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Some results in first passage percolation and related models

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Chapter 1

Preamble

1.1 Introduction

1.1.1 Percolation

Percolation is a model for porous media. It was introduced by Broadbent and Hammersley [BH57] in 1957. It aims to understand how the porosity of a media at a macroscopic scale, *i.e.*, at the scale of a piece of rock, is created by the circulation of water at a microscopic scale. We suppose that the water can flow through microscopic tubes inside the rock that are the edges E of a given graph $G = (V, E)$ of vertices V. For instance one can choose $G = (\mathbb{Z}^d, \mathbb{E}^d)$ where $d \geq 2$ and $\mathbb{E}^d$ is the set of edges between nearest neighbors in $\mathbb{Z}^d$ for the Euclidean distance. We associate with the edges of the graph a family of i.i.d. random variables $(t(e), e \in E)$ where $t(e)$ has a Bernoulli distribution of parameter $p \in [0, 1]$. This parameter gives the level of porosity of the rock: we say that an edge e is open, *i.e.*, that the water can flow through the corresponding microscopic tube, if $t(e) = 1$, and we say that the edge e is closed otherwise. This is the model of i.i.d. bound percolation on G.

The piece of rock itself is macroscopic, *i.e.*, infinitely large in comparison with the microscopic tubes, thus the media is porous at the macroscopic scale if the water can flow through microscopic tubes between points that are arbitrarily far away one from each other. In other words, the media is porous if the graph $(V, \{e \in E : t(e) = 1\})$ of open edges has an infinite cluster. The fundamental result in percolation theory is the existence of a phase transition for percolation on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$. It states that the behavior of the system changes precisely when the parameter p reaches a critical value $p_c(d) \in (0, 1)$: the media is porous for $p > p_c(d)$, and it is not for $p < p_c(d)$. It has been proved by Broadbent and Hammersley [BH57] and Hammersley [Ham57, Ham59].

Theorem 1 (Phase transition)**.** *Let $d \geq 2$ and $G = (\mathbb{Z}^d, \mathbb{E}^d)$. There exists a critical parameter $p_c(d) \in (0, 1)$ such that*

- *if $p > p_c(d)$ then almost surely there exists a unique infinite cluster in the graph $(\mathbb{Z}^d, \{e \in \mathbb{E}^d : t(e) = 1\})$ of open edges;*
- *if $p < p_c(d)$ then almost surely there does not exist any infinite cluster in the graph $(\mathbb{Z}^d, \{e \in \mathbb{E}^d : t(e) = 1\})$ of open edges.*

In dimension $d = 2$, it has been proved by Kesten [Kes80] in 1980 that $p_c(2) = 1/2$, and by Harris [Har60] in 1960 that the media is not porous at criticality ($p = 1/2$). For a proof of these results and many others, we refer to Grimmett's book [Gri99]. Indeed, percolation has been intensively studied since the 60's, and is still a very active field of research. In particular the understanding of the behavior of the model in dimension $d = 2$ at criticality has been widely improved in the last two decades thanks to the introduction of tools from complex analysis and the SLE processes - the interested reader should study Werner's works for instance. However many questions remain open, in particular in dimension $d = 3$. What is the value of $p_c(3)$? Even a precise conjecture for the value of $p_c(3)$ is missing. Is the media porous at criticality ? It is a major open problem to prove that the answer is no in dimension 3 also.

We do not develop further the state of the art in percolation since this dissertation does not focus on this model. However, the model of first passage percolation we study is a refinement of percolation. As we will see phase transitions also occur in first passage percolation, and they are linked with phase transitions for underlying percolation models.

1.1.2 First passage percolation

First passage percolation was introduced by Hammersley and Welsh [HW65] in 1965. Consider a piece of rock. Percolation is a model to understand if the rock is porous, *i.e.*, if water can flow through it. First passage percolation is a model to understand at which speed water can flow through it. We still suppose that the water can flow through microscopic tubes inside the rock that are the edges E of a given graph $G = (V, E)$, that will be $(\mathbb{Z}^d, \mathbb{E}^d)$ in our study. Now we associate with the edges of the graph a family of i.i.d. random variables $(t(e), e \in E)$ where $t(e)$ is non-negative, and we interpret $t(e)$ as the time that water needs to cross the edge e. The variable $t(e)$ is called the passage time of e. We call path an alternative sequence $(v_0, e_1, v_1, \dots, e_n, v_n)$ of vertices $(v_i)_{0 \leq i \leq n}$ and edges $(e_i)_{1 \leq i \leq n}$ such that the vertices v_{i-1} and v_i are the endpoints of the edge e_i for $i \in \{1, \dots, n\}$. If γ is a path, we define the passage time of γ as $T(\gamma) = \sum_{e \in \gamma} t(e)$. Then the passage time between two points x and y in $\mathbb{Z}^d$ is given by

$$T(x, y) \,=\, \inf\{T(\gamma) \,:\, \gamma \text{ is a path from } x \text{ to } y\}\,.$$

This defines a random pseudo-distance on $\mathbb{Z}^d$ (the only property that can be missing is the separation property). The firsts questions we can ask are the following.

- If water is injected at the origin of the graph at time 0, how long does it take for the water to wet a point x far away from the origin, *i.e.*, how does $T(0, x)$ behave for large x ?

- If water is injected at the origin of the graph at time 0, how is growing the set of wet points in the rock asymptotically, *i.e.*, what is the behavior of $\{x \in \mathbb{Z}^d \,:\, T(0, x) \leq t\}$ for large t ?

A partial state of the art concerning these questions is given in section 1.2.1, and some new answers are proposed in Chapter 5.

In the context of the study of porous media, the variable $t(e)$ associated with an edge e, *i.e.*, with a microscopic tube inside the rock, can be interpreted differently as the maximal amount of water that can cross e per second. We call it the capacity of the edge e. This interpretation leads naturally to the study of the maximal flow through a piece of rock. This maximal flow corresponds to the maximal amount of water that can cross the piece of rock per second from its top to its bottom for instance, and is defined rigorously in Section 1.2.2, where a quite general state of the art is also given. The study of maximal flows is the purpose of Chapters 2, 3 and 4. We want to underline the fact that first passage percolation is a toy model to understand maximal flows through porous media. The limitation of the flow through an edge by a fixed capacity may not seem physically relevant. Alternatively, the graph G can also be seen as a communication network, where each edge e is a communication channel that cannot transmit more informations per second than a given limit $t(e)$.

We want to mention a third possible interpretation of $t(e)$: e is a wire whose electrical conductance is given by $t(e)$. One can study the behavior of the effective electrical resistance of a large subset of the graph G. This interpretation leads also to many interesting mathematical problems. However, we do not discuss it in the rest of this dissertation.

1.2 State of the art in first passage percolation

1.2.1 Random distance

Time constant and shape theorem

The random distance in first passage percolation has been and is still intensively studied. A reference work is Kesten's lecture notes [Kes86]. We refer also to Howard's review [How04]. Grimmett and Kesten [GK12] gives an overview on recent advances in this field. Auffinger, Damron and Hanson wrote very recently the survey [ADH15] that provides an overview on results obtained in the 80's and 90's, describes the recent advances and gives a collection of old and new open questions. From now on, we restrict ourselves to the study of first passage percolation on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ for a dimension $d \geq 2$.

Given a probability measure F on $\mathbb{R}^+$, we equip the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ with a i.i.d. family $(t(e), e \in \mathbb{E}^d)$ of random variables of common distribution F. The variable $t(e)$ is interpreted here as the passage time of e, thus we define for any path γ its passage time $T(\gamma) = \sum_{e\in\gamma} t(e)$, and we define the random pseudo-metric T on $\mathbb{Z}^d$ as

$$\forall x, y \,, \in \mathbb{Z}^d \quad T(x,y) \,=\, \inf\{T(\gamma) \,:\, \gamma \text{ is a path from } x \text{ to } y\}\,.$$

The variable $T(x,y)$ is the minimum time needed to go from x to y. A central object in the study of first passage percolation is the set of points reached from the origin 0 of the graph within a time $t \in \mathbb{R}^+$:

$$B_t^v \,=\, \{z \in \mathbb{Z}^d \,|\, T(0,z) \leq t\}\,.$$

The exponent v indicates that B_t^v is a set of vertices. It may be useful to consider a fattened set B_t by adding a small unit cube around each point of B_t^v, *i.e.*, to define the following random set:

$$\forall t \in \mathbb{R}^+\,, \quad B_t \,=\, \{z + u \,|\, z \in \mathbb{Z}^d \text{ s.t. } T(0,z) \leq t\,,\ u \in [-1/2, 1/2]^d\}\,.$$

Fix $e_1 = (1, 0, \dots, 0)$. Thanks to a subadditive argument, Hammersley and Welsh [HW65] and Kingman [Kin68] proved that if $d = 2$ and F has finite mean, then $\lim_{n\to\infty} T(0, ne_1)/n$ exists a.s. and in L^1, the limit is a constant denoted by $\mu(e_1)$ and called the time constant. The moment condition was improved some years later by several people independently, and the study was extended to any dimension $d \geq 2$ (see for instance Kesten's St Flour notes [Kes86]). The convergence to the time constant can be stated as follows.

Theorem 2 (Definition of the time constant)**.** *If* $\mathbb{E}[\min(t_1, \dots, t_{2d})] < \infty$ *where* (t_i) *are i.i.d. with distribution* F*, there exists a constant* $\mu(e_1) \in \mathbb{R}^+$ *such that*

$$\lim_{n\to\infty} \frac{T(0, ne_1)}{n} \,=\, \mu(e_1) \quad \textit{a.s. and in } L^1\,.$$

Moreover, the condition $\mathbb{E}[\min(t_1, \dots, t_{2d})] < \infty$ *is necessary for this convergence to hold a.s. or in* L^1.

