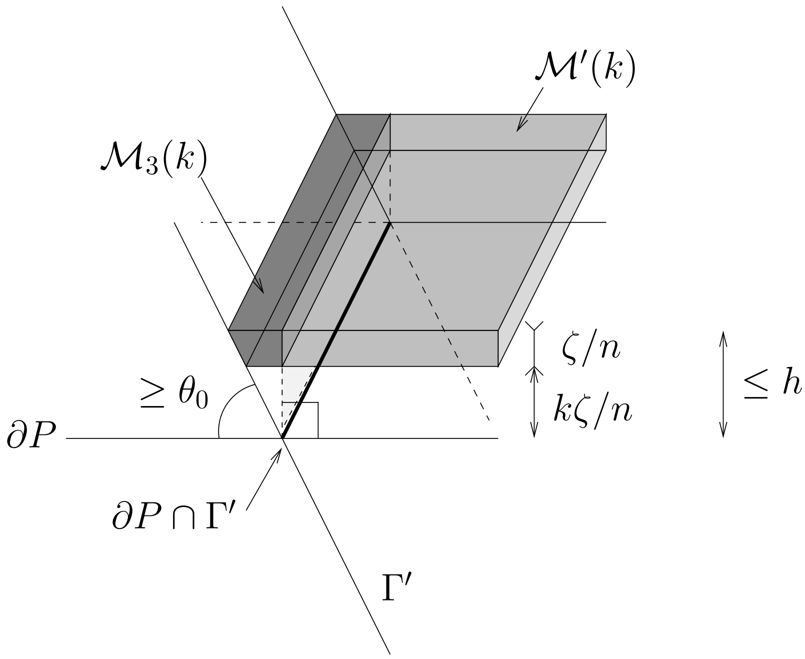

FIGURE 15. The set $\mathcal{M}_3(k)$.

$c_5(d, P, \Omega, \Omega')$ such that for all $h \leq h_2$,

$$\mathrm{card}(M_3)(k) \;=\; c_5 h n^{d-1}\,.$$

We conclude that there exists a positive constant $c_6(d, P, \Omega, \Omega')$ such that

$$\mathrm{card}\, M(k) \;\leq\; c_6(\varepsilon + l^{-1}\eta + h^{d-2} + h)n^{d-1}\,.$$

3.7. Calibration of the constants. We remark that the sets $W(k)$ (resp., the sets $M(k)$) are pairwise disjoint for different k. Then we obtain that

$$\begin{aligned}
\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \,\leq\, & \mathbb{P}\left[\phi_n \geq (1+s/2)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
\leq\, & \mathbb{P}\left[\sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{N_i}\phi_{B_{i,j}} \geq (1+s/4)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
& + \mathbb{P}\left[\min_{k_1} V(W(k_1)) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
& + \mathbb{P}\left[\min_{k_2} V(M(k_2)) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right] \\
\leq\, & \sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{N_i}\left(\max_{i,j}\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]\right) \\
& + \mathbb{P}\left[\sum_{i=1}^{c_2 l^{-1} h n^{d-1}} t(e_i) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right]^{\lfloor \eta n/\zeta\rfloor} \\
& + \mathbb{P}\left[\sum_{i=1}^{c_6(\varepsilon + l^{-1}\eta + h^{d-2}+h)n^{d-1}} t(e_i) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)\right]^{2\lfloor hn/\zeta\rfloor} .
\end{aligned}$$

The terms

$$\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]$$

have already been studied in Chapter 3, where we proved the following theorem for the standard model of first passage percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$:

THEOREM 33. *We suppose that the law Λ of the capacities of the edges admits an exponential moment. Let A be a non degenerate hyperrectangle in $\mathbb{R}^d$, of normal unit vector v, and let $B = \mathrm{cyl}(nA, h(n))$ where $h : \mathbb{N} \to \mathbb{R}^+$ is such that $\lim_{n\to\infty} h(n) = +\infty$. For every $\lambda > \nu(v)$, we have*

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \log \mathbb{P}[\phi_B \geq \lambda \mathcal{H}^{d-1}(A) n^{d-1}] \,<\, 0\,.$$

We can obviously apply Theorem 33 to our rescaled lattice $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$, we are in the particular case where $h(n)/n$ is constant since actually our cylinders $B_{i,j}$ are fixed and the mesh of the lattice goes to zero at the same speed in each direction.

It remains to study two terms of the type

$$\mathcal{P}(n) \,=\, \mathbb{P}\left(\sum_{i=1}^{\alpha n^{d-1}} t(e_i) \geq \beta n^{d-1}\right).$$

As soon as $\beta > \alpha\mathbb{E}(t)$ and the law of the capacity of the edges admits an exponential moment, the Cramér theorem in $\mathbb{R}$ allows us to affirm that

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}} \log \mathcal{P}(n) \,<\, 0\,.$$

Moreover, for all

$$\varepsilon \,\leq\, \varepsilon_0 \,=\, \frac{1}{2\nu_{\max}} \int_{\mathcal{P}\cap\Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x)\,,$$

we have

$$\begin{aligned}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i) \,&\geq\, \int_{\partial P\cap\Omega'} \nu(v_P(x))d\mathcal{H}^{d-1}(x) - \varepsilon\nu_{max}\\ &\geq\, \frac{1}{2}\int_{\partial P\cap\Omega'} \nu(v_P(x))d\mathcal{H}^{d-1}(x)\\ &\geq\, \frac{\nu_{\min}}{2}\mathcal{H}^{d-1}(\partial P\cap\Omega')\,.\end{aligned}$$

Thus, for all $\varepsilon < \varepsilon_0$ and $h < \min(h_0, h_1, h_2)$, if the constants satisfy the two following conditions:

$$c_2 l^{-1} h \,<\, \mathcal{H}^{d-1}(\partial P\cap\Omega')\nu_{\min}\mathbb{E}(t(e))s/16\,, \tag{7.4}$$

and

$$c_6(\varepsilon + l^{-1}\eta + h^{d-2} + h) \,<\, \mathcal{H}^{d-1}(\partial P\cap\Omega')\nu_{\min}\mathbb{E}(t(e))s/16\,, \tag{7.5}$$

thanks to lemma 33 (applied in the rescaled graph, so $h(n)/n$ is constant) and the Cramér theorem in $\mathbb{R}$, we obtain that

$$\limsup_{n\rightarrow\infty}\frac{1}{n^d}\log\mathbb{P}[\phi_n\geq\lambda n^{d-1}] \,<\, 0\,,$$

and theorem 24 is proved. We claim that it is possible to choose the constants such that conditions (7.4) and (7.5) are satisfied. Indeed, we first choose $\varepsilon < \varepsilon_0$ such that

$$\varepsilon \,<\, \frac{1}{4}\frac{\mathcal{H}^{d-1}(\partial P\cap\Omega)\nu_{\min}\mathbb{E}(t(e))s}{16 c_6}\,.$$

To this fixed ε corresponds a l. Knowing ε and l, we choose $h\leq\min(h_0, h_1, h_2)$ and η such that

$$\max(h, h^{d-2}, l^{-1}h, l^{-1}\eta) \,<\, \frac{1}{4}\frac{\mathcal{H}^{d-1}(\partial P\cap\Omega')\nu_{\min}\mathbb{E}(t(e))s}{16\max(c_2, c_6)}\,.$$

This ends the proof of theorem 24.

REMARK 33. We try here to explain why we built several sets $W(k_1)$ and $M(k_2)$, and not only one couple of such sets, that would have been sufficient to construct a cutset from Γ_n^1 to Γ_n^2 in Ω_n. To use estimates of upper large deviations of maximal flows in cylinder we already know, we want to compare ϕ_n with $\sum_{i,j}\phi_{B_{i,j}}$. Heuristically, to construct a (Γ_n^1, Γ_n^2)-cut in Ω_n from the union of cutsets in each cylinder $B_{i,j}$, we have to add edges to glue together the different cutsets at the common boundary of the small cylinders, and to extend these cutsets to $(\partial P\cap\Omega_n)\smallsetminus\bigcup_{i=1}^{\mathcal{N}}\bigcup_{j=1}^{N_i} R_{i,j}$. Yet we want to prove that the upper large deviations of ϕ_n are of volume order. If we only consider one possible set E of edges such that

$$\phi_n \,\leq\, \sum_{i,j}\phi_{B_{i,j}} + V(E)\,,$$

we will obtain that

$$\begin{aligned}\mathbb{P}[\phi_n\geq\lambda n^{d-1}] \,\leq\, &\sum_{i,j}\mathbb{P}[\phi_{B_{i,j}}\geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]\\ &+\mathbb{P}\left[V(E)\geq n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1}\nu(v_i)s/4\right]\,.\end{aligned}$$

We can choose such a set E so that it contains less than δn^{d-1} edges for a small δ (E is equal to $W(k_1)\cup M(k_2)$ for a fixed couple (k_1, k_2) for example), but the probability

$$\mathbb{P}\left[\sum_{i=1}^{\delta n^{d-1}} t(e_i)\geq C n^{d-1}\right]$$

does not decay exponentially fast with n^d in general. To obtain this speed of decay, we have to make an optimization over the possible choices of the set E, i.e., we choose E among a set of $C'n$ possible disjoint sets of edges $E_1, ..., E_{C'n}$; in this case, we obtain that

$$\phi_n \,\leq\, \sum_{i,j} \phi_{B_{i,j}} + \min_{k=1,\dots,C'n} V(E_k)\,,$$

and so

$$\begin{aligned} \mathbb{P}[\phi_n \geq \lambda n^{d-1}] \,\leq\, & \sum_{i,j} \mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}] \\ & + \prod_{k=1}^{C'n} \mathbb{P}\left[V(E_k) \geq n^{d-1} \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i) s/4 \right]. \end{aligned} \tag{7.6}$$

It is then sufficient to prove that for all k, $\mathbb{P}[V(E_k) \geq C'' n^{d-1}]$ decays exponentially fast with n^{d-1} to conclude that the last term in (7.6) decays exponentially fast with n^d. The results we have proved in Chapter 3 control the terms

$$\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1+s/4)n^{d-1}]\,.$$

The conclusion is that to obtain the volume order of the upper large deviations, the optimization over the different possible values of k_1 and k_2 is really important, even if it is not needed if we only want to prove that $\mathbb{P}(\phi_n \geq \lambda n^{d-1})$ goes to zero when n goes to infinity.

4. Geometric part of the proof

4.1. Polyhedral approximation. We consider an open bounded domain Ω in $\mathbb{R}^d$. We denote its topological boundary by $\Gamma = \partial\Omega$. Let also Γ^1, Γ^2 be two disjoint subsets of Γ.

Hypothesis on Ω: We suppose that Ω is a Lipschitz domain, i.e., its boundary Γ can be locally represented as the graph of a Lipschitz function defined on some open ball of $\mathbb{R}^{d-1}$. Moreover there exists a finite number of oriented hypersurfaces $S_1, \dots, S_p$ of class C^1 which are transerve to each other and such that Γ is included in their union $S_1 \cup \cdots \cup S_p$.

This hypothesis is automatically satisfied when Ω is a bounded open set with a C^1 boundary or when Ω is a polyhedral domain. The Lipschitz condition can be expressed as follows: each point x of $\Gamma = \partial\Omega$ has a neighbourhood U such that $U \cap \Omega$ is represented by the inequality $x_n < f(x_1, \cdots, x_{n-1})$ in some cartesian coordinate system where f is a function satisfying a Lipschitz condition. Such domains are usually called Lipschitz domains in the literature. The boundary Γ of a Lipschitz domain is $d-1$ rectifiable (in the terminology of Federer's book [**29**]), so that its Minkowski content is equal to $\mathcal{H}^{d-1}(\Gamma)$. In addition, a Lipschitz domain Ω is admissible (in the terminology of Ziemer's book [**60**]) and in particular $\mathcal{H}^{d-1}(\Gamma \smallsetminus \partial^*\Omega) = 0$. Moreover, each point of Γ is accessible from Ω through a rectifiable arc.

Hypothesis on Γ^1, Γ^2: The sets Γ^1, Γ^2 are open subsets of Γ. The relative boundaries $\partial_\Gamma\, \Gamma^1$, $\partial_\Gamma\, \Gamma^2$ of Γ^1, Γ^2 in Γ have null $\mathcal{H}^{d-1}$ measure. The distance between Γ^1 and Γ^2 is positive.