This convergence can be generalized by the same arguments, and under the same hypothesis, to rational directions : there exists an homogeneous function $\mu : \mathbb{Q}^d \to \mathbb{R}^+$ such that for all $x \in \mathbb{Z}^d$, we have $\lim_{n\to\infty} T(0, nx)/n = \mu(x)$ a.s. and in L^1. The function μ can be extended to $\mathbb{R}^d$ by continuity (see [Kes86]). A simple convexity argument proves that either $\mu(x) = 0$ for all $x \in \mathbb{R}^d$, or $\mu(x) > 0$ for all $x \neq 0$. Kesten [Kes86] proved the following result on the positivity of the time constant.

Theorem 3 (Positivity of the time constant)**.** *If* $\mathbb{E}[\min(t_1, \dots, t_{2d})] < \infty$, *thus* $\mu(e_1)$ *is well defined, then* $\mu(e_1) > 0$ *if and only if* $F(\{0\}) < p_c(d)$, i.e., *if and only if the percolation* $(\mathbb{1}_{\{t(e)=0\}}, e \in \mathbb{E}^d)$ *is subcritical.*

If $F(\{0\}) < p_c(d)$, μ is a norm on $\mathbb{R}^d$, and the unit ball for this norm

$$\mathcal{B}_\mu = \{x \in \mathbb{R}^d \,|\, \mu(x) \leq 1\}$$

is compact. A natural question at this stage is whether the convergence $\lim_{n\to\infty} T(0, nx)/n = \mu(x)$ is uniform in all directions. The shape theorem, inspired by Richardson's work [Ric73], answers positively this question under a stronger moment condition. It can be stated as follows (see Cox and Durrett [CD81] in dimension $d = 2$ and Kesten [Kes86] in higher dimension).

Theorem 4 (Shape theorem)**.** *If* $\mathbb{E}[\min(t_1^d, \dots, t_{2d}^d)] < \infty$ *where* (t_i) *are i.i.d. with distribution* F, *and if* $F(\{0\}) < p_c(d)$, *then for all* $\varepsilon > 0$, *a.s., there exists* $t_0 \in \mathbb{R}^+$ *such that*

$$\forall t \geq t_0 \,, \quad (1-\varepsilon)\mathcal{B}_\mu \subset \frac{B_t}{t} \subset (1+\varepsilon)\mathcal{B}_\mu \,. \tag{1.1}$$

Moreover, the condition $\mathbb{E}[\min(t_1^d, \dots, t_{2d}^d)] < \infty$ *is necessary for this convergence to hold a.s.*

An equivalent shape theorem can be stated when $F(\{0\}) \geq p_c(d)$, but the "shape" appearing in this case is $\mathbb{R}^d$ itself.

Generalizations

A first direction in which these results can be extended is by considering a law F on $[0, +\infty[$ which does not satisfy any moment condition, at the price of obtaining weaker convergences. This work was performed successfully by Cox and Durrett [CD81] in dimension $d = 2$ and then by Kesten [Kes86] in any dimension $d \geq 2$. More precisely, they proved that there always exists a function $\hat{\mu} : \mathbb{R}^d \to \mathbb{R}^+$ such that for all $x \in \mathbb{Z}^d$, we have $\lim_{n\to\infty} T(0, nx)/n = \hat{\mu}(x)$ in probability. If $\mathbb{E}[\min(t_1, \dots, t_{2d})] < \infty$ then $\hat{\mu} = \mu$. The function $\hat{\mu}$ is built as the a.s. limit of a more regular sequence of times $\hat{T}(0, nx)/n$ that we now describe roughly. They consider a $M \in \mathbb{R}^+$ large enough so that $F([0, M])$ is very close to 1. Thus the percolation $(\mathbb{1}_{\{t(e)\leq M\}}, e \in \mathbb{E}^d)$ is highly supercritical, so if we denote by $\mathcal{C}_M$ its infinite cluster, each point $x \in \mathbb{Z}^d$ is a.s. surrounded by a small contour $S(x) \subset \mathcal{C}_M$. They define $\hat{T}(x, y) = T(S(x), S(y))$ for $x, y \in \mathbb{Z}^d$. The times $\hat{T}(0, x)$ have good moment properties, thus $\hat{\mu}(x)$ can be defined as the a.s. and L^1 limit of $\hat{T}(0, nx)/n$ for all $x \in \mathbb{Z}^d$ by a classical subadditive argument; then $\hat{\mu}$ can be extended to $\mathbb{Q}^d$ by homogeneity, and finally to $\mathbb{R}^d$ by continuity. The convergence of $T(0, nx)/n$ towards $\hat{\mu}(x)$ in probability is a consequence of the fact that T and $\hat{T}$ are close enough. Kesten's result on the positivity of the time constant, Theorem

3, remains valid for $\hat{\mu}$. Moreover, Cox and Durrett [CD81] and Kesten [Kes86] proved an a.s. shape theorem for

$$\hat{B}_t \,=\, \{z+u \,|\, z\in\mathbb{Z}^d \text{ s.t. } \hat{T}(0,z)\leq t\,,\; u\in[-1/2,1/2]^d\}$$

with shape limit $\mathcal{B}_{\hat{\mu}} = \{x\in\mathbb{R}^d \,|\, \hat{\mu}(x)\leq 1\}$ when $\hat{\mu}$ is a norm, *i.e.*, when $F(\{0\})<p_c(d)$ (and an equivalent shape result with shape limit equal to $\mathcal{B}_{\hat{\mu}}=\mathbb{R}^d$ when $F(\{0\})\geq p_c(d)$). In dimension $d=2$, Cox and Durrett [CD81] also deduced a weak shape theorem for B_t:

$$\forall K\in\mathbb{R}^+\,,\quad \lim_{t\to\infty}\mathcal{L}^2\left(\left(\frac{B_t}{t}\triangle\mathcal{B}_{\hat{\mu}}\right)\cap\{x\in\mathbb{R}^d\,|\,\|x\|_1\leq K\}\right) \,=\, 0 \quad \text{a.s.}\,, \tag{1.2}$$

where $\mathcal{L}^2$ denotes the Lebesgue measure in $\mathbb{R}^2$ and $\triangle$ denotes the symmetric difference between two sets, and

$$\forall\varepsilon>0\,,\ \text{a.s.}\,,\ \exists t_0\in\mathbb{R}^+\,,\ \forall t\geq t_0 \quad \frac{B_t}{t}\,\subset\,\mathcal{B}_{\mu}\,. \tag{1.3}$$

In fact, in the case $F(\{0\})<p_c(d)$, the intersection with $\{x\in\mathbb{R}^d\,|\,\|x\|_1\leq K\}$ is not needed in (1.2), since $\mathcal{B}_{\hat{\mu}}$ is compact. The inclusion in (1.3) follows directly from the a.s. shape theorem for $\hat{B}_t$ since $\hat{T}(0,x)\leq T(0,x)$ for all $x\in\mathbb{Z}^d$. Kesten did not write the generalization to any dimension $d\geq 2$ of this weak shape theorem for B_t without moment condition on F but all the required ingredients are present in [Kes86].

A second direction in which these results can be extended is by considering random passage times $(t(e),e\in\mathbb{E}^d)$ that are not i.i.d. but only stationary and ergodic. Boivin [Boi90] defined a time constant in this case and proved a corresponding shape theorem under some moment assumptions on F. We do not present these results in details since this generalization is not directly linked with our works.

A third possible way to generalize these results is to consider infinite passage time. This case has been studied by Garet and Marchand [GM04]. They present it as a model of first passage percolation in random environment: they consider first a supercritical Bernoulli percolation on $\mathbb{E}^d$, and then they associate with each remaining edge e a finite passage time $t(e)$ such that the family $(t(e),e\in\mathbb{E}^d)$ is stationary and ergodic. If x and y are two vertices that do not belong to the same cluster of the Bernoulli percolation, there is no path from x to y and $T(x,y)=+\infty$. To define a time constant $\mu'(x)$ in a rational direction x, they first consider the probability $\overline{\mathbb{P}}$ conditioned by the event $\{0\in\mathcal{C}_\infty\}$, where $\mathcal{C}_\infty$ is the infinite cluster of the supercritical percolation mentioned above. In the direction of x they only take into account the points $(x_n)_{n\in\mathbb{N}}$ that belong to $\mathcal{C}_\infty$, with $\lim_{n\to\infty}\|x_n\|_1=\infty$. Then under a moment condition on the law of the passage times, they prove that $\overline{\mathbb{P}}$-a.s., the times $T(0,x_n)$ properly rescaled converge to a constant $\mu'(x)$. They also prove a shape theorem for B_t when μ' is a norm (*i.e.*, when $\mu'(e_1)>0$):

$$\lim_{t\to\infty}\mathcal{D}_H\left(\frac{B_t}{t},\mathcal{B}_{\mu'}\right) \,=\, 0 \quad \overline{\mathbb{P}}\text{-a.s.}\,,$$

where $\mathcal{D}_H$ denotes the Hausdorff distance between two sets, and $\mathcal{B}_{\mu'}=\{x\in\mathbb{R}^d\,|\,\mu'(x)\leq 1\}$. Let us remark that the infinite cluster $\mathcal{C}_\infty$ has holes, and so does the set B_t, thus a shape theorem as stated in (1.1) cannot hold. The use of the Hausdorff distance allows to fill the small holes in B_t. Garet and Marchand's results are all the more general since they did not consider i.i.d. passage times but the ergodic stationary case as in Boivin [Boi90]. However, their moment condition on the finite passage times is quite restrictive. In the i.i.d. case, the existence of μ' is proved with a moment of order $2+\varepsilon$, and the shape theorem is proved with a moment of order $2(d^2+2d-1)+\varepsilon$. We emphasize that these hypotheses are of course fulfilled if the finite passage times are bounded, which is the case in particular if the finite passage times are equal to 1. In this case $T(x, y)$, $x, y \in \mathbb{Z}^d$ is equal to the length of the shortest path that links x to y in the percolation model if x and y are connected, and it is equal to $+\infty$ if x and y are not connected. The variable $T(x, y)$ is called the *chemical distance* between x and y and is usually denoted by $D(x, y)$. This chemical distance was previously studied, see for instance Antal and Pisztora's article [AP96]. To finish with the presentation of Garet and Marchand's works, we should say that the generality of the stationary ergodic setting they chose makes it also difficult to give a characterization of the positivity of the time constant in terms of the law of the passage times. Garet and Marchand give sufficient or necessary conditions for the positivity of μ', which in the i.i.d. case correspond to:

$$[F(\{0\}) > p_c(d) \Rightarrow \mu'(e_1) = 0] \quad \text{and} \quad [F(\{0\}) < p_c(d) \Rightarrow \mu'(e_1) > 0] \,,$$

but they do not study the critical case $F(\{0\}) = p_c(d)$.

Continuity

Once the time constant is defined, a natural question is to wonder if it varies continuously with the distribution of the passage times of the edges. This question has been answered positively by Cox and Kesten [Cox80, CK81, Kes86]. In this paragraph, we will emphasize the dependance of the passage times $t(e)$ (resp. the distance T, the time contant μ) on the probability measure F by denoting it by $t_F(e)$ (resp. T_F, μ_F).

Theorem 5 (Continuity of the time constant)**.** *Let F, F_n be probability measures on $\mathbb{R}^+$. If F_n converges weakly towards F, then for every $x \in \mathbb{R}^d$,*

$$\lim_{n\to\infty} \mu_{F_n}(x) \;=\; \mu_F(x) \,.$$

Cox [Cox80] proved first this result in dimension $d = 2$ with an additional hypothesis of uniform integrability: he supposed that all the probability measures F_n were stochastically dominated by a probability measure H with finite mean. To remove this hypothesis of uniform integrability in dimension $d = 2$, Cox and Kesten [CK81] used the regularized passage times and the technology of the contours introduced by Cox and Durrett [CD81]. The key step of their proof is the following lemma.

Lemma 6 (Truncated passage times)**.** *Let $d = 2$. Let F be a probability measure on $\mathbb{R}^+$, and let $F^K = \mathbb{1}_{[0,K)} F + F([K, +\infty))\delta_K$ be the distribution of the passage times $t_F(e)$ truncated at K. Then for every $x \in \mathbb{R}^2$,*

$$\lim_{K\to\infty} \mu_{F^K}(x) \;=\; \mu_F(x) \,.$$

To prove this lemma, they consider a geodesic γ from 0 to a fixed vertex x for the truncated passage times $\inf(t_F(e), K)$. When looking at the original passage times $t_F(e)$, some edges along γ may have an arbitrarily large passage time: to recover a path γ' from 0 to x such that $T_F(\gamma')$ is not too large in comparison with $T_{F^K}(\gamma)$, they need to bypass these bad edges. They construct the bypass of a bad edge e inside the contour $S(e) \subset \mathcal{C}_M$ of the edge e, thus they bound the passage time of this bypass by $M|S(e)|$ where $|S(e)|$ denotes the cardinality of $S(e)$. Kesten [Kes86] extended these results to any dimension $d \geq 2$.

Other directions

A lot more was proved concerning the random distance in first passage percolation, and many works are in progress. We have no hope to be exhaustive here, so we refer to Grimmett and Kesten [GK12] and Auffinger, Damron and Hanson [ADH15] for more informations. We try to give just an idea of the directions in which research is active:

- properties of the limit shape $\mathcal{B}_\mu$ (strict convexity of μ, link between the probability measure F and the corresponding shape $\mathcal{B}_\mu$...)
- fluctuations and concentration of $T(0,x)$ (study of $\mathrm{Var}(T(0,x))$, large deviations...)
- geodesics (existence and properties of infinite geodesics, wandering exponent, link with Busemann functions...)
- first passage percolation as a competition model (competition with same or different speed, competition interface...).

We chose to present in this dissertation only the results on which we rely in this dissertation (see Section 5).

1.2.2 Maximal flow

Maximal stream and minimal cutset

The study of maximal flows in first passage percolation on $\mathbb{Z}^d$ has been initiated by Grimmett and Kesten [GK84] in 1984 in dimension 2 and Kesten [Kes87] in 1987 in higher dimension. This interpretation of the model of first passage percolation has been a lot less studied than the one in terms of random distance we discussed in Section 1.2.1. One of the reason is the added difficulty to deal with this interpretation, in which the study of the random paths that are the geodesics is replaced by the study of some random hypersurfaces that we present in this section.

As previously, given a probability measure F on $\mathbb{R}^+$, we equip the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ with a i.i.d. family $(t(e), e \in \mathbb{E}^d)$ of random variables of common distribution F. The variable $t(e)$ is now interpreted as the maximal amount of water that can cross the edge e par second. Consider a finite subset Ω of the graph (or a bounded subset of $\mathbb{R}^d$ that we intersect with $(\mathbb{Z}^d, \mathbb{E}^d)$ to obtain a finite graph), which represents the piece of rock through which the water flows, and let Γ^1 and Γ^2 be two disjoint subsets of vertices in Ω: Γ^1 (resp. Γ^2) represents the sources (resp. the sinks) through which the water can enter in (resp. escapes from) Ω. A possible stream inside Ω between Γ^1 and Γ^2 is a function $\vec{f} : \mathbb{E}^d \mapsto \mathbb{R}^d$ such that for all $e \in \mathbb{E}^d$,

- $\|\vec{f}(e)\|_2$ is the amount of water that flows through e per second,
- $\vec{f}(e)/\|\vec{f}(e)\|_2$ is the direction in which the water flows through e.

For instance, if the endpoints of e are the vertices x and y, which are at Euclidean distance 1, then $\vec{f}(e)/\|\vec{f}(e)\|_2$ can be either the unit vector $\vec{xy}$ or the unit vector $\vec{yx}$. A stream $\vec{f}$ inside Ω between Γ^1 and Γ^2 is admissible if and only if it satisfies the following constraints :

- *the node law:* for every vertex x in $\Omega \smallsetminus (\Gamma^1 \cap \Gamma^2)$, we have

$$\sum_{y \in \mathbb{Z}^d \,:\, e=(x,y) \in \mathbb{E}^d \cap \Omega} \|\vec{f}(e)\|_2 \left(\mathbb{1}_{\vec{f}(e)/\|\vec{f}(e)\|_2 = \vec{xy}} - \mathbb{1}_{\vec{f}(e)/\|\vec{f}(e)\|_2 = \vec{yx}} \right) \,=\, 0\,,$$

i.e., there is no loss of fluid inside Ω;

- *the capacity constraint:* for every edge e in Ω, we have
$$0 \le \|\vec{f}(e)\|_2 \le t(e)\,,$$
i.e., the amount of water that flows through e per second cannot exceed its capacity $t(e)$.

Since the capacities are random, the set of admissible streams inside Ω between Γ^1 and Γ^2 is also random. With each such admissible stream $\vec{f}$, we associate its flow defined by

$$\mathrm{flow}(\vec{f}) \,=\, \sum_{x\in\Gamma^1}\ \sum_{y\in\Omega\setminus\Gamma^1\,:\,e=(x,y)\in\mathbb{E}^d} \|\vec{f}(e)\|_2 \left(\mathbb{1}_{\vec{f}(e)/\|\vec{f}(e)\|_2=\vec{xy}} - \mathbb{1}_{\vec{f}(e)/\|\vec{f}(e)\|_2=\vec{yx}}\right)\,.$$

This is the amount of water that enters in Ω through Γ^1 per second (we count it negatively if the water escapes from Ω). By the node law, equivalently, $\mathrm{flow}(\vec{f})$ is equal to the amount of water that escapes from Ω through Γ^2 per second:

$$\mathrm{flow}(\vec{f}) \,=\, \sum_{x\in\Gamma^2}\ \sum_{y\in\Omega\setminus\Gamma^2\,:\,e=(x,y)\in\mathbb{E}^d} \|\vec{f}(e)\|_2 \left(\mathbb{1}_{\vec{f}(e)/\|\vec{f}(e)\|_2=\vec{yx}} - \mathbb{1}_{\vec{f}(e)/\|\vec{f}(e)\|_2=\vec{xy}}\right)\,.$$

The maximal flow from Γ^1 to Γ^2 in Ω, denoted by $\phi(\Gamma^1 \to \Gamma^2 \text{ in } \Omega)$, is the supremum of the flows of all admissible streams through Ω:

$$\phi(\Gamma^1 \to \Gamma^2 \text{ in } \Omega) \,=\, \sup\{\mathrm{flow}(\vec{f}) \,:\, \vec{f} \text{ is an admissible inside } \Omega \text{ between } \Gamma^1 \text{ and } \Gamma^2\}\,.$$

The maximal flow through large domains Ω is the first object to study. If it exists, one can also want to study a maximal stream, *i.e.*, an admissible stream whose flow is maximal and thus equal to $\phi(\Gamma^1 \to \Gamma^2 \text{ in } \Omega)$.

It is not so easy to deal with admissible streams, but hopefully there is an alternative description of maximal flows we can work with. We say that a set of edges $E \subset \Omega$ cuts Γ^1 from Γ^2 in Ω (or is a cutset, for short) if there is no path from Γ^1 to Γ^2 in $\Omega \setminus E$. We associate with any set of edges E its capacity $T(E)$ defined by $T(E) = \sum_{e\in E} t(e)$. The max-flow min-cut theorem (see [Bol79, Ful75]), a result of graph theory, states that

$$\phi(\Gamma^1 \to \Gamma^2 \text{ in } \Omega) \,=\, \inf\{T(E) \,:\, E \text{ cuts } \Gamma^1 \text{ from } \Gamma^2 \text{ in } \Omega\}\,.$$

The idea of this theorem is quite intuitive: the maximal flow is limited by edges that are jammed, *i.e.*, that are crossed by an amount of water per second which is equal to their capacities. These jammed edges form a cutset, otherwise there would be a path of edges from Γ^1 to Γ^2 through which a higher amount of water could circulate. Finally, some of the jammed edges may not limit the flow since other edges, before or after them on the trajectory of water, already limit the flow, thus the maximal flow is given by the minimal capacity of a cutset. A third object of interest is a minimal cutset, *i.e.*, a cutset with minimal capacity.