We recall that the relative topology of Γ is the topology induced on Γ by the topology of $\mathbb{R}^d$. Hence each of the sets Γ^1, Γ^2 is the intersection of Γ with an open set of $\mathbb{R}^d$. For F a subset of Ω having finite perimeter in Ω, we define its capacity

$$\mathcal{I}_\Omega(F) \,=\, \int_{\Omega\cap\partial^* F} \nu(v_F(y))\, d\mathcal{H}^{d-1}(y) + \int_{\Gamma^2\cap\partial^* F} \nu(v_F(y))\, d\mathcal{H}^{d-1}(y) + \int_{\Gamma^1\cap\partial^*(\Omega\smallsetminus F)} \nu(v_{\Omega\smallsetminus F}(y))\, d\mathcal{H}^{d-1}(y)\,.$$

For all $A \subset \mathbb{R}^d$, $\overline{A}$ is the closure of A, $\overset{\circ}{A}$ its interior and $A^c = \mathbb{R}^d \smallsetminus A$. We will prove the following theorem:

THEOREM 34. *Let F be a subset of Ω having finite perimeter. For any $\varepsilon > 0$, there exists a polyhedral set P whose boundary ∂P is transverse to Γ and such that*

$$\overline{\Gamma^1} \subset \overset{\circ}{P}\,, \quad \overline{\Gamma^2} \subset \widehat{\mathbb{R}^d \setminus \overset{\circ}{P}}\,, \quad \mathcal{L}^d(F\Delta(P\cap\Omega)) < \varepsilon\,,$$
$$\int_{\partial^* P\cap\Omega} \nu(v_P(x))d\mathcal{H}^{d-1}(x) = \mathcal{I}_\Omega(P) \leq \mathcal{I}_\Omega(F)+\varepsilon\,.$$

First we notice that theorem 34 implies theorem 25. It is obvious since

$$\phi_\Omega \leq \widetilde{\phi_\Omega}\,,$$

and theorem 34 implies that

$$\phi_\Omega \geq \widetilde{\phi_\Omega}\,.$$

The main difficulty of the proof of theorem 34 is to handle properly the approximation close to Γ in order to push back inside Ω all the interfaces. The essential tools of the proof are the Besicovitch differentiation theorem, the Vitali covering theorem and an approximation technique due to De Giorgi. Let us summarise the global strategy.

Sketch of the proof: We fix $\gamma > 0$. We cover $\partial^*\Omega$ up to a set of $\mathcal{H}^{d-1}$ measure less than γ by a finite collection of disjoint balls $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$, centered on Γ, whose radii are sufficiently small to ensure that the surface and volume estimates within the balls are controlled by the factor γ. The indices of I_1 correspond to balls centered on $\Gamma^1 \cap \partial^*(\Omega \setminus F)$, the indices of I_2 to balls centered on $\Gamma^2 \cap \partial^* F$, the indices of I_3 to balls centered on $(\Gamma \setminus \Gamma^2) \cap \partial^* F$, the indices of I_4 to balls centered on $(\Gamma \setminus \Gamma^1) \cap \partial^*(\Omega \setminus F)$ (see figure 16). The remaining part of Γ is covered

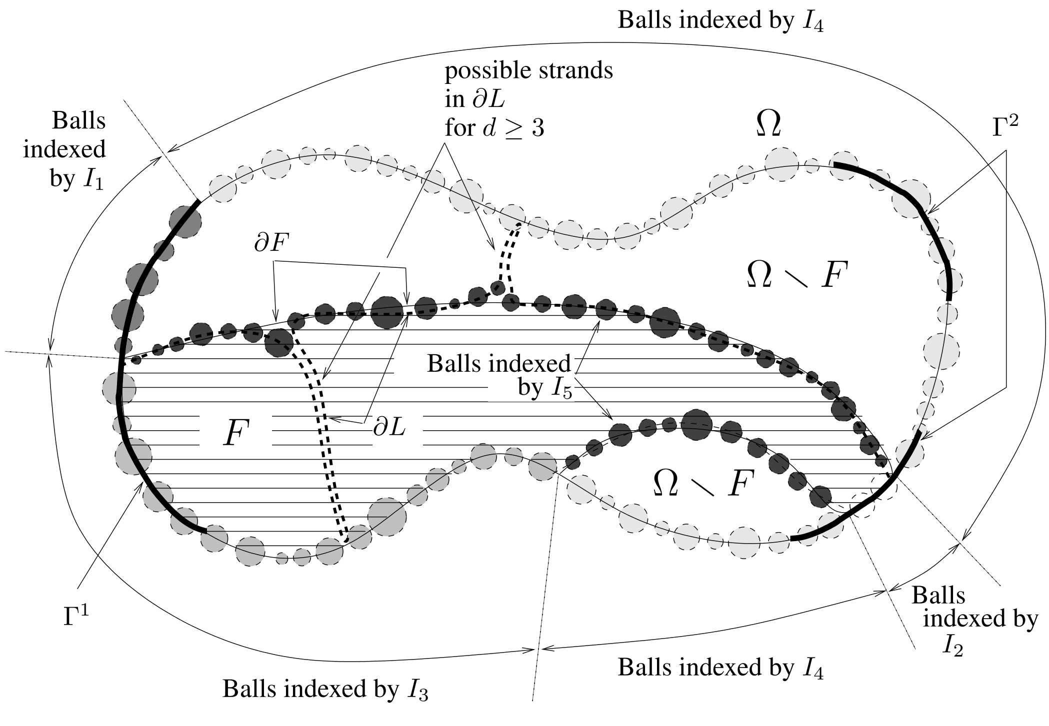

FIGURE 16. The balls indexed by I_i for $i = 1, ..., 5$.

by a finite collection of balls $B(y_j, s_j)$, $j \in J_0 \cup J_1 \cup J_2$. The indices of J_1 correspond to balls covering the remaining part of $\overline{\Gamma}_1$, the indices of J_2 correspond to balls covering the remaining part of $\overline{\Gamma}_2$. We choose $\varepsilon > 0$ sufficiently small, depending on γ and on the previous families of balls and we approximate the set F by a smooth set L inside Ω, whose capacity and volume are at distance less than ε from those of F. We build then two further family of balls:

- $B(x_i, r_i)$, $i \in I_5$, cover $\Omega \cap \partial L$, up to a set of $\mathcal{H}^{d-1}$ measure ε.
- $B(y_j, s_j)$, $j \in J_3$, cover the remaining set $\Omega \cap \partial L \smallsetminus \bigcup_{i \in I_5} B(x_i, r_i)$.

Inside each ball $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$, up to a small fraction, the interfaces are located on hypersurfaces and the radii of the balls are so small that these hypersurfaces are almost flat. Hence we can enclose the interfaces into small flat polyhedral cylinders D_i, $i \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$, and by aggregating adequately the cylinders to the set F or to its complement $\Omega \smallsetminus F$, we move these interfaces on the boundaries of these cylinders. The remaining interfaces are enclosed in the balls $B(y_j, s_j)$, $j \in J_0 \cup J_1 \cup J_2 \cup J_3$ and we approximate these balls from the outside by polyhedra.

We have to define delicately the whole process, in order not to lose too much capacity, and to control the possible interaction between interfaces close to Γ and interfaces in Ω. The presence of boundary conditions creates a substantial additional difficulty compared to the polyhedral approximation performed in [**19**]. Indeed, the most difficult interfaces to handle are those corresponding to D_i, $i \in I_3 \cup I_4$. We first choose the balls $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$, corresponding to γ. We cover the remaining portion of Γ with the balls $B(y_j, s_j)$, $j \in J_0 \cup J_1 \cup J_2$. At this point we can already in principle define the cylinders D_i, $i \in I_1 \cup I_2$. Then we choose ε small enough, depending on γ and the balls $B(x_i, r_i)$, $i \in I_1 \cup I_2 \cup I_3 \cup I_4$, to ensure that the perturbation of volume ε caused when smoothing the set F inside Ω will not alter significantly the situation inside the balls $B(x_i, r_i)$, $i \in I_3 \cup I_4$. Then we move inside Ω and we build the cylinders D_i, $i \in I_5$. Then we come back to the boundary and we build the cylinders D_i, $i \in I_3 \cup I_4$. We cover the remaining interfaces in Ω by the balls $B(y_j, s_j)$, $j \in J_3$. Finally we aggregate successively each flat polyhedral cylinder D_i to the set L or to its complement.

Preparation of the proof. Let us consider a subset F of Ω having finite perimeter. Let γ belong to $]0, 1/16[$. We start by handling the boundary Γ, for which we make locally flat approximations controlled by the factor γ. By hypothesis, there exists a finite number of oriented hypersurfaces $S_1, \dots, S_p$ of class C^1 such that Γ is included in their union $S_1 \cup \cdots \cup S_p$. In particular, we have

$$\Gamma \smallsetminus \partial^* \Omega \subset S = \bigcup_{1 \le k < l \le p} S_k \cap S_l \,.$$

Since the hypersurfaces $S_1, \dots, S_r$ are transverse to each other, this implies that $\mathcal{H}^{d-1}(S) = 0$.
• **Continuity of the normal vectors.** The hypersurfaces $S_1, \dots, S_p$ being C^1 and the set Γ compact, the maps $x \in \Gamma \mapsto v_{S_k}(x)$, $1 \le k \le p$ (where $v_{S_k}(x)$ is the unit normal vector to S_k at x) are uniformly continuous:

$$\forall \delta > 0 \quad \exists \eta > 0 \quad \forall k \in \{1, \dots, p\} \quad \forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \le \eta \Rightarrow \left| v_{S_k}(x) - v_{S_k}(y) \right|_2 < \delta \,.$$

Let η^* be associated to $\delta = 1$ by this property. We will use also a more refined property.
• **Localisation of the interfaces.** We first prove a geometric lemma:

LEMMA 21. *Let Γ be an hypersurface (that is a C^1 submanifold of $\mathbb{R}^d$ of codimension 1) and let K be a compact subset of Γ. There exists a positive $M = M(\Gamma, K)$ such that:*

$$\forall \varepsilon > 0 \quad \exists\, r > 0 \quad \forall x, y \in K \qquad |x - y|_2 \le r \quad \Rightarrow \quad d_2(y, \tan(\Gamma, x)) \le M \, \varepsilon \, |x - y|_2 \,.$$

($\tan(\Gamma, x)$ is the tangent hyperplane of Γ at x).

PROOF. By a standard compactness argument, it is enough to prove the following local property:

$$\forall x \in \Gamma \quad \exists\, M(x) > 0 \quad \forall \varepsilon > 0 \quad \exists\, r(x, \varepsilon) > 0 \quad \forall y, z \in \Gamma \cap B(x, r(x, \varepsilon))$$
$$d_2(y, \tan(\Gamma, z)) \le M(x) \, \varepsilon \, |y - z|_2 \,.$$

Indeed, if this property holds, we cover K by the open balls $\overset{\circ}{B}(x, r(x,\varepsilon)/2)$, $x \in K$, we extract a finite subcovering $\overset{\circ}{B}(x_i, r(x_i,\varepsilon)/2)$, $1 \le i \le k$, and we set

$$M = \max\{\, M(x_i) : 1 \le i \le k \,\}\,, \quad r = \min\{\, r(x_i,\varepsilon)/2 : 1 \le i \le k \,\}\,.$$

Let now y, z belong to K with $|y - z|_2 \le r$. Let i be such that y belongs to $B(x_i, r(x_i,\varepsilon)/2)$. Since $r \le r(x_i,\varepsilon)/2$, then both y, z belong to the ball $B(x_i, r(x_i,\varepsilon))$ and it follows that

$$d_2(y, \tan(\Gamma, z)) \;\le\; M(x_i)\,\varepsilon\,|y - z|_2 \le M\,\varepsilon\,|y - z|_2\,.$$

We turn now to the proof of the above local property. Since Γ is an hypersurface, for any x in Γ there exists a neighbourhood V of x in $\mathbb{R}^d$, a diffeomorphism $f : V \mapsto \mathbb{R}^d$ of class C^1 and a $(d-1)$ dimensional vector space Z of $\mathbb{R}^d$ such that $Z \cap f(V) = f(\Gamma \cap V)$ (see for instance [**29**], 3.1.19). Let A be a compact neighbourhood of x included in V. Since f is a diffeomorphism, the maps $y \in A \mapsto df(y) \in \mathrm{End}(\mathbb{R}^d)$, $u \in f(A) \mapsto df^{-1}(u) \in \mathrm{End}(\mathbb{R}^d)$ are continuous. Therefore they are bounded:

$$\exists M > 0 \quad \forall y \in A \quad ||df(y)|| \le M\,, \quad \forall u \in f(A) \quad ||df^{-1}(u)|| \le M$$

(here $||df(x)|| = \sup\{\, |df(x)(y)|_2 : |y|_2 \le 1 \,\}$ is the standard operator norm in $\mathrm{End}(\mathbb{R}^d)$). Since $f(A)$ is compact, the differential map df^{-1} is uniformly continuous on $f(A)$:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u, v \in f(A) \qquad |u - v|_2 \le \delta \quad \Rightarrow \quad ||df^{-1}(u) - df^{-1}(v)|| \le \varepsilon\,.$$