If the distribution F of the capacities of the edges is a Bernoulli law, the maximal flow $\phi(\Gamma^1 \to \Gamma^2 \text{ in } \Omega)$ is simply the maximal number of disjoint open paths from Γ^1 to Γ^2 in Ω, where a path is open if all its edges have capacity 1 (as in the percolation setting) and two paths are disjoint if they share no common edge. By the max-flow min-cut theorem, $\phi(\Gamma^1 \to \Gamma^2 \text{ in } \Omega)$ is also equal to the minimal number of edges that have to be deleted to disconnect completely Γ^1 from Γ^2 in the subgraph of Ω made of the open edges.

Kesten [Kes87] presented this interpretation of first passage percolation as a higher dimensional version of classical first passage percolation. To understand this point of view, let us associate with each edge e a small plaquette e^*, *i.e.*, a $(d-1)$-dimensional hypersquare whose sides have length 1, are parallel to the edges of the graph, such that e^* is normal to e and cuts e in its middle (see Figure 1.1). We associate with the plaquette e^* the capacity $t(e)$ of the edge e to which it corresponds.

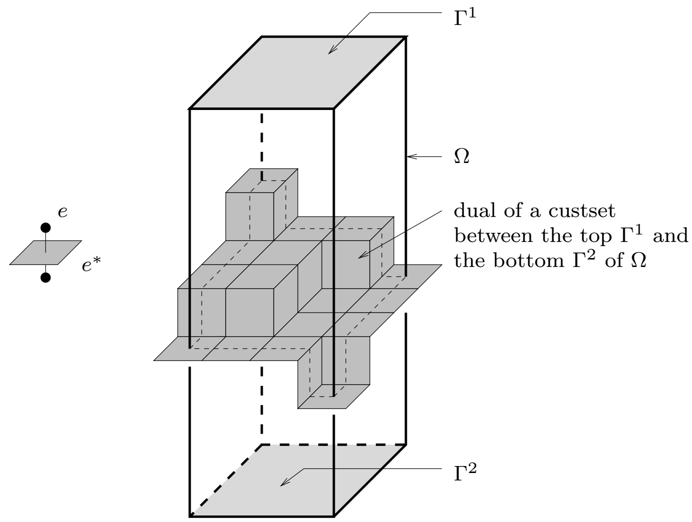

Figure 1.1: Plaquette and cutset.

With a set of edges E we associate the set of the corresponding plaquettes $E^* = \{e^* : e \in E\}$. Roughly speaking, if E cuts Γ^1 from Γ^2 in Ω then E^* is a "surface" of plaquettes that disconnects Γ^1 from Γ^2 in Ω - we do not try to give a rigorous definition of the term surface here, but the reader can look Figure 1.1 to fix ideas. The study of maximal flows in first passage percolation is equivalent, through the max-flow min-cut theorem, to the study of the minimal capacities of cutsets. When we compare this to the classical interpretation of first passage percolation, the study of geodescis (*i.e.*, paths of dimension 1) is replaced by the study of minimal cutsets (*i.e.*, hypersurfaces of dimension $d-1$). In this sense, the study of maximal flow is a higher dimensional version of classical first passage percolation.

A few more should be said about dimension $d = 2$, where we can use the notion of duality of planar graphs. The graph $(\mathbb{Z}^2, \mathbb{E}^2)$ is self-dual, up to a translation of vector $(1/2, 1/2)$. The plaquette e^* associated with an edge $e \in \mathbb{E}^2$ is simply the dual edge of e, thus the dual of a custset E is a random object of dimension 1, exactly as geodesics are. Let us consider a precise example to fix ideas. We consider $\Omega = [k_1, k_2] \times [k_3, k_4]$ a rectangle in $\mathbb{Z}^2$ ($k_i \in \mathbb{Z}$ for $i \in \{1, \dots, 4\}$, $k_1 < k_2$ and $k_3 < k_4$), $\Gamma^1 = \{(i, k_4) : i \in \{k_1, \dots, k_2\}\}$ its top and $\Gamma^2 = \{(i, k_3) : i \in \{k_1, \dots, k_2\}\}$ its bottom. Then consider a cutset E inside Ω, and remove from E any edge e which is unnecessary, *i.e.*, such that $E \smallsetminus \{e\}$ is still a cutset. Then E^* is exactly a path from the left side to the right side of $\Omega^* = [k_1 - 1/2, k_2 + 1/2] \times [k_3 + 1/2, k_4 - 1/2]$, the dual rectangle of Ω (see Figure 1.2). This implies that the maximal flow $\phi(\Gamma^1 \to \Gamma^2 \text{ in } \Omega)$ in this setting is equal to the minimal capacity of a left-right path in Ω^* in the dual graph. If the capacities of the dual edges are interpreted as passage times, then maximal flows in the initial graph correspond to minimal passage times in the dual graph, thus

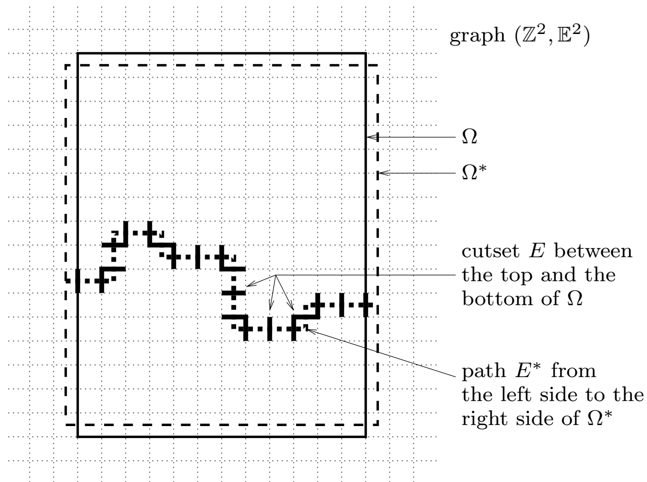

Figure 1.2: Dimension two and duality.

the two interpretations of first passage percolation lead to the study of the same objects. However, this correspondance does not hold anymore in dimension $d \geq 3$. Except in Sections 2.4 and 3.2, all the results we present about first passage percolation in this dissertation hold for any dimension $d \geq 2$. In all likelihood, many of them could be proved more easily in dimension $d = 2$, but our goal is to study the model in dimension 3 or more. An exception is made in Sections 2.4 and 3.2 where we study specifically maximal flows in dimension 2, using precisely arguments that hold only in dimension 2 to get more informations than in higher dimensions.

Maximal flow through cylinders : the subadditive case

We consider a general dimension $d \geq 3$. We now define two specific maximal flows through cylinders that are of particular interest. Let A be a non-degenerate hypersquare, *i.e.*, a rectangle of dimension $d-1$ in $\mathbb{R}^d$. Let $\vec{v}$ be one of the two unit vectors normal to A. For a positive real h, denote by $\mathrm{cyl}(A,h)$ the cylinder of basis A and height $2h$ defined by

$$\mathrm{cyl}(A,h) \,=\, \{x + t\vec{v} \,:\, x \in A\,,\, t \in [-h,h]\}\,.$$

Let $B_1(A,h)$ (resp. $B_2(A,h)$) be (a discrete version of) the top (resp. the bottom) of this cylinder, more precisely defined by

$$\begin{aligned} B_1(A,h) \,&=\, \{x \in \mathbb{Z}^d \cap \mathrm{cyl}(A,h) \,:\, \exists y \notin \mathrm{cyl}(A,h)\,,\, (x,y) \in \mathbb{E}^d \text{ and } (xy) \text{ intersects } A + h\vec{v}\}\,,\\ B_2(A,h) \,&=\, \{x \in \mathbb{Z}^d \cap \mathrm{cyl}(A,h) \,:\, \exists y \notin \mathrm{cyl}(A,h)\,,\, (x,y) \in \mathbb{E}^d \text{ and } (xy) \text{ intersects } A - h\vec{v}\}\,. \end{aligned}$$

We denote by $\phi(A,h)$ the maximal flow from the top to the bottom of the cylinder $\mathrm{cyl}(A,h)$ in the direction $\vec{v}$, defined by

$$\phi(A,h) \,=\, \phi(B_1(A,h) \to B_2(A,h) \text{ in } \mathrm{cyl}(A,h))\,.$$

The maximal flow $\phi(A,h)$ is the maximal flow through cylinders we want to study - if $\mathrm{cyl}(A,h)$ is a layer of rock, $\phi(A,h)$ corresponds to the amount of water that can cross the piece of rock from its top to its bottom per second. We denote by $\mathcal{H}^{d-1}$ the Hausdorff measure in dimension $d-1$: for

$A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{c\}$ with $k_i < l_i, c \in \mathbb{Z}$, we have $\mathcal{H}^{d-1}(A) = \prod_{i=1}^{d-1}(l_i - k_i)$. We expect that $\phi(A,h)$ grows asymptotically linearly in $\mathcal{H}^{d-1}(A)$ when the dimensions of the cylinder goes to infinity, since $\mathcal{H}^{d-1}(A)$ is the surface of the area through which the water can enter in the cylinder or escapes from it. However, $\phi(A,h)$ is not easy to deal with. Indeed, by the max-flow min-cut theorem, $\phi(A,h)$ is equal to the minimal capacity of a set of edges that cuts $B_1(A,h)$ from $B_2(A,h)$ in the cylinder. The dual of this set of edges is a surface of plaquettes whose boundary on the sides of $\mathrm{cyl}(A,h)$ is completely free. This implies that the union of cutsets between the top and the bottom of two adjacent cylinders is not a cutset itself between the top and the bottom of the union of the two cylinders. The maximal flow $\phi(A,h)$ does not have a property of subadditivity, which is the key tool in the study of classical first passage percolation. This is the reason why we define another maximal flow through $\mathrm{cyl}(A,h)$, for which subadditivity is recovered. The set $\mathrm{cyl}(A,h) \smallsetminus A$ has two connected components, denoted by $C_1(A,h)$ and $C_2(A,h)$. For $i = 1,2$, we denote by $C_i'(A,h)$ the discrete boundary of $C_i(A,h)$ defined by

$$C_i'(A,h) \,=\, \{x \in \mathbb{Z}^d \cap C_i(A,h) \,:\, \exists y \notin \mathrm{cyl}(A,h)\,,\, (x,y) \in \mathbb{E}^d\}\,.$$

We denote by $\tau(A,h)$ the maximal flow from the upper half part of the boundary of the cylinder to its lower half part, *i.e.*,

$$\tau(A,h) \,=\, \phi(C_1'(A,h) \rightarrow C_2'(A,h) \text{ in } \mathrm{cyl}(A,h))$$

(see Figure 1.3). By the max-flow min-cut theorem $\tau(A,h)$ is equal to the minimal capacity of a set of edges that cuts $C_1'(A,h)$ from $C_2'(A,h)$ inside the cylinder. To such a cutset E corresponds a dual set of plaquettes E^* whose boundary has to be very close to ∂A, the boundary of the hyperrectangle A. We say that a cylinder is straight if $\vec{v} = \vec{v}_0 := (0,0,\dots,1)$ and if there exists $k_i < l_i, c \in \mathbb{Z}$ such that $A = A(\vec{k},\vec{l}) = \prod_{i=1}^{d-1}[k_i,l_i] \times \{c\}$. In this case, for $c = 0$ and $k_i \leq 0 < l_i$, the family of variables $(\tau(A(\vec{k},\vec{l}),h))_{\vec{k},\vec{l}}$ is subadditive, since the minimal cutsets in adjacent cylinders can be glued together along the common side of these cylinders (see Figure 1.4). Thus a straightforward application of ergodic subadditive theorems in the multiparameter case (see Krengel and Pyke [KP87] and Smythe [Smy76]) leads to the following result.

Lemma 7 (Definition of the asymptotic rescaled flow)**.** *Let $A = A(\vec{k},\vec{l}) = \prod_{i=1}^{d-1}[k_i,l_i] \times \{0\}$ with $k_i \leq 0 < l_i \in \mathbb{Z}$. Let $h : \mathbb{N} \rightarrow \mathbb{R}^+$ such that $\lim_{n\rightarrow\infty} h(n) = +\infty$. Suppose that the distribution F of the capacities of the edges admits a finite mean,* i.e., $\int_{\mathbb{R}^+} x\, dF(x) < \infty$. *Then there exists a constant $\nu(\vec{v}_0)$, that does not depend on A and h, such that*

$$\lim_{n\rightarrow\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\vec{v}_0) \quad \textit{a.s. and in } L^1\,.$$

This result has been stated in a slightly different way by Kesten in [Kes87]. He considered there the more general case of flows through cylinders whose dimensions goes to infinity at different speeds in each direction, but in dimension $d = 3$. The constant $\nu(\vec{v}_0)$ obtained here is the equivalent of the time constant $\mu(e_1)$ defined in the context of random distances. In dimension $d = 2$, thanks to duality, it is easy to see that $\nu(\vec{v}_0) = \mu(e_1)$.

As suggested by classical first passage percolation, a constant $\nu(\vec{v})$ can be defined in any direction $\vec{v} \in \mathbb{S}^{d-1}$. This is not that trivial, since a lack of subadditivity appears when we look at tilted cylinders, due to the discretization of the boundary of the cylinders. Moreover, classical ergodic subadditive theorems cannot be used if the direction $\vec{v}$ is not rational, *i.e.*, if there does not exists

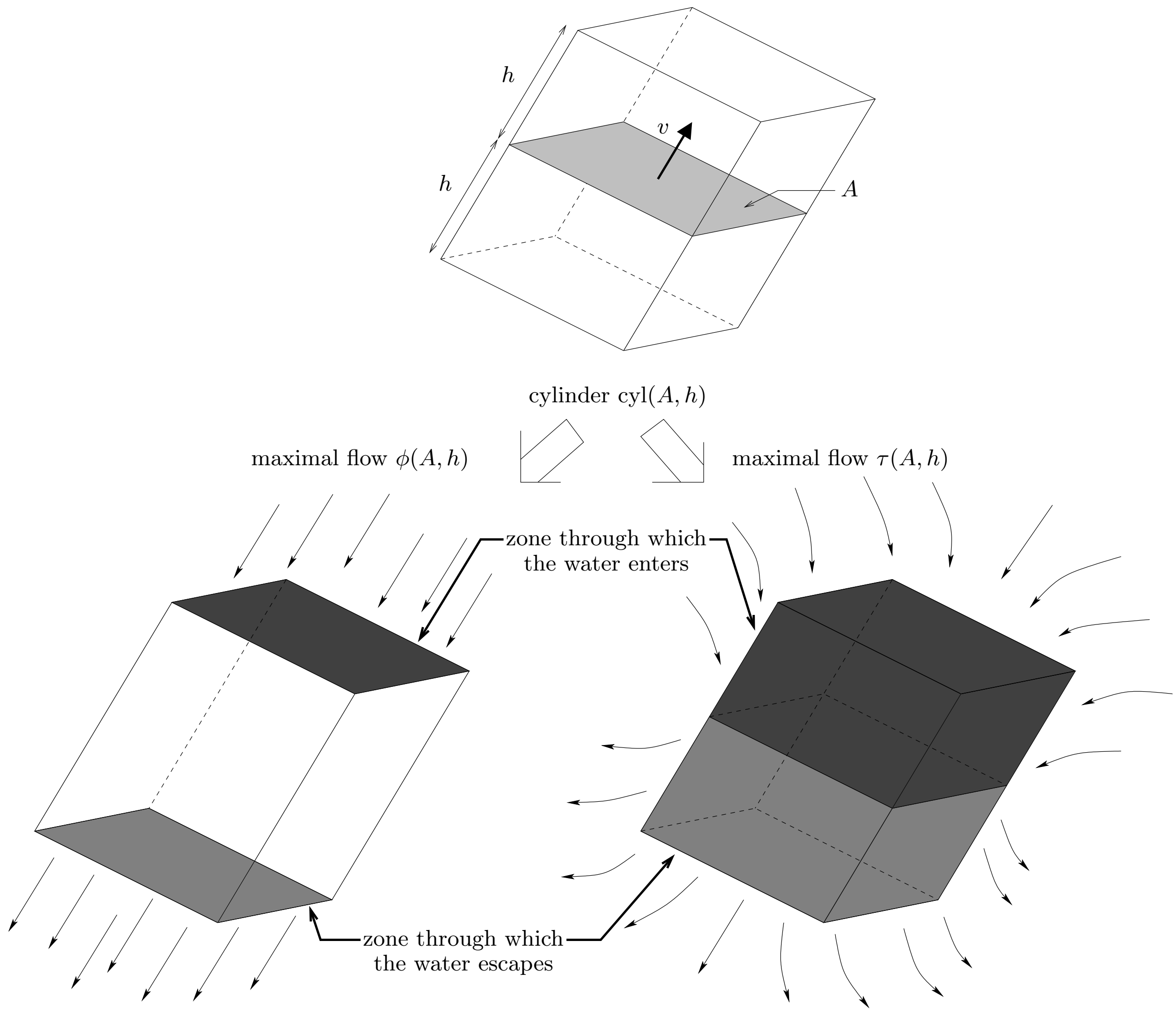

Figure 1.3: The cylinder $\operatorname{cyl}(A, h)$ and the maximal flows $\phi(A, h)$ and $\tau(A, h)$.

an integer M such that $M\vec{v}$ has integer coordinates. However, it is quite easy to recover at least the convergence of the expectation of the maximal flows τ through tilted cylinders.

Lemma 8 (Generalization of the definition of the asymptotic rescaled flow)**.** *Let A be a non-degenerate hyperrectangle and $\vec{v}$ a unit vector normal to A. Let $h : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{n\to\infty} h(n) = +\infty$. Suppose that the distribution F of the capacities of the edges admits a finite mean,* i.e., *$\int_{\mathbb{R}^+} x\, dF(x) < \infty$. Then there exists a constant $\nu(\vec{v})$, that does not depend on A and h, such that*

$$\lim_{n\to\infty} \frac{\mathbb{E}[\tau(nA, h(n))]}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}) .$$

As we will see in Section 2.2, $\nu(\vec{v})$ is in fact the limit of the rescaled maximal flows a.s. and in L^1 under additional hypotheses, but these convergences are less obvious to prove. The constant $\nu(\vec{v})$ can be seen as the asymptotic rescaled maximal flow that can flow through the media in the

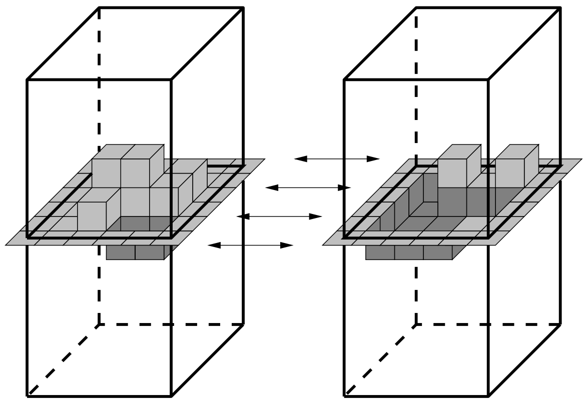

Figure 1.4: Subadditivity of τ in straight cylinders.

direction $\vec{v}$. It is also the asymptotic rescaled minimal capacity of a unit surface of plaquettes globally normal to $\vec{v}$.

Maximal flow through straight cylinders

Kesten [Kes87] studied the maximal flow $\phi(A,h)$ from the top to the bottom of a straight cylinder. Kesten considered cylinders whose sides have lengths that go to infinity at different speeds, but we present its result only in the particular case of the study of $\phi(nA,h(n))$ where $\lim_{n\to\infty} h(n) = +\infty$. The main result of [Kes87] is the following.