Let ε be positive and let δ be associated to ε as above. Let ρ be positive and small enough so that $\rho < \delta/2$ and $B(f(x),\rho) \subset f(A)$ (since f is a C^1 diffeomorphism, $f(A)$ is a neighbourhood of $f(x)$). Let r be such that $0 < r < \rho/M$ and $B(x, r) \subset A$. We claim that M associated to x and r associated to ε, x answer the problem. Let y, z belong to $\Gamma \cap B(x, r)$. Since $[y, z] \subset B(x, r) \subset A$, and $||df(\zeta)|| \le M$ on A, then

$$|f(y) - f(x)|_2 \le M|y - x|_2 \le Mr < \rho\,, \quad |f(z) - f(x)|_2 < \rho\,,$$
$$|f(y) - f(z)|_2 < \delta\,, \quad |f(y) - f(z)|_2 < M|y - z|_2\,.$$

We apply next a classical lemma of differential calculus (see [**45**], I, 4, Corollary 2) to the map f^{-1} and the interval $[f(z), f(y)]$ (which is included in $B(f(x),\rho) \subset f(A)$) and the point $f(z)$:

$$|y - z - df^{-1}(f(z))(f(y) - f(z))|_2 \;\le$$
$$|f(y) - f(z)|_2 \sup\{\, ||df^{-1}(\zeta) - df^{-1}(f(z))|| : \zeta \in [f(z), f(y)] \,\}\,.$$

The right–hand member is less than $M|y - z|_2\,\varepsilon$. Since $z + df^{-1}(f(z))(f(y) - f(z))$ belongs to $\tan(\Gamma, z)$, we are done. $\square$

We come back to our case. Let $k \in \{\, 1, \dots, p \,\}$. The set $S_k \cap \Gamma$ is a compact subset of the hypersurface S_k. Applying lemma 21, we get:

$$\exists M_k \;\forall \delta_0 > 0 \;\exists\, \eta_k > 0 \;\forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \le \eta_k \;\Rightarrow\; d_2\big(y, \tan(S_k, x)\big) \le M_k \delta_0 |x - y|_2\,.$$

Let $M_0 = \max_{1 \le k \le p} M_k$ and let δ_0 in $]0, 1/2[$ be such that $M_0\delta_0 < \gamma$. For each k in $\{\, 1, \dots, p \,\}$, let η_k be associated to δ_0 as in the above property and let

$$\eta_0 \;=\; \min\Big(\min_{1 \le k \le p} \eta_k,\; \eta^*,\; \frac{1}{8d}\,\mathrm{dist}(\Gamma^1, \Gamma^2)\Big)\,.$$

• **Covering of Γ by transverse cubes.** This technique was already used in section 3.2. We build a family of cubes $Q(x, r)$, indexed by $x \in \Gamma$ and $r \in]0, r_\Gamma[$ such that $Q(x, r)$ is a cube centered at x of side length r which is transverse to Γ. For $x \in \mathbb{R}^d$ and $k \in \{\, 1, \dots, p \,\}$, let $p_k(x)$ be a point of $S_k \cap \Gamma$ such that

$$|x - p_k(x)|_2 \;=\; \inf\;\Big\{\, |x - y|_2 : y \in S_k \cap \Gamma \,\Big\}\,.$$

Such a point exists since $S_k \cap \Gamma$ is compact. We define then for $k \in \{1, \dots, p\}$

$$\forall x \in \mathbb{R}^d \qquad v_k(x) \,=\, v_{S_k}(p_k(x))\,.$$

We define also

$$d_r \,=\, \inf_{v_1,\dots,v_p \in S^{d-1}} \max_{b \in \mathcal{B}_d} \min_{\substack{1 \le k \le r \\ e \in b}} \Big(|e - v_i|_2, |-e - v_i|_2\Big)$$

where $\mathcal{B}_d$ is the collection of the orthonormal basis of $\mathbb{R}^d$ and S^{d-1} is the unit sphere of $\mathbb{R}^d$. Let η be associated to $d_r/4$ as in the above continuity property. We set

$$r_\Gamma = \frac{\eta}{2d}\,.$$

Let $x \in \Gamma$. By the definition of d_r, there exists an orthonormal basis b_x of $\mathbb{R}^d$ such that

$$\forall e \in b_x \quad \forall k \in \{1, \dots, p\} \quad \min\Big(|e - v_k(x)|_2, |-e - v_k(x)|_2\Big) \,>\, \frac{d_r}{2}\,.$$

Let $Q(x,r)$ be the cube centered at x of sidelength r whose sides are parallel to the vectors of b_x. We claim that $Q(x,r)$ is transverse to Γ for $r < r_\Gamma$. Indeed, let $y \in Q(x,r) \cap \Gamma$. Suppose that $y \in S_k$ for some $k \in \{1, \dots, p\}$, so that $v_k(y) = v_{S_k}(y)$ and $|x - p_k(x)|_2 < dr_\Gamma$. In particular, we have $|y - p_k(x)|_2 < 2dr_\Gamma < \eta$ and $|v_{S_k}(y) - v_k(x)|_2 < d_r/4$. For $e \in b_x$,

$$\frac{d_r}{2} \,\le\, |e - v_k(x)|_2 \,\le\, |e - v_{S_k}(y)|_2 + |v_{S_k}(y) - v_k(x)|_2$$

whence

$$|e - v_{S_k}(y)|_2 \,\ge\, \frac{d_r}{2} - \frac{d_r}{4} \,=\, \frac{d_r}{4}\,.$$

This is also true for $-e$, therefore the faces of the cube $Q(x,r)$ are transverse to S_k.

Start of the main argument. We first handle the interfaces along Γ. Let $\mathcal{R}(\Gamma)$ be the set of the points x of $\Gamma \setminus S$ such that

$$\lim_{r \to 0} \quad (\alpha_d r^d)^{-1} \mathcal{L}^d(B(x,r) \setminus \Omega) \,=\, 1/2\,,$$

$$\lim_{r \to 0} \quad (\alpha_{d-1} r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x,r) \cap \Gamma) \,=\, 1\,.$$

Let $\mathcal{R}(\Omega \setminus F)$ be the set of the points x belonging to $\partial^*(\Omega \setminus F) \cap \mathcal{R}(\Gamma)$ such that

$$\lim_{r \to 0} \quad (\alpha_{d-1} r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x,r) \cap \partial^*(\Omega \setminus F)) \,=\, 1\,,$$

$$\lim_{r \to 0} \quad (\alpha_d r^d)^{-1} \mathcal{L}^d(B(x,r) \cap (\Omega \setminus F)) \,=\, 1/2\,,$$

$$\lim_{r \to 0} \quad (\alpha_{d-1} r^{d-1})^{-1} \int_{B(x,r) \cap \partial^*(\Omega \setminus F)} \nu(v_{\Omega \setminus F}(y))\, d\mathcal{H}^{d-1}(y) \,=\, \nu(v_\Omega(x))\,.$$

Let $\mathcal{R}(F)$ be the set of the points x belonging to $\partial^* F \cap \mathcal{R}(\Gamma)$ such that

$$\lim_{r \to 0} \quad (\alpha_{d-1} r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x,r) \cap \partial^* F) \,=\, 1\,,$$

$$\lim_{r \to 0} \quad (\alpha_d r^d)^{-1} \mathcal{L}^d(B(x,r) \cap F) \,=\, 1/2\,,$$

$$\lim_{r \to 0} \quad (\alpha_{d-1} r^{d-1})^{-1} \int_{B(x,r) \cap \partial^* F} \nu(v_F(y))\, d\mathcal{H}^{d-1}(y) \,=\, \nu(v_\Omega(x))\,.$$

Thanks to the hypothesis on Γ and the structure of the sets of finite perimeter (see either Lemma 1, section 5.8 of [**27**], Lemma 5.9.5 in [**60**] or Theorem 3.61 of [**5**]), we have

$$\mathcal{H}^{d-1}\Big(\Gamma \setminus (\mathcal{R}(F) \cup \mathcal{R}(\Omega \setminus F))\Big) \,=\, 0\,.$$

For x in $\mathcal{R}(\Gamma)$, there exists a positive $r_0(x,\gamma)$ such that, for any $r < r_0(x,\gamma)$,

$$|\mathcal{L}^d(B(x,r) \smallsetminus \Omega) - \alpha_d r^d/2| \,\leq\, \gamma\, \alpha_d r^d\,,$$
$$|\mathcal{H}^{d-1}(B(x,r) \cap \Gamma) - \alpha_{d-1} r^{d-1}| \,\leq\, \gamma\, \alpha_{d-1} r^{d-1}\,.$$

For x in $\mathcal{R}(\Omega \smallsetminus F)$, there exists a positive $r(x,\gamma) < r_0(x,\gamma)$ such that, for any $r < r(x,\gamma)$,

$$|\mathcal{H}^{d-1}(B(x,r) \cap \partial^*(\Omega \smallsetminus F)) - \alpha_{d-1} r^{d-1}| \,\leq\, \gamma\, \alpha_{d-1} r^{d-1}\,,$$
$$|\mathcal{L}^d(B(x,r) \cap (\Omega \smallsetminus F)) - \alpha_d r^d/2| \,\leq\, \gamma\, \alpha_d r^d\,,$$
$$\left|(\alpha_{d-1} r^{d-1})^{-1} \int_{B(x,r)\cap\partial^*(\Omega\smallsetminus F)} \nu(v_{\Omega\smallsetminus F}(y))\, d\mathcal{H}^{d-1}(y) - \nu(v_\Omega(x))\right| \,\leq\, \gamma\,.$$

For x in $\mathcal{R}(F)$, there exists a positive $r(x,\gamma) < r_0(x,\gamma)$ such that, for any $r < r(x,\gamma)$,

$$|\mathcal{H}^{d-1}(B(x,r) \cap \partial^* F) - \alpha_{d-1} r^{d-1}| \,\leq\, \gamma\, \alpha_{d-1} r^{d-1}\,,$$
$$|\mathcal{L}^d(B(x,r) \cap F) - \alpha_d r^d/2| \,\leq\, \gamma\, \alpha_d r^d\,,$$
$$\left|(\alpha_{d-1} r^{d-1})^{-1} \int_{B(x,r)\cap\partial^* F} \nu(v_F(y))\, d\mathcal{H}^{d-1}(y) - \nu(v_\Omega(x))\right| \,\leq\, \gamma\,.$$

Let us define the sets

$$\Gamma^{1*} \,=\, \Gamma^1 \cap \mathcal{R}(\Omega \smallsetminus F)\,, \quad \Gamma^{2*} \,=\, \Gamma^2 \cap \mathcal{R}(F)\,,$$
$$\Gamma^{3*} \,=\, (\Gamma \smallsetminus \overline{\Gamma}_2) \cap \mathcal{R}(F)\,, \quad \Gamma^{4*} \,=\, (\Gamma \smallsetminus \overline{\Gamma}_1) \cap \mathcal{R}(\Omega \smallsetminus F)\,.$$

The family of balls

$$B(x,r)\,, \quad x \in \Gamma^{1*} \cup \Gamma^{2*}\,, \quad r < \min\Big(r(x,\gamma), \gamma, \eta_0, \frac{1}{2}\mathrm{dist}(x,S)\Big)\,,$$
$$B(x,r)\,, \quad x \in \Gamma^{3*}\,, \quad r < \min\Big(r(x,\gamma), \gamma, \eta_0, \frac{1}{2}\mathrm{dist}(x,S), \frac{1}{2}\mathrm{dist}(x,\overline{\Gamma}_2)\Big)\,,$$
$$B(x,r)\,, \quad x \in \Gamma^{4*}\,, \quad r < \min\Big(r(x,\gamma), \gamma, \eta_0, \frac{1}{2}\mathrm{dist}(x,S), \frac{1}{2}\mathrm{dist}(x,\overline{\Gamma}_1)\Big)$$

is a Vitali relation for $\Gamma^{1*} \cup \Gamma^{2*} \cup \Gamma^{3*} \cup \Gamma^{4*}$. Recall that S is the set of the points belonging to two or more of the hypersurfaces $S_1, \dots, S_p$ and since S is disjoint from $\Gamma^{1*}, \Gamma^{2*}, \Gamma^{3*}, \Gamma^{4*}$, then $\mathrm{dist}(x,S) > 0$ for $x \in \Gamma^{1*} \cup \Gamma^{2*} \cup \Gamma^{3*} \cup \Gamma^{4*}$. By the standard Vitali covering Theorem (see theorem 31), we may select a finite or countable collection of disjoint balls $B(x_i,r_i)$, $i \in I$, such that: for $i \in I$, $x_i \in \Gamma^{1*} \cup \Gamma^{2*} \cup \Gamma^{3*} \cup \Gamma^{4*}$, $r_i < \min(r(x_i,\gamma), \gamma, \eta_0, \frac{1}{2}\mathrm{dist}(x_i,S))$ and

$$\text{either} \qquad \mathcal{H}^{d-1}\Big(\Gamma \smallsetminus \bigcup_{i\in I} B(x_i,r_i)\Big) \,=\, 0 \qquad \text{or} \qquad \sum_{i\in I} r_i^{d-1} \,=\, \infty\,.$$