Theorem 9 (Maximal flow through straight cylinders - first result)**.** *Let $d=3$. Let $A=[0,k]\times[0,l]\times\{0\}$ with $k,l\in\mathbb{N}^*$. Let $h:\mathbb{N}\to\mathbb{R}^+$ such that $\lim_{n\to\infty} h(n)=+\infty$.*
i) Suppose that $F(\{0\}) < p_0$ for a fixed $p_0 \geq 1/27$, and suppose that F admits an exponential moment:

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) \,<\, \infty\,.$$

Suppose that there exists $\delta > 0$ satisfying

$$\lim_{n\to\infty} \frac{\log h(n)}{n^{1-\delta}} \,=\, 0\,. \tag{1.4}$$

Then

$$\lim_{n\to\infty} \frac{\phi(nA,h(n))}{\mathcal{H}^2(nA)} \,=\, \nu(\vec{v}_0) \,>\, 0 \quad a.s. \text{ and in } L^1\,. \tag{1.5}$$

ii) Suppose that $F(\{0\}) > 1-p_c(3)$ and F admits a moment of order 6. Suppose that there exists a constant $C<\infty$ such that

$$\liminf_{n\to\infty} \frac{h(n)}{\log n} \,>\, C\,.$$

Then for all n sufficiently large, we have

$$\phi(nA,h(n)) \,=\, 0 \quad a.s.\,.$$

Kesten [Kes87] emphasizes the fact that the hypotheses of this theorem are not optimal. He conjectures that an hypothesis of second moment for F should be sufficient for the convergence (1.5) to hold. He also conjectures that the condition (1.4) on the height of the cylinder can be improved. He conjectures that the condition $F(\{0\}) < p_0$ could be replaced by $F(\{0\}) < 1 - p_c(3)$, and thus that $F(\{0\}) < 1 - p_c(3)$ implies $\nu(\vec{v}_0) > 0$ and $F(\{0\}) > 1 - p_c(3)$ implies $\nu(\vec{v}_0) = 0$. As we will see, Zhang [Zha00, Zha07] answers some of these questions, whereas we give other answers in this dissertation.

We want to say a few words about Kesten's proof of the convergence (1.5). As we said previously, the variable $\phi(nA, h(n))$ is not subadditive, since minimal cutsets corresponding to these flows in adjacent cylinders cannot be glued together to get a bigger cutset in the union of the cylinders. The key idea of the proof is the following: instead of looking at all the possible cutsets between the top and the bottom of a cylinder, we restrict ourselves to cutsets that have a prescribed boundary condition, *i.e.*, hypersurfaces of plaquettes whose intersection with the vertical faces of the cylinder is given by a certain curve $\mathcal{C}$. Such a surface of plaquettes can be glued together with a surface of plaquettes in an adjacent cylinder which has symmetric boundary conditions $\mathcal{C}^*$ (see Figure 1.5). Thanks to the invariance of the model by symmetries with regards to the hyperplane spanned by

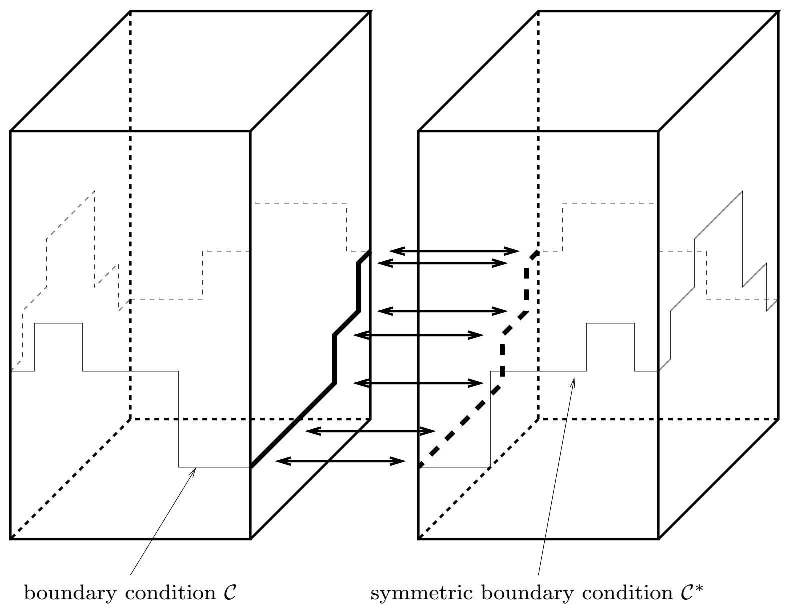

Figure 1.5: Glueing cutsets having symmetric boundary conditions.

the faces of a straight cylinder, a boundary condition $\mathcal{C}$ for a cutset and any symmetric boundary condition $\mathcal{C}^*$ has the same probability to be satisfied by a minimal cutset in the cylinder. Thus consider

$$\phi^{\mathcal{C}}(nA, h(n)) \,=\, \min\{T(E) \,:\, E \text{ is a cutset in } \mathrm{cyl}(nA, h(n)) \text{ with fixed boundary condition } \mathcal{C}\}\,,$$

where the boundary condition $\mathcal{C}$ is chosen such that the probability to observe a minimal cutset with boundary condition $\mathcal{C}$ is maximal. Then by the symmetry arguments we mentioned, $\phi^{\mathcal{C}}(nA, h(n))$ is a quasi-subadditive object that we can compare to $\tau(nA, h(n))$. Kesten uses the hypothesis $F(\{0\}) < p_0$ to control the cardinality of a minimal cutset, and thus the number of possible boundary conditions for the minimal cutsets, in order to compare $\phi^{\mathcal{C}}(nA, h(n))$ with $\phi(nA, h(n))$. We want to emphasize that the ideas we give are very rough, and the work of Kesten is in fact a lot more subtle.

The critical case

One of the questions left open by Kesten is the precise study of the positivity of $\nu(\vec{v}_0)$. Intuitively, one can think that if the percolation $(\mathbb{1}_{t(e)>0}, e \in \mathbb{E}^d)$ is subcritical, *i.e.*, if $F(\{0\}) > 1 - p_c(d)$, then water cannot flow through the graph at a macroscopic scale thus $\nu(\vec{v}_0) = 0$, whereas if the percolation $(\mathbb{1}_{t(e)>0}, e \in \mathbb{E}^d)$ is supercritical, *i.e.*, if $F(\{0\}) < 1 - p_c(d)$, then the opposite happens and $\nu(\vec{v}_0) > 0$. Kesten's work [Kes87] implies that in dimension $d = 3$, under some moment conditions, $F(\{0\}) > 1 - p_c(3)$ implies $\nu(\vec{v}_0) = 0$ and $F(\{0\}) < p_0$ implies $\nu(\vec{v}_0) > 0$. Zhang [Zha00] studies the behavior of maximal flows when $F(\{0\}) = 1 - p_c(d)$, *i.e.*, in the critical case. Like Kesten, Zhang works in dimension $d = 3$ and considers cylinders whose dimensions go to infinity at different speed, but we choose to present its result in the same framework as we did for Kesten's work. He obtains a general version of the following result.

Theorem 10 (Critical case). *Let $d = 3$. Suppose that $F(\{0\}) = 1 - p_c(3)$ and that F admits a finite mean*

$$\int_{\mathbb{R}^+} x\, dF(x) \,<\, \infty\,.$$

Let $A = [0, k] \times [0, l] \times \{0\}$ with $k, l \in \mathbb{N}^*$. Let $h : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{n\to\infty} h(n) = +\infty$. Then

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^2(nA)} \,=\, \lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^2(nA)} \,=\, 0 \quad \textit{a.s. and in } L^1\,,$$

We do not give details about the proof of this result, but we add two comments. The first one is that this proof relies on non trivial percolation properties and is sophisticated. The second one is that the proof works well in any dimension $d \geq 2$, as Zhang [Zha00] noticed it himself.

Bound on the cardinality of a minimal cutset

A more recent work of Zhang [Zha07] states a control on the cardinality of a minimal cutset under the relevant hypothesis $F(\{0\}) < 1 - p_c(d)$ in any dimension $d \geq 2$. We choose again to present its results in the same framework as previously, even if Zhang's results are more general.

Theorem 11 (Bound on the cardinality of a minimal cutset). *Let $A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{0\}$ with $k_i \leq 0 < l_i \in \mathbb{Z}$. Let $h : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{n\to\infty} h(n) = +\infty$ and $\log h(n) \leq \mathcal{H}^{d-1}(nA)$. Suppose that $F(\{0\}) < 1 - p_c(d)$ and that F admits an exponential moment:*

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) \,<\, \infty\,.$$

Let $E(nA, h(n))$ be a cutset between the top and the bottom of $\mathrm{cyl}(nA, h(n))$ *of minimal capacity, and of minimal cardinality $|E(nA, h(n))|$ among those custsets. There exists constants $\beta(F, d)$, $n_0(F, d)$ and $C_i(F, d)$ such that for all $n \geq n_0$, for every $x > \beta\mathcal{H}^{d-1}(nA)$, we have*

$$\mathbb{P}\left[|E(nA, h(n))| \geq x\right] \,\leq\, C_1 e^{-C_2 x}\,.$$

Zhang controls in fact the cardinality of a minimal cutset that cuts any large box from infinity, and uses this control to obtain Theorem 11. This result allows him to fulfill the gap between the two regimes $F(\{0\}) < p_0$ and $F(\{0\}) \geq 1 - p_c(d)$ previously studied, in any dimension $d \geq 2$.

Theorem 12 (Maximal flow through straight cylinders - second result)**.** *Let* $A = \prod_{i=1}^{d-1} [k_i, l_i] \times \{0\}$ *with* $k_i \leq 0 < l_i \in \mathbb{Z}$. *Let* $h : \mathbb{N} \to \mathbb{R}^+$ *such that* $\lim_{n\to\infty} h(n) = +\infty$. *Suppose that there exists* $\delta > 0$ *satisfying*

$$\log h(n) \leq n^{1-\delta} .$$

Suppose that F *admits an exponential moment:*

$$\exists \lambda > 0 , \quad \int_{\mathbb{R}^+} e^{\lambda x} \, dF(x) < \infty .$$

Then

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}_0) \quad a.s. \text{ and in } L^1 .$$

Moreover $\nu(\vec{v}_0) > 0$ *if and only if* $F(\{0\}) < 1 - p_c(d)$.

Zhang's proof of Theorem 12 relies on the same ideas as Kesten's proof of Theorem 9. The subadditivity for maximal flow through straight cylinders is recovered by imposing some fixed boundary conditions to the cutsets he considers, and by gluing together cutsets in adjacent cylinders with symmetric boundary conditions. Zhang adds mainly two new ingredients. First he uses a concentration inequality to reduce the problem to prove the convergence of $\mathbb{E}[\phi(nA, h(n))]/\mathcal{H}^{d-1}(nA)$, which simplifies its work. Secondly he uses Theorem 11 to control the cardinality of a cutset, and thus the number of possible boundary conditions for cutsets in a cylinder, under the hypothesis $F(\{0\}) < 1 - p_c(d)$ instead of the (technical but not relevant) condition $F(\{0\}) < p_0$ that appeared in Kesten's Theorem 9.

The major contribution of [Zha07] is thus Theorem 11. We try to give a glimpse on the arguments involved. Consider a minimal cutset E of minimal cardinality that cuts a box D from infinity. Fix a $\eta > 0$ very small. If $F((0, \eta])$ is small enough, we can control the number of edges $e \in E$ such that $t(e) \in (0, \eta]$. The number of edges $e \in E$ such that $t(e) > \eta$ is bounded by $T(E)/\eta$, thus can be controlled too. The hard part is to bound the number of edges of capacity equal to 0. It reduces to the study of properties of the percolation $(\mathbb{1}_{t(e)>0}, e \in \mathbb{E}^d)$. Up to flipping to 0 the capacities of all the edges of E, it is the case that the connected component of open edges connected to the box D is finite, thus its boundary is a cutset itself between D and infinity. Zhang regularizes this boundary, which is too tangled, by a rescaling argument. Then, using percolation estimates, he controls the cardinality of this regularized boundary, and obtains this way an upper bound on the cardinality of E which was supposed to be minimal.

Other directions

We want to mention two others works on maximal flows or minimal cutsets in first passage percolation. Boivin [Boi90] extends the study of minimal cutsets to the case of ergodic and stationary capacities instead of i.i.d. capacities. In dimension $d = 3$, he proves the convergence of the rescaled minimal capacity of cutsets that are not constrained to stay inside a cylinder but must have a boundary equals to a deterministic curve $\mathcal{C}$ included in a plane. Moreover, he gives some necessary and sufficient moment conditions on the distribution of the capacities for this convergence to hold uniformly in all directions, moment conditions that depend on the regularity of the curve $\mathcal{C}$.

Garet [Gar06] studies the maximal flow in dimension $d = 2$ from a convex bounded subset A of $\mathbb{R}^2$ to infinity. He proves that the asymptotic rescaled maximal flow, when the dimensions of A go to infinity, behaves like

$$\mathrm{capa}^{\mathrm{cont}}(A) \,=\, \int_{\partial^* A} \nu(\vec{v}_A(x)) d\mathcal{H}^2(x)\,,$$

where $\vec{v}_A(x)$ a the exterior unit vector normal to A at x and $\partial^* A$ is the subset of A where such a normal vector exists. He also obtains large deviation estimates for this maximal flow. Garet's proof relies heavily on the fact that in dimension $d = 2$, minimal cutsets are geodesics in the dual graph. We do not get into the details of this proof. However, we want to say a few words about the limit $\mathrm{capa}^{\mathrm{cont}}(A)$ appearing here. We recall that $\nu(\vec{v})$ can be interpreted as the asymptotic rescaled minimal capacity of a unit surface globally normal to $\vec{v}$. The limit $\mathrm{capa}^{\mathrm{cont}}(A)$ can thus be interpreted as the continuous capacity of the surface $\partial^*(A)$. Moreover, Garet needs to prove that

$$\mathrm{capa}^{\mathrm{cont}}(A) \,=\, \inf\{\mathrm{capa}^{\mathrm{cont}}(A') \,:\, A \subset A'\,,\, A' \text{ is polygonal}\}\,.$$

The limit of the rescaled maximal flows is thus the solution of a variational problem. These ideas will be used again in this dissertation, see Section 2.4 and Chapter 4.

We finish here our quick scan of the results concerning maximal flows in first passage percolation. We are not exhaustive, one could cite for instance previous related works of Aizenman, Chayes, Chayes, Fröhlich and Russo [ACC+83], but a lot less is missing than concerning the random distance in first passage percolation since maximal flows have been a lot less studied.

1.3 Related models

This dissertation, as indicated by its title, mainly deals with first passage percolation. However, two other models or objects will appear in Section 5.3. Indeed, the study of first passage percolation is interesting in itself, but it is also a laboratory in which we develop techniques that can be used to understand other models.

1.3.1 Cheeger constant

For a finite graph $G = (V, E)$, the isoperimetric constant φ_G is defined as

$$\varphi_G = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V\,,\, 0 < |A| \leq \frac{|V|}{2} \right\}\,,$$

where ∂A is the edge boundary of A, $\partial A = \{e = (x, y) \in E : x \in A, y \notin A\}$, and $|B|$ denotes as previously the cardinality of the finite set B.

For a dimension $d \geq 2$, let $p \in (p_c(d), 1]$ and denote by $\mathcal{C}_p$ the a.s. unique infinite cluster of the i.i.d. bound Bernoulli percolation of parameter p on $(\mathbb{Z}^d, \mathbb{E}^d)$. We consider the isoperimetric constant $\varphi_n(p)$ of $\mathcal{C}_p \cap [-n, n]^d$, the intersection of this infinite cluster with the box $[-n, n]^d$:

$$\varphi_n(p) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset \mathcal{C}_p \cap [-n, n]^d, 0 < |A| \leq \frac{|\mathcal{C}_p \cap [-n, n]^d|}{2} \right\}\,.$$

In several papers (e.g. [BM03, MR04, Pet08, BBHK08]), it was shown that there exist constants $c, C > 0$ such that $c < n\varphi_n(p) < C$, with probability tending rapidly to 1. This led Benjamini

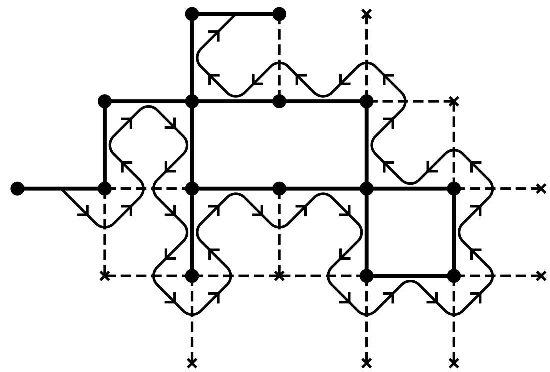

Figure 1.6: A right most path.

to conjecture the existence of $\lim_{n\to+\infty} n\varphi_n(p)$. Rosenthal and Procaccia [PR11] proved that the variance of $n\varphi_n(p)$ is smaller than Cn^{2-d}, which implies that $n\varphi_n(p)$ is concentrated around its mean for $d \geq 3$. Biskup, Louidor, Procaccia and Rosenthal [BLPR12] proved the existence of $\lim_{n\to+\infty} n\varphi_n(p)$ for $d = 2$. This constant is called the Cheeger constant. In addition, a shape theorem was obtained: any set yielding the isoperimetric constant converges in the Hausdorff metric to the normalized Wulff shape $\widehat{W}_p$, with respect to a specific norm given in an implicit form, see Theorem 15 below. For additional background and a wider introduction on Wulff construction in this context, the reader is referred to [BLPR12].

In dimension $d = 2$, the Cheeger constant can also be represented as the solution of a continuous isoperimetric problem with respect to some norm. To define this norm, we first require some definitions. For a path $r = (v_0, e_1, \dots, e_n, v_n)$, and $i \in \{2, \dots, n-1\}$, an edge $e = (v_i, z)$ is said to be a right-boundary edge if z is a neighbor of v_i between v_{i+1} and v_{i-1} in the clockwise direction. The right boundary $\partial^+ r$ of r is the set of its right-boundary edges. A path is called right-most if it uses every edge at most once in every orientation and it doesn't contain right-boundary edges. See Figure 1.6; the solid lines represent the path, dashed lines represent the right-boundary edges, and the curly line is a path in the medial graph which shows the orientation (see [BLPR12] for a thorough discussion). For $x, y \in \mathbb{Z}^2$, let $\mathcal{R}(x,y)$ be the set of right-most paths from x to y. For a path $r \in \mathcal{R}(x,y)$, define $\mathbf{b}_p(r) = |\{e \in \partial^+ r : e \text{ is } p\text{-open}\}|$. For $x, y \in \mathcal{C}_p$ we define the right boundary distance, $b_p(x,y) = \inf\{\mathbf{b}_p(r) : r \in \mathcal{R}(x,y),\, r \text{ is } p\text{-open}\}$. For any $x \in \mathbb{R}^2$, define $\widetilde{x}^{\mathcal{C}_p}$ as the vertex of $\mathcal{C}_p$ which minimizes $\|x - \widetilde{x}^{\mathcal{C}_p}\|_1$, with a deterministic rule to break ties. The next result, proved in [BLPR12], yields uniform convergence of the right boundary distance to a norm on $\mathbb{R}^2$.

Theorem 13 (Definition of the norm)**.** *Let $d = 2$. For any $p > p_c(2)$, there exists a norm β_p on $\mathbb{R}^2$ such that for any $x \in \mathbb{R}^2$,*

$$\beta_p(x) = \lim_{n\to\infty} \frac{b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p})}{n} \quad \textit{a.s. and in } L^1 .$$

Moreover, the convergence is uniform on $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$.