Because for each i in I, r_i is smaller than $r(x_i,\gamma)$,

$$\alpha_{d-1}(1-\gamma)\sum_{i\in I} r_i^{d-1} \,\leq\, \mathcal{H}^{d-1}(\Gamma) \,<\, \infty$$

and therefore the first case occurs, so that we may select four finite subsets I_1, I_2, I_3, I_4 of I such that

$$\forall k \in \{\,1,\dots,4\,\} \quad \forall i \in I_k \quad x_i \in \Gamma^{k*}\,,$$
$$\mathcal{H}^{d-1}\Big(\Gamma \smallsetminus \bigcup_{1\leq k\leq 4}\,\bigcup_{i\in I_k} B(x_i,r_i)\Big) \,<\, \gamma\,.$$

Let i belong to $I_1 \cup I_2 \cup I_3 \cup I_4$. We have

$$\begin{aligned}\mathcal{H}^{d-1}(\Gamma \cap B(x_i,r_i) \smallsetminus B(x_i,r_i(1-2\sqrt{\gamma}))) &= \mathcal{H}^{d-1}(\Gamma \cap B(x_i,r_i)) - \mathcal{H}^{d-1}(\Gamma \cap B(x_i,r_i(1-2\sqrt{\gamma})))\\ &\leq (1+\gamma)\alpha_{d-1} r_i^{d-1} - (1-\gamma)\alpha_{d-1} r_i^{d-1}(1-2\sqrt{\gamma})^{d-1}\\ &= \alpha_{d-1} r_i^{d-1}(1+\gamma-(1-\gamma)(1-2\sqrt{\gamma})^{d-1})\end{aligned}$$

$$\leq \alpha_{d-1} r_i^{d-1} 2d\sqrt{\gamma}\,.$$

Hence

$$\sum_{i\in I_1\cup I_2\cup I_3\cup I_4} \mathcal{H}^{d-1}(\Gamma\cap B(x_i,r_i)\smallsetminus B(x_i,r_i(1-2\sqrt{\gamma})))$$
$$\leq\ 2d\sqrt{\gamma}\sum_{i\in I_1\cup I_2\cup I_3\cup I_4}\alpha_{d-1}r_i^{d-1}\ \leq\ 4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)$$

and

$$\mathcal{H}^{d-1}\Big(\Gamma\smallsetminus\bigcup_{i\in I_1\cup I_2\cup I_3\cup I_4} B(x_i,r_i(1-2\sqrt{\gamma}))\Big)\ <\ \gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)\,.$$

We have a finite number of disjoint closed balls $B(x_i,r_i(1-2\sqrt{\gamma}))$, $i\in I_1\cup I_2\cup I_3\cup I_4$. By increasing slightly all the radii r_i, we can keep the balls disjoint, ensure that each radius r_i satisfies the same strict inequalities for i in $I_1\cup I_2\cup I_3\cup I_4$, and get the inequality

$$\mathcal{H}^{d-1}\Big(\Gamma\smallsetminus\bigcup_{i\in I_1\cup I_2\cup I_3\cup I_4} \overset{\circ}{B}(x_i,r_i(1-2\sqrt{\gamma}))\Big)\ <\ 2\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)\,.$$

The above set is a compact subset of Γ. For $k=1,2$, we define

$$R_k\ =\ \overline{\Gamma}_k\smallsetminus\bigcup_{i\in I_1\cup I_2\cup I_3\cup I_4} \overset{\circ}{B}(x_i,r_i(1-2\sqrt{\gamma}))\,.$$

The sets R_1 and R_2 are compact and their $\mathcal{H}^{d-1}$ measure is less than $2\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)$ (recall that $\partial_\Gamma\Gamma^1$ and $\partial_\Gamma\Gamma^2$ have a null $\mathcal{H}^{d-1}$ measure). For $k=1,2$, by the definition of the Hausdorff measure $\mathcal{H}^{d-1}$, there exists a collection of balls $B(y_j,s_j)$, $j\in J_k$ such that:

$$\forall j\in J_k\qquad 0<s_j<\min\Big(\eta_0,\frac{r_\Gamma}{2}\Big)\,,\qquad B(y_j,s_j)\cap R_k\neq\varnothing\,,$$
$$\sum_{j\in J_k}\alpha_{d-1}s_j^{d-1}\ <\ 3\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)\,,$$
$$R_k\ \subset\ \bigcup_{j\in J_k}\overset{\circ}{B}(y_j,s_j)\,.$$

By compactness of R_1 and R_2, the sets J_1 and J_2 can be chosen to be finite. It remains to cover

$$R_0\ =\ \Gamma\smallsetminus\bigcup_{i\in I_1\cup I_2\cup I_3\cup I_4}\overset{\circ}{B}(x_i,r_i(1-2\sqrt{\gamma}))\smallsetminus\bigcup_{j\in J_1\cup J_2}\overset{\circ}{B}(y_j,s_j)\,.$$

The set R_0 is a closed subset of Γ which is at a positive distance from Γ^1 and Γ^2. There exists a collection of balls $B(y_j,s_j)$, $j\in J_0$ such that:

$$\forall j\in J_0\qquad 0<s_j<\min\Big(\eta_0,\frac{r_\Gamma}{2},\frac{1}{8d}\operatorname{dist}(R_0,\Gamma^1\cup\Gamma^2)\Big)\,,\qquad B(y_j,s_j)\cap R_0\neq\varnothing\,,$$
$$\sum_{j\in J_0}\alpha_{d-1}s_j^{d-1}\ <\ 3\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)\,,$$
$$R_0\ \subset\ \bigcup_{j\in J_0}\overset{\circ}{B}(y_j,s_j)\,.$$

Now the collection of balls

$$\overset{\circ}{B}(x_i,r_i(1-2\sqrt{\gamma})),\ i\in I_1\cup I_2\cup I_3\cup I_4,\quad B(y_j,s_j),\ j\in J_0\cup J_1\cup J_2$$

covers completely Γ. We will next replace these balls by polyhedra. For $j\in J_0\cup J_1\cup J_2$, let x_j belong to $B(y_j,s_j)\cap\Gamma$ and let Q_j be the cube $Q(x_j,4s_j)$. For i in $I_1\cup I_2\cup I_3\cup I_4$, the point x_i belongs to exactly one hypersurface among $S_1,\dots,S_p$, which we denote by $S_{s(i)}$. In particular Γ admits a normal vector $v_\Omega(x_i)$ at x_i in the classical sense. For each i in $I_1\cup I_2\cup I_3\cup I_4$, let P_i be a convex open polygon inside the hyperplane $\mathrm{hyp}(x_i, v_\Omega(x_i))$ such that

$$\mathrm{disc}(x_i, r_i(1-2\sqrt{\gamma}), v_\Omega(x_i)) \subset P_i \subset \mathrm{disc}(x_i, r_i(1-\sqrt{\gamma}), v_\Omega(x_i))\,,$$
$$|\mathcal{H}^{d-2}(\partial P_i) - \alpha_{d-2} r_i^{d-2}(1-\sqrt{\gamma})^{d-2}| \,\leq\, \delta_0 \alpha_{d-2} r_i^{d-2}(1-\sqrt{\gamma})^{d-2}\,,$$
$$|\mathcal{H}^{d-1}(P_i) - \alpha_{d-1} r_i^{d-1}(1-\sqrt{\gamma})^{d-1}| \leq \delta_0 \alpha_{d-1} r_i^{d-1}(1-\sqrt{\gamma})^{d-1}\,.$$

Thanks to the choices of the radius r_i and the constants M_0, η_0, we have then

$$\Gamma \cap B(x_i, r_i(1-2\sqrt{\gamma})) \,\subset\, S_{s(i)} \cap B(x_i, r_i(1-2\sqrt{\gamma})) \,\subset\, \overset{\circ}{\mathrm{cyl}}(P_i, 2\gamma r_i)\,,$$
$$\Gamma \cap B(x_i, r_i) \,\subset\, S_{s(i)} \cap B(x_i, r_i) \,\subset\, \mathrm{cyl}(\mathrm{disc}(x_i, r_i, v_\Omega(x_i)), M_0\delta_0 r_i)\,,$$
$$\forall x \in B(x_i, r_i) \cap \Gamma \qquad |v_\Omega(x) - v_\Omega(x_i)|_2 \,<\, 1\,.$$

The choice of δ_0 guarantees that $M_0\delta_0(1+\delta_0)r_i < 2\gamma r_i$. Let t be such that

$$M_0\delta_0(1+\delta_0)r_i \leq t < \sqrt{\gamma} r_i\,.$$

We have

$$-tv_\Omega(x_i) + P_i \subset \Omega \cap B(x_i, r_i)\,, \qquad \Gamma \cap (-tv_\Omega(x_i) + P_i) = \varnothing\,.$$

In particular, the set Γ can intersect the cylinder $\mathrm{cyl}(P_i, t)$ only along its lateral sides, which are parallel to $v_\Omega(x_i)$. Let x belong to $\Gamma \cap \partial\, \mathrm{cyl}(P_i, t)$. Then

$$|v_{\mathrm{cyl}(P_i,t)}(x) - v_\Omega(x)|_2 \,\geq\, |v_{\mathrm{cyl}(P_i,t)}(x) - v_\Omega(x_i)|_2 - |v_\Omega(x_i) - v_\Omega(x)|_2 \,\geq\, \sqrt{2} - 1\,.$$

Therefore the cylinder $\mathrm{cyl}(P_i, t)$ is transverse to Γ. We will replace the ball $\overset{\circ}{B}(x_i, r_i(1-2\sqrt{\gamma}))$ by the cylinder $\mathrm{cyl}(P_i, t_i)$, for a carefully chosen value of t_i in the interval $[M_0\delta_0(1+\delta_0)r_i, \sqrt{\gamma} r_i[$. However, we must delay the choices of the values t_i, $i \in I_3 \cup I_4$ until we have modified the set F inside Ω. We deal next with the interfaces inside Ω and we make an approximation of F controlled by a factor ε. We choose ε sufficiently small compared to γ so that, when we perturb the set F by a volume ε, the resulting effect close to Γ is still of order γ. Let ε be such that $0 < \varepsilon < \gamma$ and

$$\varepsilon \,<\, \gamma \alpha_d \min_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} r_i^d\,.$$

We use next a classical approximation result: there exists a relatively closed subset L of Ω having finite perimeter such that $\Omega \cap \partial L$ is an hypersurface of class C^∞ and

$$\mathcal{L}^d(F \Delta L) \,<\, \varepsilon\,, \qquad \left| \int_{\Omega \cap \partial^* F} \nu(v_F(y))\, d\mathcal{H}^{d-1}(y) - \int_{\Omega \cap \partial L} \nu(v_L(y))\, d\mathcal{H}^{d-1}(y) \right| \,<\, \varepsilon\,.$$

In the case where ν is constant, this result is stated in Lemma 4.4 of [**50**]. In the non constant case, the argument should be slightly modified, as explained in the proof of proposition 14.8 of [**19**], where the approximation is performed in $\mathbb{R}^d$ instead of Ω. When working inside Ω, the extra difficulty is to deal with regions close to the boundary (see the proof of Proposition 4.3 of [**50**]). For $r > 0$, we define

$$\partial L_r \,=\, \left\{ x \in \partial L : d(x, \Gamma) \geq r \right\}.$$

By continuity of the measure $\mathcal{H}^{d-1}|_{\partial L}$, there exists $r^* > 0$ such that

$$\mathcal{H}^{d-1}(\Omega \cap \partial L \setminus \partial L_{2r^*}) \,\leq\, \varepsilon\,.$$

We apply lemma 21 to the set ∂L_{r^*} and the hypersurface $\Omega \cap \partial L$:

$$\exists M > 0 \quad \forall \delta > 0 \quad \exists\, \eta > 0 \quad \forall x, y \in \partial L_{r^*} \quad |x-y|_2 \leq \eta \,\Rightarrow\, d_2\big(y, \tan(\partial L, x)\big) \leq M\delta |x-y|_2\,.$$

For a point x belonging to ∂L_{r^*}, the tangent hyperplane of $\Omega \cap \partial L$ at x is precisely $\mathrm{hyp}(x, v_L(x))$. Let M be as above. We can assume that $M > 1$. Let δ in $]0, \delta_0[$ be such that $2\delta M < \varepsilon$. Let η be associated to δ as in the above property. For $x \in \partial L_{2r^*}$,

$$\lim_{r \to 0} \quad (\alpha_{d-1} r^{d-1})^{-1} \mathcal{H}^{d-1}(B(x, r) \cap \partial L) \,=\, 1\,,$$

$$\lim_{r\to 0} \quad (\alpha_{d-1}r^{d-1})^{-1}\int_{B(x,r)\cap\partial L} \nu(v_L(y))\, d\mathcal{H}^{d-1}(y) \;=\; \nu(v_L(x))\,.$$

For any x in ∂L_{2r^*}, there exists a positive $r(x,\varepsilon)$ such that, for any $r < r(x,\varepsilon)$,

$$|\mathcal{H}^{d-1}(B(x,r)\cap\partial L) - \alpha_{d-1}r^{d-1}| \;\leq\; \varepsilon\,\alpha_{d-1}r^{d-1}\,,$$

$$\left|(\alpha_{d-1}r^{d-1})^{-1}\int_{B(x,r)\cap\partial L} \nu(v_L(y))\, d\mathcal{H}^{d-1}(y) - \nu(v_L(x))\right| \;\leq\; \varepsilon\;.$$

The family of balls $B(x,r)$, $x\in\partial L_{2r^*}$, $r < \min(r^*,\eta_0,r(x,\varepsilon),\varepsilon,\eta)$, is a Vitali relation for ∂L_{2r^*}. By the standard Vitali covering Theorem, we may select a finite or countable collection of disjoint balls $B(x_i,r_i)$, $i\in I'$, such that: for any i in I', $x_i\in\partial L_{2r^*}$,

$$r_i \;<\; \min(r^*,\eta_0,r(x_i,\varepsilon),\varepsilon,\eta)$$

and

$$\text{either}\qquad \mathcal{H}^{d-1}\Big(\partial L_{2r^*}\smallsetminus\bigcup_{i\in I'} B(x_i,r_i)\Big) \;=\; 0 \qquad\text{or}\qquad \sum_{i\in I'} r_i^{d-1} \;=\; \infty\,.$$

Because for each i in I', r_i is smaller than $r(x_i,\varepsilon)$,

$$\alpha_{d-1}(1-\varepsilon)\sum_{i\in I'} r_i^{d-1} \;\leq\; \mathcal{H}^{d-1}(\Omega\cap\partial L) \;<\; \infty$$

and therefore the first case occurs, so that we may select a finite subset I_5 of I' such that

$$\mathcal{H}^{d-1}\Big(\partial L_{2r^*}\smallsetminus\bigcup_{i\in I_5} B(x_i,r_i)\Big) \;<\; \varepsilon\,.$$

We have a finite number of disjoint closed balls $B(x_i,r_i)$, $i\in I_5$. By increasing slightly all the radii r_i, we can keep the balls disjoint, each r_i strictly smaller than $\min(r^*,\eta_0,r(x_i,\varepsilon),\varepsilon,\eta)$ for i in I_5, and get the stronger inequality

$$\mathcal{H}^{d-1}\Big(\partial L_{2r^*}\smallsetminus\bigcup_{i\in I_5} \overset{\circ}{B}(x_i,r_i)\Big) \;<\; \varepsilon\,.$$

For each i in I_5, let P_i be a convex open polygon inside the hyperplane $\mathrm{hyp}(x_i,v_L(x_i))$ such that

$$\mathrm{disc}(x_i,r_i,v_L(x_i))\subset P_i\subset \mathrm{disc}(x_i,r_i(1+\delta),v_L(x_i))\,,$$
$$|\mathcal{H}^{d-2}(\partial P_i) - \alpha_{d-2}r_i^{d-2}| \;\leq\; \delta\alpha_{d-2}r_i^{d-2}\,,$$
$$|\mathcal{H}^{d-1}(P_i) - \alpha_{d-1}r_i^{d-1}| \leq \delta\alpha_{d-1}r_i^{d-1}\;.$$

We set $\psi = M\delta(1+\delta)$ (hence $\psi<\varepsilon<1$). Let i belong to I_5. Let D_i be the cylinder

$$D_i \;=\; \mathrm{cyl}(P_i, M\delta(1+\delta)r_i)$$

of basis P_i and height $2\psi r_i$. The point x_i belongs to ∂L_{2r^*}, the radius r_i is smaller than η and r^*, so that

$$\forall x\in\partial L\cap B(x_i,r_i)\quad d_2\Big(x,\mathrm{hyp}(x_i,v_L(x_i))\Big)\leq M\delta|x-x_i|_2\,,$$

whence

$$\partial L\cap B(x_i,r_i)\;\subset\;\mathrm{cyl}\Big(\mathrm{disc}(x_i,r_i,v_L(x_i)),M\delta r_i\Big)\;\subset\;\overset{\circ}{D}_i\,.$$

We will approximate F by L inside Ω and we will push the interfaces $\Gamma^1\cap\partial^*(\Omega\smallsetminus F)$ and $\Gamma^2\cap\partial^* F$ into Ω. We next handle the regions close to Γ inside the family of balls $B(x_i,r_i)$, $i\in I_1\cup I_2\cup I_3\cup I_4$. We will modify adequately the set F to ensure that no significant interface is created within these balls. Our technique consists in building a small flat cylinder centered on Γ which we add (for indices in $I_1\cup I_3$) or remove (for indices in $I_2\cup I_4$) to the set F. We have to design carefully this operation in order not to create any significant additional interface. This is the place where we tie together the covering of the boundary and the inner approximation. Recall that we already chose a family of polygons P_i, $i\in I_1\cup I_2\cup I_3\cup I_4$. For $i\in I_1\cup I_2$, we simply define D_i to be the cylinder

$$D_i \,=\, \mathrm{cyl}(P_i, M_0\delta_0(1+\delta_0)r_i)\,,$$

see figure 17. The construction of the cylinders associated to the indices $i \in I_3 \cup I_4$ is more

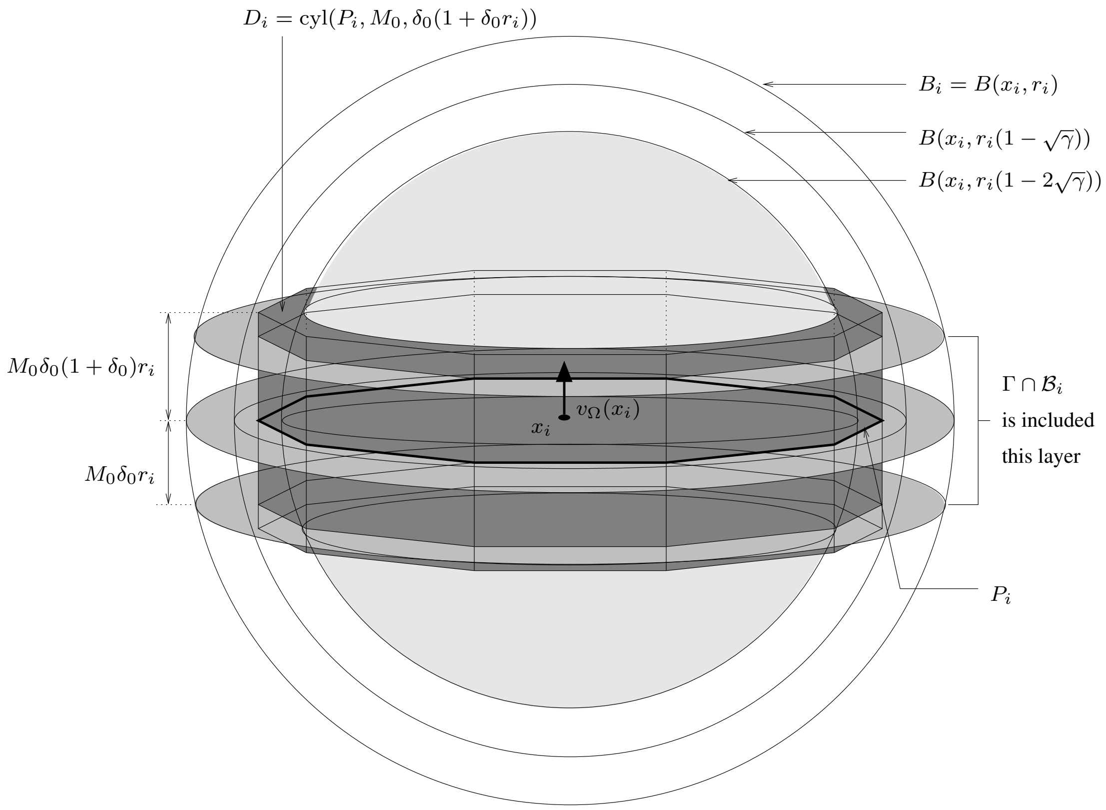

FIGURE 17. The cylinder D_i for $i \in I_1 \cup I_2$.

complicated. Our technique consists in choosing carefully the height t_i of the cylinders $\mathrm{cyl}(P_i, t_i)$ for $i \in I_3 \cup I_4$. We examine separately the indices in I_3 and I_4.

• **Balls indexed by** I_3**.** Let i belong to I_3. Because of the condition imposed on ε, we have

$$|\mathcal{L}^d(B(x_i,r_i)\cap L) - \alpha_d r_i^d/2| \,\leq\, \gamma\,\alpha_d r_i^d + \varepsilon \,\leq\, 2\gamma\,\alpha_d r_i^d\,.$$

Since in addition

$$|\mathcal{L}^d(B(x_i,r_i)\setminus\Omega) - \alpha_d r_i^d/2| \,\leq\, \gamma\,\alpha_d r_i^d\,,$$

it follows that

$$\mathcal{L}^d(B(x_i,r_i)\cap(\Omega\setminus\overset{\circ}{L})) \,\leq\, 3\gamma\,\alpha_d r_i^d\,.$$

Thanks to the choice of the polygon P_i, we have then

$$\int_{2\gamma r_i<t<\sqrt{\gamma}r_i} \mathcal{H}^{d-1}((-tv_\Omega(x_i)+P_i)\setminus\overset{\circ}{L})\,dt \,\leq\, \mathcal{L}^d(B(x_i,r_i)\cap(\Omega\setminus\overset{\circ}{L})) \,\leq\, 3\gamma\alpha_d r_i^d\,.$$

The condition on γ yields in particular $\sqrt{\gamma}-2\gamma \geq \sqrt{\gamma}/2$. Hence there exists $t_i \in]2\gamma r_i, \sqrt{\gamma}r_i[$ such that

$$\mathcal{H}^{d-1}((-t_iv_\Omega(x_i)+P_i)\setminus\overset{\circ}{L}) \,\leq\, 6\sqrt{\gamma}\alpha_d r_i^{d-1}\,.$$

Let D_i be the cylinder $D_i = \mathrm{cyl}(P_i, t_i)$.

• **Balls indexed by** I_4**.** Let i belong to I_4. Because of the condition imposed on ε, we have

$$|\mathcal{L}^d(B(x_i,r_i)\cap(\Omega\setminus L)) - \alpha_d r_i^d/2| \,\leq\, \gamma\,\alpha_d r_i^d + \varepsilon \,\leq\, 2\gamma\,\alpha_d r_i^d\,.$$

Since in addition
$$|\mathcal{L}^d(B(x_i,r_i)\smallsetminus\Omega)-\alpha_d r_i^d/2|\;\leq\;\gamma\,\alpha_d r_i^d\,,$$
it follows that
$$\mathcal{L}^d(B(x_i,r_i)\cap L)\;\leq\;3\gamma\,\alpha_d r_i^d\,.$$
Thanks to the choice of the polygon P_i, we have then
$$\int_{2\gamma r_i<t<\sqrt{\gamma}r_i}\mathcal{H}^{d-1}((-tv_\Omega(x_i)+P_i)\cap L)\,dt\;\leq\;\mathcal{L}^d(B(x_i,r_i)\cap L)\;\leq\;3\gamma\alpha_d r_i^d\,.$$
The condition on γ yields in particular $\sqrt{\gamma}-2\gamma\;\geq\;\sqrt{\gamma}/2$. Hence there exists $t_i\in]2\gamma r_i,\sqrt{\gamma}r_i[$ such that
$$\mathcal{H}^{d-1}((-t_iv_\Omega(x_i)+P_i)\cap L)\;\leq\;6\sqrt{\gamma}\alpha_d r_i^{d-1}\,.$$
Let D_i be the cylinder $D_i=\mathrm{cyl}(P_i,t_i)$ (see figure 18). We have now built the whole family of

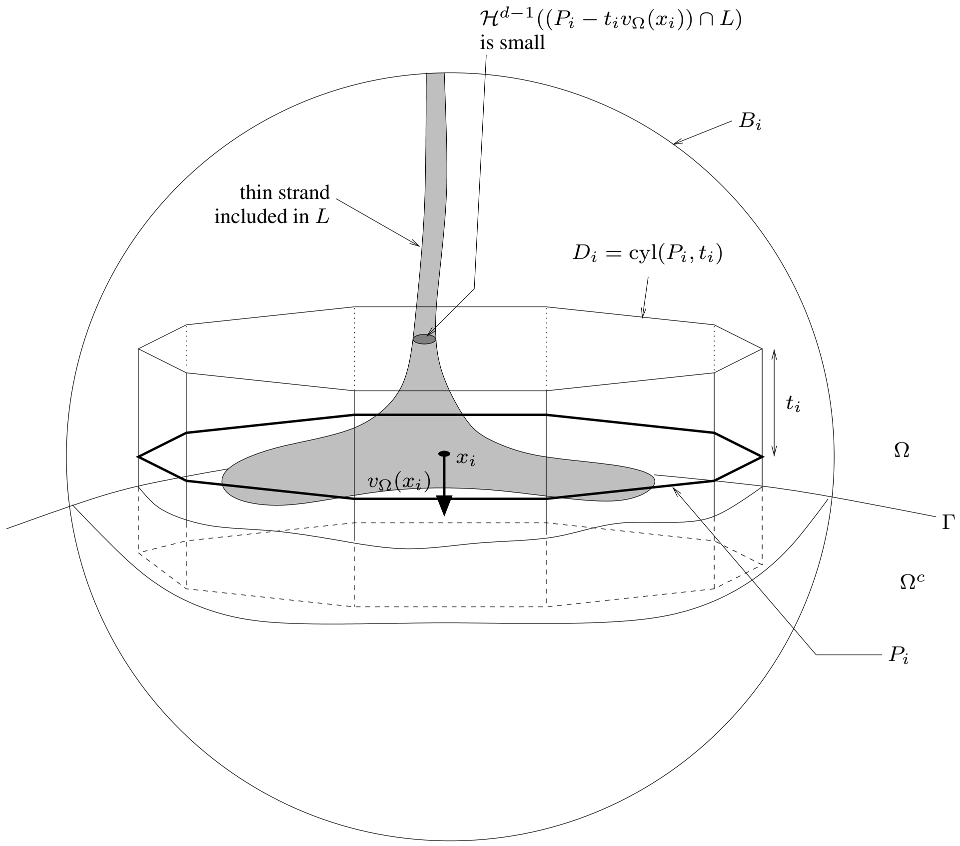

FIGURE 18. The cylinder D_i for $i\in I_4$.

cylinders D_i, $i\in I_1\cup I_2\cup I_3\cup I_4\cup I_5$. Moreover, the sets
$$\overset{\circ}{D}_i\,,\quad i\in I_1\cup I_2\cup I_3\cup I_4\,,\qquad\overset{\circ}{B}(y_j,s_j)\,,\quad j\in J_0\cup J_1\cup J_2\,,$$
cover completely Γ. It remains now to cover the region
$$R_3\;=\;\Omega\cap\partial L\;\smallsetminus\bigcup_{i\in I_1\cup I_2\cup I_3\cup I_4\cup I_5}\overset{\circ}{D}_i\;\smallsetminus\bigcup_{j\in J_0\cup J_1\cup J_2}\overset{\circ}{B}(y_j,s_j)\,.$$
Since R_3 does not intersect Γ, the distance
$$\rho=\frac{1}{8d}\,\mathrm{dist}(\Gamma,R_3)$$

is positive and also R_3 is compact. From the preceding inequalities, we deduce that

$$\mathcal{H}^{d-1}(R_3) \le \mathcal{H}^{d-1}(\Omega\cap\partial L \smallsetminus \partial L_{2r^*}) + \mathcal{H}^{d-1}\Big(\partial L_{2r^*} \smallsetminus \bigcup_{i\in I_5} \overset{\circ}{D}_i\Big)$$
$$\le \varepsilon + \mathcal{H}^{d-1}\Big(\partial L_{2r^*} \smallsetminus \bigcup_{i\in I_5} \overset{\circ}{B}(x_i,r_i)\Big) \le 2\varepsilon\,.$$

By the definition of the Hausdorff measure $\mathcal{H}^{d-1}$, there exists a collection of balls $B(y_j,s_j)$, $j\in J_3$, such that:

$$\forall j\in J_3 \qquad 0<s_j<\rho, \qquad B(y_j,s_j)\cap R_3\neq\varnothing\,,$$
$$R_3 \subset \bigcup_{j\in J_3} \overset{\circ}{B}(y_j,s_j)\,,$$
$$\sum_{j\in J_3} \alpha_{d-1} s_j^{d-1} \le 3\varepsilon\,.$$

By compactness, we might assume in addition that J_3 is finite. For $j\in J_3$, let x_j belong to $B(y_j,s_j)\cap R_3$ and let Q_j be the cube $Q(x_j,4s_j)$. We set

$$P = \Bigg((\Omega\cap L)\cup \bigcup_{i\in I_1\cup I_3\cup I_5} D_i \cup \bigcup_{j\in J_1} Q_j\Bigg) \smallsetminus \bigcup_{i\in I_2\cup I_4} D_i \smallsetminus \bigcup_{j\in J_0\cup J_2\cup J_3} Q_j\,.$$

The sets $\overset{\circ}{Q}_j$, $j\in J_0\cup J_1\cup J_2\cup J_3$, $\overset{\circ}{D}_i$, $i\in I_1\cup I_2\cup I_3\cup I_4\cup I_5$ cover $\partial L\cup\Gamma$, therefore

$$\partial P \subset \bigcup_{i\in I_1\cup I_2\cup I_3\cup I_4\cup I_5} \partial D_i \cup \bigcup_{j\in J_0\cup J_1\cup J_2\cup J_3} \partial Q_j\,,$$

thus P is polyhedral and ∂P is transverse to Γ. Since the sets

$$\overset{\circ}{D}_i\,,\quad i\in I_1\cup I_3\,,\qquad \overset{\circ}{Q}_j\,,\quad j\in J_1$$

cover completely $\overline{\Gamma}^1$, while the sets

$$D_i\,,\quad i\in I_2\cup I_4\cup I_5\,,\qquad Q_j\,,\quad j\in J_0\cup J_2\cup J_3$$

do not intersect $\overline{\Gamma}^1$, then $\overline{\Gamma}^1$ is included in the interior of P. Similarly, the sets

$$\overset{\circ}{D}_i\,,\quad i\in I_2\cup I_4\,,\qquad \overset{\circ}{Q}_j\,,\quad j\in J_2$$

cover completely $\overline{\Gamma}^2$, while the sets

$$D_i\,,\quad i\in I_1\cup I_3\cup I_5\,,\qquad Q_j\,,\quad j\in J_0\cup J_1\cup J_3$$

do not intersect $\overline{\Gamma}^2$, thus $\overline{\Gamma}^2$ is included in the interior of the complement of P. We next check that the set $P\cap\Omega$ approximates the initial set F with respect to the volume. We have

$$(P\cap\Omega)\Delta F \subset (L\Delta F)\cup \bigcup_{i\in I_1\cup I_2\cup I_3\cup I_4\cup I_5} D_i \cup \bigcup_{j\in J_0\cup J_1\cup J_2\cup J_3} Q_j$$

whence

$$\mathcal{L}^d((P\cap\Omega)\Delta F) \le \varepsilon + \sum_{i\in I_1\cup I_2\cup I_3\cup I_4} 2\alpha_{d-1} r_i^{d-1}(1+\delta_0)\sqrt{\gamma} r_i + \sum_{i\in I_5} 2\alpha_{d-1} r_i^{d-1}(1+\delta)\psi r_i + \sum_{j\in J_0\cup J_1\cup J_2\cup J_3} \alpha_d (2s_j)^d\,.$$

Yet each r_i is smaller than γ,

$$\sum_{i\in I_1\cup I_2\cup I_3\cup I_4} \alpha_{d-1} r_i^{d-1} \le 2\mathcal{H}^{d-1}(\Gamma)\,,$$

$$\sum_{i\in I_5} \alpha_{d-1} r_i^{d-1} \le 2\mathcal{H}^{d-1}(\Omega\cap\partial L) \le \frac{2}{\nu_{\text{min}}}(\nu_{\text{max}}\mathcal{H}^{d-1}(\partial^* F\cap\Omega)+\varepsilon)\,,$$

$$\sum_{j\in J_0\cup J_1\cup J_2\cup J_3} \alpha_{d-1}s_j^{d-1} \leq 3\big(3\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)\big)+3\varepsilon\,,$$

so that

$$\begin{aligned}\mathcal{L}^d((P\cap\Omega)\Delta F) \;\leq\; &\varepsilon+6\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)+\frac{6\varepsilon}{\nu_{\min}}(\nu_{\max}\mathcal{H}^{d-1}(\partial^*F\cap\Omega)+\varepsilon)\\ &+3\cdot 2^d\frac{\alpha_d}{\alpha_{d-1}}(3\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)+\varepsilon)\,.\end{aligned}$$

We estimate next the capacity of P. To do this, we examine the intersection of $\partial P\cap\Omega$ with each polyhedral cylinder. For $i\in I_1\cup I_2$, we use the obvious inclusion

$$P\cap\Omega\cap\partial D_i\;\subset\;\Omega\cap\partial D_i\,.$$

For $i\in I_3\cup I_4\cup I_5$, the sets $\partial P\cap\Omega\cap\partial D_i$ require more attention. We consider separately the indices of I_3, I_4 and I_5.

• **Cylinders indexed by** I_3**.** Let i in I_3. We have

$$\Omega\cap\partial P\cap\partial D_i\;\subset\;\Omega\cap(\partial D_i\smallsetminus\overset{\circ}{L})\cup\bigcup_{j\in J_0\cup J_1\cup J_2\cup J_3}\partial Q_j\,.$$

Yet, thanks to the construction of the cylinder D_i,

$$\begin{aligned}\mathcal{H}^{d-1}(\Omega\cap\partial D_i\smallsetminus\overset{\circ}{L})\;&\leq\;\mathcal{H}^{d-1}((-t_iv_\Omega(x_i)+P_i)\smallsetminus\overset{\circ}{L})+\mathcal{H}^{d-2}(\partial P_i)2\sqrt{\gamma}r_i\\ &\leq\;6\sqrt{\gamma}\alpha_dr_i^{d-1}+2\alpha_{d-2}r_i^{d-2}2\sqrt{\gamma}r_i\;\leq\;6\sqrt{\gamma}(\alpha_d+\alpha_{d-2})r_i^{d-1}\,.\end{aligned}$$

• **Cylinders indexed by** I_4**.** Let i in I_4. We have

$$\Omega\cap\partial P\cap\partial D_i\;\subset\;\Omega\cap(\partial D_i\cap L)\cup\bigcup_{j\in J_0\cup J_1\cup J_2\cup J_3}\partial Q_j\,.$$

Yet, thanks to the construction of the cylinder D_i,

$$\begin{aligned}\mathcal{H}^{d-1}(\Omega\cap\partial D_i\cap L)\;&\leq\;\mathcal{H}^{d-1}((-t_iv_\Omega(x_i)+P_i)\cap L)+\mathcal{H}^{d-2}(\partial P_i)2\sqrt{\gamma}r_i\\ &\leq\;6\sqrt{\gamma}\alpha_dr_i^{d-1}+2\alpha_{d-2}r_i^{d-2}2\sqrt{\gamma}r_i\;\leq\;6\sqrt{\gamma}(\alpha_d+\alpha_{d-2})r_i^{d-1}\,.\end{aligned}$$

• **Cylinders indexed by** I_5**.** Let i in I_5. We set

$$G_i\;=\;\mathrm{disc}\left(x_i-\psi r_iv_L(x_i),\sqrt{1-\psi^2}r_i,v_L(x_i)\right).$$

We claim that the set G_i is included in the interior of L. Indeed, $G_i\subset B(x_i,r_i)\cap\partial D_i$, yet $\partial L\cap B(x_i,r_i)\subset\overset{\circ}{D}_i$, therefore G_i does not intersect ∂L. Since $v_L(x_i)$ is the exterior normal vector to L at x_i, then G_i is included in $\overset{\circ}{L}$. The definition of the set P implies that

$$\partial P\cap G_i\;\subset\;\bigcup_{j\in J_0\cup J_1\cup J_2\cup J_3}\partial Q_j\,,$$

whence

$$\Omega\cap\partial P\cap\partial D_i\;\subset\;(\partial D_i\smallsetminus G_i)\cup\bigcup_{j\in J_0\cup J_1\cup J_2\cup J_3}\partial Q_j\,.$$

Yet

$$\begin{aligned}\mathcal{H}^{d-1}\Big(\partial D_i\smallsetminus(P_i+\psi r_iv_L(x_i))\smallsetminus G_i\Big)\;&\leq\;2\alpha_{d-2}r_i^{d-2}2\psi r_i+\alpha_{d-1}r_i^{d-1}\Big(1+\delta-(1-\psi^2)^{(d-1)/2}\Big)\\ &\leq\;\alpha_{d-1}r_i^{d-1}\Big(4\frac{\alpha_{d-2}}{\alpha_{d-1}}\psi+1+\delta-(1-\psi^2)^{(d-1)/2}\Big)\,.\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}\Omega\cap\partial P\;\subset\;&\bigcup_{i\in I_1\cup I_2}(\Omega\cap\partial D_i)\cup\bigcup_{i\in I_3}(\Omega\cap D_i\smallsetminus\overset{\circ}{L})\cup\bigcup_{i\in I_4}(\Omega\cap\partial D_i\cap L)\\ &\cup\bigcup_{i\in I_5}(\partial D_i\smallsetminus G_i)\cup\bigcup_{j\in J_0\cup J_1\cup J_2\cup J_3}\partial Q_j\,.\end{aligned}$$

Therefore

$$
\begin{aligned}
\mathcal{I}_\Omega(P) \;\le\; & \sum_{i\in I_1\cup I_2} \int_{\Omega\cap\partial D_i} \nu(v_P(x))\,d\mathcal{H}^{d-1}(x) + \nu_{\max}\sum_{i\in I_3}\mathcal{H}^{d-1}(\Omega\cap\partial D_i\smallsetminus \overset{\circ}{L})\\
& + \nu_{\max}\sum_{i\in I_4}\mathcal{H}^{d-1}(\Omega\cap\partial D_i\cap L)\\
& + \sum_{i\in I_5}\Big(\nu(v_L(x_i))\mathcal{H}^{d-1}(P_i)+\nu_{\max}\mathcal{H}^{d-1}\big(\partial D_i\smallsetminus (P_i+\psi r_i v_L(x_i))\smallsetminus G_i\big)\Big)\\
& + \nu_{\max}\sum_{j\in J_0\cup J_1\cup J_2\cup J_3}\mathcal{H}^{d-1}(\partial Q_j)\,.
\end{aligned}
$$

We use now the various estimates obtained in the course of the approximation. We get

$$
\begin{aligned}
\mathcal{I}_\Omega(P) \;\le\; & \sum_{i\in I_1\cup I_2}\Big(\alpha_{d-1}r_i^{d-1}(1+\delta_0)\nu(v_\Omega(x_i))+\nu_{\max}\alpha_{d-2}r_i^{d-1}2M_0\delta_0(1+\delta_0)^2\Big)\\
& + \sum_{i\in I_3\cup I_4}\nu_{\max}\Big(6\sqrt{\gamma}(\alpha_d+\alpha_{d-2})r_i^{d-1}\Big)\\
& + \sum_{i\in I_5}\Big(\alpha_{d-1}r_i^{d-1}(1+\delta)\nu(v_L(x_i))\\
& \quad + \nu_{\max}\alpha_{d-1}r_i^{d-1}\Big(4\frac{\alpha_{d-2}}{\alpha_{d-1}}\psi+1+\delta-(1-\psi^2)^{(d-1)/2}\Big)\Big)\\
& + \sum_{j\in J_0\cup J_1\cup J_2\cup J_3}\nu_{\max}\alpha_{d-1}2^{d-1}s_j^{d-1}\\
\le\; & \frac{1+\delta_0}{1-\gamma}\sum_{i\in I_1}\int_{B(x_i,r_i)\cap\partial^*(\Omega\smallsetminus F)}\nu(v_\Omega(y))\,d\mathcal{H}^{d-1}(y)\\
& + \frac{1+\delta_0}{1-\gamma}\sum_{i\in I_2}\int_{B(x_i,r_i)\cap\partial^* F}\nu(v_\Omega(y))\,d\mathcal{H}^{d-1}(y)\\
& + \frac{1+\delta}{1-\varepsilon}\sum_{i\in I_5}\int_{B(x_i,r_i)\cap\partial L}\nu(v_L(y))\,d\mathcal{H}^{d-1}(y)\\
& + \sum_{i\in I_1\cup I_2\cup I_3\cup I_4\cup I_5}\nu_{\max}\alpha_{d-1}r_i^{d-1}\Big(\frac{\alpha_{d-2}}{al_{d-1}}5\gamma+6\sqrt{\gamma}\frac{\alpha_d+\alpha_{d-2}}{\alpha_{d-1}}+4\frac{\alpha_{d-2}}{\alpha_{d-1}}\psi\\
& \quad +1+\delta-(1-\psi^2)^{(d-1)/2}\Big)+\nu_{\max}2^{d-1}3\big(3\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)+\varepsilon\big)\\
\le\; & \frac{1+\delta_0}{1-\gamma}\bigg(\int_{\Gamma^1\cap\partial^*(\Omega\smallsetminus F)}\nu(v_\Omega(y))\,d\mathcal{H}^{d-1}(y)+\int_{\Gamma^2\cap\partial^* F}\nu(v_\Omega(y))\,d\mathcal{H}^{d-1}(y)\\
& + \int_{\Omega\cap\partial L}\nu(v_L(y))\,d\mathcal{H}^{d-1}(y)\bigg)\\
& + 2(\mathcal{H}^{d-1}(\Gamma)+\mathcal{H}^{d-1}(\Omega\cap\partial L))\nu_{\max}\Big(\frac{\alpha_{d-2}}{\alpha_{d-1}}5\gamma+6\sqrt{\gamma}\frac{\alpha_d+\alpha_{d-2}}{\alpha_{d-1}}+4\frac{\alpha_{d-2}}{\alpha_{d-1}}\psi\\
& \quad +1+\delta-(1-\psi^2)^{(d-1)/2}\Big)+\nu_{\max}\Big(2^{d-1}3\big(3\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)\big)+3\varepsilon\Big)\\
\le\; & \frac{1+\delta_0}{1-\gamma}\big(\mathcal{I}_\Omega(F)+\varepsilon\big)\\
& + 2\Big(\mathcal{H}^{d-1}(\Gamma)+\frac{\nu_{\max}\mathcal{I}_\Omega(F)+\varepsilon}{\nu_{\min}}\Big)\nu_{\max}\Big(\frac{\alpha_{d-2}}{\alpha_{d-1}}5\gamma+6\sqrt{\gamma}\frac{\alpha_d+\alpha_{d-2}}{\alpha_{d-1}}+\delta\varepsilon+4\frac{\alpha_{d-2}}{\alpha_{d-1}}\varepsilon\Big)
\end{aligned}
$$

$$+\,\nu_{\max}\Big(2^{d-1}3\big(3\gamma+4d\sqrt{\gamma}\mathcal{H}^{d-1}(\Gamma)\big)+3\varepsilon\Big)$$

where we have used the inequality $\psi<\varepsilon$ in the last step. We have also use the inclusions

$$\begin{aligned}\forall i\in I_1 \qquad B(x_i,r_i)\cap\partial^*(\Omega\smallsetminus F)\;\subset\;\Gamma^1\cap\partial^*(\Omega\smallsetminus F)\,,\\ \forall i\in I_2 \qquad B(x_i,r_i)\cap\partial^* F\;\subset\;\Gamma^2\cap\partial^* F\,.\end{aligned}$$

Since $\delta_0,\delta,\gamma,\varepsilon$ can be chosen arbitrarily small, we have obtained the desired approximation. $\square$

4.2. Positivity of $\widetilde{\phi_\Omega}$. We will prove that as soon as $\Lambda(0)<1-p_c(d)$ and

$$\int_{[0,+\infty[} x\,d\Lambda(x)\;<\;\infty\,, \tag{7.7}$$

we have $\phi_\Omega>0$. In fact we know that if the condition (7.7) is satisfied,

$$\Lambda(0)<1-p_c(d)\quad\iff\quad\exists v\,,\;\nu(v)>0\quad\iff\quad\forall v\,,\;\nu(v)>0\,.$$

Since ν satisfies the weak triangle inequality, the function $v\mapsto\nu(v)$ is continuous, and so as soon as $\Lambda(0)<1-p_c(d)$ and (7.7) is satisfied, we have

$$\nu_{\min}\;=\;\min_{\mathbb{S}^1}\nu\;>\;0\,.$$

If P is a polyhedral set, then $\mathcal{H}^{d-1}((\partial P\cap\Omega)\smallsetminus(\partial^*P\cap\Omega))=0$. We then obtain that

$$\widetilde{\phi_\Omega}\;\geq\;\nu_{\min}\times\inf\{\mathcal{H}^{d-1}(\mathcal{S}\cap\Omega)\,|\,\mathcal{S}\text{ hypersurface that cuts }\Gamma^1\text{ from }\Gamma^2\text{ in }\overline{\Omega}\,,\;d(\mathcal{S},\Gamma^1\cup\Gamma^2)>0\}\,.$$

We recall that the hypersurface $\mathcal{S}$ cuts Γ^1 from Γ^2 in $\overline{\Omega}$ if $\mathcal{S}$ intersects any continuous path from a point in Γ^1 to a point in Γ^2 that is included in $\overline{\Omega}$. We consider such a hypersurface $\mathcal{S}\subset\mathbb{R}^d$, and we want to bound from below the quantity $\mathcal{H}^{d-1}(\mathcal{S}\cap\Omega)$ independently on $\mathcal{S}$.

For $i=1,2$, we can find x_i in Γ^i and $r_i>0$ such that $\Gamma\cap B(x_i,r_i)\subset\Gamma^i$ and $\Gamma\cap B(x_i,r_i)$ is a $\mathcal{C}^1$ hypersurface. We denote by $v_\Omega(x_i)$ the exterior normal unit vector to Ω at x_i, and by $T_\Omega(x_i)$ the hyperplane tangent to Γ at x_i. Since Γ is of class $\mathcal{C}^1$ in a neighbourhood of x_i and Ω is a Lipschitz domain, applying lemma 21, we know that for all $\theta>0$, there exists $\varepsilon>0$ depending on $(\Omega,\Gamma,\Gamma^1,\Gamma^2,x_1,x_2)$ such that for $i=1,2$ we have

$$\begin{cases}\Omega\cap B(x_i,2\varepsilon)\text{ is connected}\,,\\ \Gamma\cap B(x_i,2\varepsilon)\;\subset\;\mathcal{V}_2(T_\Omega(x_i),2\varepsilon\sin\theta)\cap B(x_i,2\varepsilon)\,,\\ \Gamma\cap B(x_i,2\varepsilon)\;\subset\;\Gamma^i\,.\end{cases}$$

We fix θ small enough to have $2\varepsilon\sin\theta<\varepsilon/2$. We define

$$A_i\;=\;T_\Omega(x_i)\cap B(x_i,\varepsilon)\quad\text{ and }\quad D_i\;=\;\mathrm{cyl}(A_i,\varepsilon)\,,$$

and then

$$\widehat{\Omega}=\Omega\cup\mathring{D}_1\cup\mathring{D}_2\,,$$

where $\mathring{D}_i$ is the interior of D_i for $i=1,2$. We define

$$X_i\;=\;\{z\in\mathring{D}_i\,|\,\overrightarrow{x_iz}\cdot v_\Omega(x_i)>\varepsilon/2\}\;\subset\;\widehat{\Omega}\,.$$

Then $X_i\subset\widehat{\Omega}\smallsetminus\Omega$. Each path r from a point $y_1\in X_1$ to a point $y_2\in X_2$ contains a path r' from a point $y_1'\in\Gamma^1$ to a point $y_2'\in\Gamma^2$ such that $r'\subset\overline{\Omega}$, thus $\mathcal{S}$ intersects r. We consider the set

$$V_i\;=\;\{z\in X_i\,|\,d_2(z,\partial X_i)>\varepsilon/8\}\,.$$

Let $\hat{y}_1\in V_1$, $\hat{y}_2\in V_2$ such that $d_2(\hat{y}_i,\partial X_i)>\varepsilon/4$ for $i=1,2$. Since $\widehat{\Omega}$ is obviously connected by arc, there exists a path $\hat{r}$ from $\hat{y}_1$ to $\hat{y}_2$ in $\widehat{\Omega}$. The path $\hat{r}$ is compact and $\widehat{\Omega}$ is open, so $\delta=d_2(\hat{r},\partial\widehat{\Omega})>0$. We thus can find a path r included in $\mathcal{V}_2(\hat{r},\min(\delta/2,\varepsilon/8))$ which is a $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^d$ of dimension 1 and which has one endpoint, denoted by y_1, in V_1, and the other one, denoted by y_2, in V_2.

As we explained previously, $d_2(r, \partial\widehat{\Omega}) > 0$, so there exists a positive η_1 such that $\mathcal{V}_2(r, \eta_1) \subset \widehat{\Omega}$. We can suppose that $\eta_1 < \varepsilon/16$, to obtain that $B(y_i, \eta_1) \subset X_i$ for $i = 1, 2$. For all z in r we denote by $N_r(z)$ the hyperplane orthogonal to r at z, and by $N_r^\eta(z)$ the subset of $N_r(z)$ composed of the points of $N_r(z)$ that are at distance smaller than or equal to η of z. The tubular neighbourhood of r of radius η, denoted by $\mathrm{tub}(r, \eta)$, is the set of all the points z in $\mathbb{R}^d$ such that there exists a geodesic of length smaller than or equal to η from z that meets r orthogonally, i.e.,

$$\mathrm{tub}(r, \eta) = \bigcup_{z \in r} N_r^\eta(z)\,,$$

(see for example [**33**]). We have a picture of this tubular neighbourhood on figure 19. Since r is a

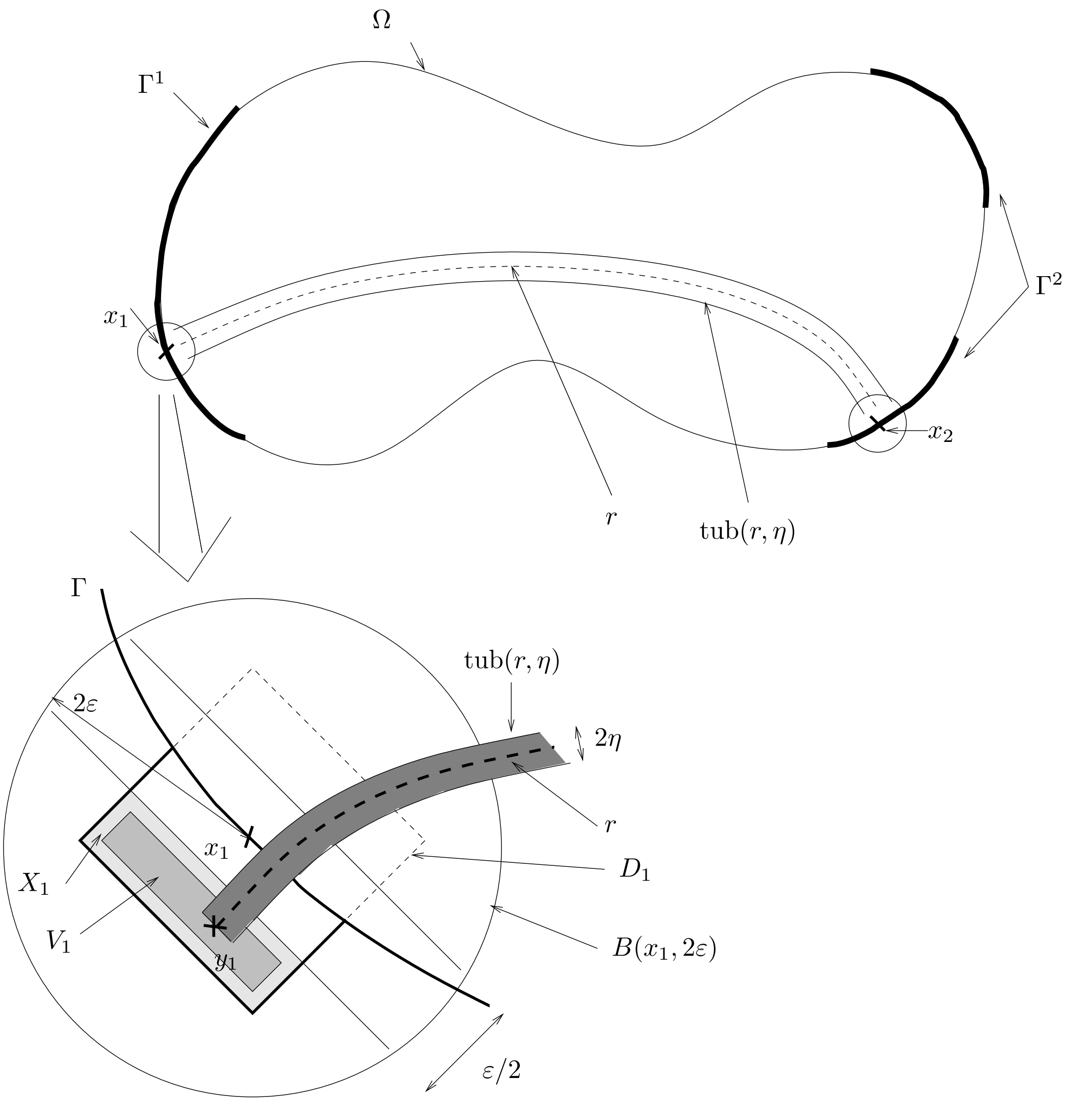

FIGURE 19. Construction of $\mathrm{tub}(r, \eta)$.

compact $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^d$ which is complete, there exists a $\eta_2 > 0$ small enough such that for all $\eta \le \eta_2$, the tubular neighbourhood of r of diameter η is well defined by a $\mathcal{C}^\infty$-diffeomorphism (see for example [**9**], Theorem 2.7.12, or [**33**]), i.e., there exists a $\mathcal{C}^\infty$-diffeomorphism ψ from

$$Nr^\eta = \{(z, v)\,,\, z \in r\,,\, v \in N_r^\eta(z)\}$$

to $\mathrm{tub}(r, \eta)$. We choose a positive η smaller than $\min(\eta_1, \eta_2)$. We stress the fact that this η depends on $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ but not on $\mathcal{S}$.

Let (I, h) be a parametrisation of class $\mathcal{C}^\infty$ of r, i.e., $I = [a, b]$ is a closed interval of $\mathbb{R}$, $h : I \to r$ is a $\mathcal{C}^\infty$-diffeomorphism which is an immersion. Let z be in r, and $u_z = h^{-1}(z) \in I$. The vector $h'(u_z)$ is tangent to r at z, and there exists some vectors $(e_2(z), ..., e_d(z))$ such that $(h'(u_z), e_2(z), ..., e_d(z))$ is a direct basis of $\mathbb{R}^d$. There exists a neighbourhood U_z of u_z in I such that for all $u \in U_z$, $(h'(u), e_2(z), ..., e_d(z))$ is still a basis of $\mathbb{R}^d$, since h' is continuous. Indeed the condition for a family of vectors $(\alpha_1, ..., \alpha_d)$ to be a basis of $\mathbb{R}^d$ is an open condition, because it corresponds to $\det((\alpha_1, ..., \alpha_d)) > 0$ where $\det$ is the determinant of the matrix. We apply the Gram-Schmidt process to the basis $(h'(u), e_2(z), ..., e_d(z))$ to obtain a direct orthonormal basis $(h'(u)/\|h'(u)\|, v_2(u, z), ..., v_d(u, z))$ of $\mathbb{R}^d$ for all $u \in U_z$, such that the dependence of $(h'(u)/\|h'(u)\|, v_2(u, z), ..., v_d(u, z))$ on $u \in U_z$ is of class $\mathcal{C}^\infty$. We remark that the family $(v_2(u, z), ..., v_d(u, z))$ is a direct orthonormal basis of $N_r(h(u))$ for all $u \in U_z$. We have associated with each $z \in r$ a neighbourhood U_z of $u_z = h^{-1}(z)$ in I, we can obviously suppose that U_z is an interval which is open in I. Since $(U_z, z \in r)$ is a covering of the compact I, we can extract a finite covering $(U_j, j = 1, ..., n)$ from it. We can choose this family to be minimal, i.e., such that $(U_j, j \in \{1, ..., n\} \smallsetminus j_0)$ is not a covering of I for any $j_0 \in \{1, ..., n\}$. We then reorder the $(U_j, j = 1, ..., n)$ (keeping the same notation) by the increasing order of their left end point in $I \subset \mathbb{R}$. Since the family (U_j) is minimal, each point of I belongs either to a unique set U_j, $j \in \{1, ..., n\}$, or to exactly two sets U_j and U_{j+1} for $j \in \{1, ..., n-1\}$. We denote by a_j the middle of the non-empty open interval $U_j \cap U_{j+1}$ for $j \in \{1, ..., n-1\}$, and by $(h'(u)/\|h'(u)\|, v_2(u, j), ..., v_d(u, j))$ the direct orthonormal basis defined previously on U_j for $j \in \{1, ..., n\}$. We want to construct a family of direct orthonormal basis $(h'(u)/\|h'(u)\|, f_2(u), ..., f_d(u))$ of $\mathbb{R}^d$ such that the function:

$$\psi : u \in I \mapsto (h'(u)/\|h'(u)\|, f_2(u), ..., f_d(u))$$

is of class $\mathcal{C}^\infty$. We have to define a concatenation of the $(h'(u)/\|h'(u)\|, v_2(u, j), ..., v_d(u, j))$ over the different sets U_j. For $u \in [a, a_1]$, we define

$$\psi(u) = (h'(u)/\|h'(u)\|, v_2(u, 1), ..., v_d(u, 1)) .$$

Thus the function ψ defined on $[a, a_1]$ is of class $\mathcal{C}^\infty$. On $U_1 \cap U_2$ we have defined two different direct orthonormal basis $(h'(u)/\|h'(u)\|, v_2(u, j), ..., v_d(u, j))$ for $j = 1$ and $j = 2$ that have the same first vector. Let $\phi_1 : U_1 \cap U_2 \to SO_{d-1}(\mathbb{R})$ be the function of class $\mathcal{C}^\infty$ that associates to each $u \in U_1 \cap U_2$ the matrix of change of basis from $(v_2(u, 2), ..., v_d(u, 2))$ to $(v_2(u, 1), ..., v_d(u, 1))$.

If b_1 is the right end point of $U_1 \cap U_2$, then ϕ_1 is in particular defined on $[a_1, b_1[$. Let g_1 be a $\mathcal{C}^\infty$-diffeomorphism from $[a_1, b_1[$ to $[a_1, \infty[$ which is strictly increasing (so $g_1(a_1) = a_1$) and such that all the derivatives of g_1 at a_1 are null. Then $\phi_1 \circ g_1^{-1}$ is defined on $[a_1, +\infty[$ and all its derivatives at a_1 are equal to those of ϕ_1. We then transform all the orthonormal basis $(v_2(u, j), ..., v_d(u, j))$ of $\mathbb{R}^{d-1}$ for $j \geq 2$ and $u \geq a_1$ by the change of basis $\phi_1 \circ g_1^{-1}$, and we denote the new direct orthonormal basis of $\mathbb{R}^{d-1}$ obtained this way by $(\widetilde{v}_2(u, j), ..., \widetilde{v}_d(u, j))$. We then define ψ on $]a_1, a_2]$ by

$$\psi(u) = (h'(u)/\|h'(u)\|, \widetilde{v}_2(u, 2), ..., \widetilde{v}_d(u, 2)) ,$$

and we remark that $\psi(u)$ still defines a direct orthonormal basis of $\mathbb{R}^d$. The function ψ is of class $\mathcal{C}^\infty$ on $[a, a_2]$, including at a_1. We iterate this process with the family of basis

$$(h'(u)/\|h'(u)\|, \widetilde{v}_2(u, j), ..., \widetilde{v}_d(u, j)) , \; j = 2, ..., n$$

at a_2, etc..., finitely many times since we work with a finite covering of I. We obtain in the end a function

$$\psi \circ h^{-1} : r \to SO_{d-1}(\mathbb{R})$$

which is of class $\mathcal{C}^\infty$, and for all $z \in r$, the set of the points of $\mathbb{R}^d$ that have for first coordinate 0 in the basis $\psi \circ h^{-1}(z)$ is exactly the hyperplane $N_r(z)$.

For each $t = (t_2, ..., t_{d-1}) \in \{z \in \mathbb{R}^{d-1} \,|\, d(z,0) \leq \eta\}$, the set

$$r_t \,=\, \{y \in \mathbb{R}^d \,|\, \exists z \in r\,,\, y \text{ has coordinates } (0, t_2, ..., t_{d-1}) \text{ in the basis } \psi \circ h^{-1}(z)\}$$

is a continuous path (even of class $\mathcal{C}^\infty$) from a point in X_1 to a point in X_2, therefore

$$r_t \cap \mathcal{S} \cap \overline{\Omega} \,\neq\, \varnothing\,.$$

Moreover, since $d(\mathcal{S}, \Gamma^1 \cup \Gamma^2) > 0$, we obtain that

$$r_t \cap \mathcal{S} \cap \Omega \,\neq\, \varnothing\,. \tag{7.8}$$

For each $y \in \mathrm{tub}(r,\eta)$, there exists a unique $z_y \in r$ such that $y \in N_r(z_y)$, so we can associate to y its coordinates $(0, t_2(y), ..., t_d(y))$ in the basis $\psi \circ h^{-1}(z_y)$. We define the projection p of $\mathrm{tub}(r,\eta)$ on $N_r^\eta(y_1)$ that associates to each y in $\mathrm{tub}(r,\eta)$ the point of coordinate $(0, t_2(y), ..., t_d(y))$ in the basis $\psi \circ h^{-1}(y_1)$. Then p is of class $\mathcal{C}^\infty$ as is $\psi \circ h^{-1}$. If z belongs to $N_r^\eta(y_1)$, and $t(z) = (t_2(z), ..., t_d(z))$, then we know by equation (7.8) that there exists a point on $r_{t(z)}$ that intersects $\mathcal{S}$ in Ω. Moreover, $r_{t(z)}$ is exactly the set of the points y of $\mathrm{tub}(r,\eta)$ whose image $p(y)$ by this projection is the point z. Thus

$$p(\mathcal{S} \cap \mathrm{tub}(r,\eta) \cap \Omega) \,=\, N_r^\eta(y_1)\,.$$

Since $\mathrm{tub}(r,\eta)$ is compact, p is a Lipschitz function on $\mathrm{tub}(r,\eta)$, and so there exists a constant K, depending on p, hence on Ω, r, η, but not on $\mathcal{S}$, such that

$$\mathcal{H}^{d-1}(\mathcal{S} \cap \Omega) \,\geq\, \mathcal{H}^{d-1}(\mathcal{S} \cap \mathrm{tub}(r,\eta) \cap \Omega) \,\geq\, K\mathcal{H}^{d-1}(p(\mathcal{S} \cap \mathrm{tub}(r,\eta))) \,\geq\, K\alpha_{d-1}\eta^{d-1}\,.$$

This ends the proof of the positivity of $\widetilde{\phi_\Omega}$.

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