This convergence is proved by a classical ergodic subadditive theorem. The connection between the Cheeger constant and the norm β_p goes through a continuous isoperimetric problem. For a continuous curve $\lambda : [0,1] \to \mathbb{R}^2$, and a norm ρ, let the ρ-length of λ be

$$\mathrm{len}_\rho(\lambda) = \sup_{N\geq 1} \sup_{0\leq t_0 < \ldots < t_N \leq 1} \sum_{i=1}^{N} \rho(\lambda(t_i) - \lambda(t_{i-1})).$$

A curve λ is said to be rectifiable if $\mathrm{len}_\rho(\lambda) < \infty$ for any norm ρ. A curve λ is called a Jordan curve if λ is rectifiable, $\lambda(0) = \lambda(1)$ and λ is injective on $[0,1)$. For any Jordan curve λ, we can define its interior $\mathrm{int}(\lambda)$ as the unique finite component of $\mathbb{R}^2 \smallsetminus \lambda([0,1])$. Denote by $\mathcal{L}^2$ the Lebesgue measure on $\mathbb{R}^2$, and by θ_p the density of $\mathcal{C}_p$, *i.e.*, $\theta_p = \mathbb{P}(0 \in \mathcal{C}_p)$. As proved in [BLPR12], the Cheeger constant can be represented as the solution of the following continuous isoperimetric problem.

Theorem 14 (Continuous isoperimetric problem)**.** *Let $d = 2$. For every $p > p_c(2)$,*

$$\lim_{n\to+\infty} n\varphi_n(p) = (\sqrt{2}\,\theta_p)^{-1} \inf\{\mathrm{len}_{\beta_p}(\lambda) : \lambda \textit{ is a Jordan curve}, \mathcal{L}^2(\mathrm{int}(\lambda)) = 1\}.$$

Moreover Biskup, Louidor, Procaccia and Rosenthal [BLPR12] obtain a limiting shape for the sets that achieve the minimum in the definition of $\varphi_n(p)$. This limiting shape is given by the Wulff construction [Wul01]. Denote by

$$W_p = \bigcap_{\hat{n}\in\mathbb{S}^1} \{x \in \mathbb{R}^2 : \hat{n}\cdot x \leq \beta_p(\hat{n})\} \text{ and } \widehat{W}_p = \frac{W_p}{\sqrt{\mathcal{L}^2(W_p)}}, \tag{1.6}$$

where $\cdot$ denotes the Euclidean inner product. The set $\widehat{W}_p$ is a minimizer for the isoperimetric problem associated with the norm β_p, and it gives the asymptotic shape of the minimizer sets in the definition of $\varphi_n(p)$. Denote by $\mathcal{U}_n(p)$ be the set of minimizers of $\varphi_n(p)$, and by d_H the Hausdorff distance between non-empty compact sets; then it is stated in [BLPR12] that the following holds.

Theorem 15 (Shape theorem for the minimizers)**.** *Let $d = 2$. For every $p > p_c(2)$,*

$$\lim_{n\to\infty} \max_{U\in\mathcal{U}_n(p)} \inf_{\xi\in\mathbb{R}^2} d_{\mathrm{H}}\left(\frac{U}{n}\,,\ \xi + \sqrt{2}\,\widehat{W}_p\right) \;=\; 0 \quad a.s.\,.$$

The norm β_p, defined via the subadditive ergodic theorem as the rescaled limit of the minimum of a certain weight among paths, is very similar to the norm μ_F associated with the distribution F of passage times in classical first passage percolation. For this reason, some methods used to prove properties of μ_F can be used to get the same properties for β_p, and thus to recover some properties of the Cheeger constant $\lim_{n\to+\infty} n\varphi_n(p)$ and the associated Wulff shape $\widehat{W}_p$.

1.3.2 Contact process

The contact process is a famous interacting particle system modeling the spread of an infection on the sites of $\mathbb{Z}^d$. The evolution in time depends on a fixed parameter $\lambda \in (0,+\infty)$ and is as follows: at each moment, an infected site becomes healthy at rate 1 while a healthy site becomes infected at a rate equal to λ times the number of its infected neighbors. We present the alternative graphical construction of the contact process proposed by Harris [Har78]. This construction is exposed in all details in [Har78]; we just give here an informal description. Independently, we associate with each edge $e \in \mathbb{E}^d$ a Poisson point process ω_e^λ on $\mathbb{R}^+$ with intensity $\lambda > 0$, and we associate with each vertex $z \in \mathbb{Z}^d$ a Poisson point process ω_z on $\mathbb{R}^+$ with intensity 1. Above each site $z \in \mathbb{Z}^d$, we draw

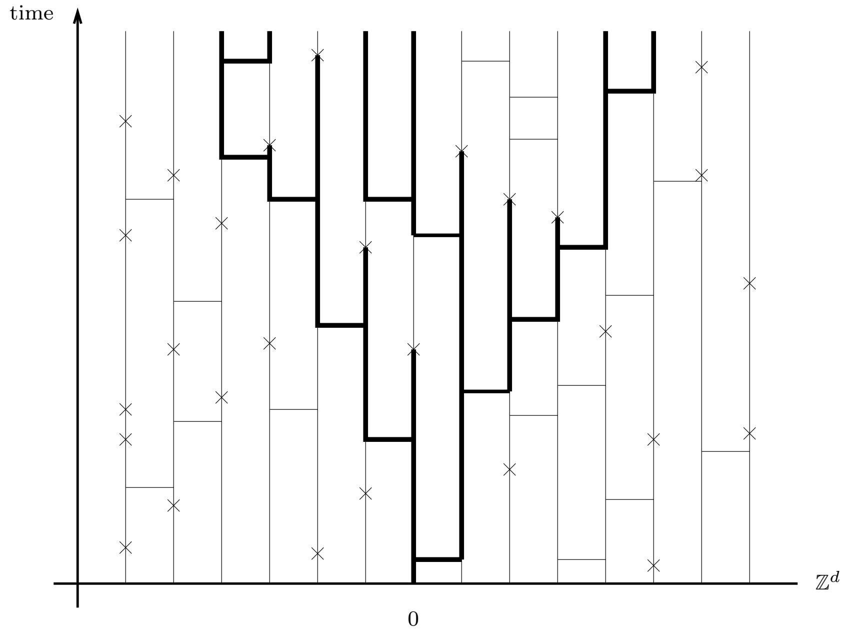

Figure 1.7: Graphical construction of the contact process (the infected region is in bold).

a time line $\mathbb{R}^+$, and we put a cross at the times given by ω_z, corresponding to potential recoveries at site z. Above each edge $e \in \mathbb{E}^d$, we draw at the times given by ω_e^λ an horizontal segment between the extremities of the edge, corresponding to a potential infection through the edge e (see Figure 1.7).

An open path is a connected oriented path which moves along the time line in the increasing time direction without passing a cross symbol, and along the horizontal segments corresponding to potential infections. In this description, the evolution of the contact process is a percolation process, oriented in time but not in space. For $x, y \in \mathbb{Z}^d$ and $t \geq 0$, we say that $y \in \xi_t^{\lambda,x}$ if and only if there exists an open path from $(x, 0)$ to (y, t), then we define:

$$\forall A \subset \mathbb{Z}^d\,, \quad \xi_t^{\lambda,A} \quad = \quad \bigcup_{x\in A} \xi_t^{\lambda,x}. \tag{1.7}$$

Harris proved that the process $(\xi_t^{\lambda,A})_{t\geq 0}$ is the contact process with infection rate λ, starting from initial configuration A.

Thanks to this graphical construction, it is easy to obtain a coupling between contact processes of different parameters λ and λ' in a common interval $(0, \lambda_{\max}]$. Indeed, equip each edge e with a family $(U_i^e)_{i\geq 1}$ of i.i.d. random variables of uniform distribution on $[0, \lambda_{\max}]$. Given $\omega_e^{\lambda_{\max}}$, the Poisson point process of parameter $\lambda_{\max}$ associated with e, construct the process ω_e^λ (resp. $\omega_e^{\lambda'}$) by keeping the i-th point of $\omega_e^{\lambda_{\max}}$ if and only if $U_i^e \leq \lambda/\lambda_{\max}$ (resp. $U_i^e \leq \lambda'/\lambda_{\max}$). Finally use the same Poisson point processes ω_z of parameter 1 associated with the vertices $z \in \mathbb{Z}^d$ and apply the graphical construction of Harris to obtain at the same time the contact processes of parameters λ and λ'. By construction, we obtain that

$$(A \subset B\,, \quad \lambda' \leq \lambda \leq \lambda_{\max}) \;\Rightarrow\; (\forall t \geq 0\,, \quad \xi_t^{\lambda',A} \subset \xi_t^{\lambda,B}).$$

For a set $A \subset \mathbb{Z}^d$, we define the life time τ_λ^A of the process starting from A by

$$\tau_\lambda^A = \inf\{t \geq 0 : \ \xi_t^{\lambda,A} = \varnothing\}.$$

With the graphical construction in mind, it is clear that $\{\tau_\lambda^{\{0\}} = +\infty\}$ if and only if there is an infinite path starting from $(0,0)$ in the graph that is built from potential infections that are present at rate λ. Then, it will be often more appealing to write $\{0 \overset{\lambda}{\leftrightarrow} \infty\}$ instead of $\{\tau_\lambda^{\{0\}} = +\infty\}$. The critical parameter for the contact process in $\mathbb{Z}^d$ is defined by

$$\begin{aligned}\lambda_c(\mathbb{Z}^d) &= \inf\{\lambda > 0 : \ \mathbb{P}(\tau_\lambda^{\{0\}} = +\infty) > 0\} \\ &= \inf\{\lambda > 0 : \ \mathbb{P}(0 \overset{\lambda}{\leftrightarrow} \infty) > 0\} \in (0, +\infty).\end{aligned}$$

The fact that $\lambda_c(\mathbb{Z}^d) < +\infty$ is due to Harris [Har74]. By definition, for $\lambda > \lambda_c$, the infection starting from the origin infinitely expands with positive probability. Define, for $\lambda > \lambda_c(\mathbb{Z}^d)$, the following conditional probability

$$\overline{\mathbb{P}}_\lambda(\cdot) = \mathbb{P}(\cdot | \tau_\lambda^{\{0\}} = +\infty) = \frac{\mathbb{P}(\ \cdot \ \cap \{0 \overset{\lambda}{\leftrightarrow} \infty\})}{\mathbb{P}(0 \overset{\lambda}{\leftrightarrow} \infty)}.$$

For $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we also define the first infection time $t_\lambda^A(x)$ of site x from set A by

$$t_\lambda^A(x) = \inf\{t \geq 0 : \ x \in \xi_t^{\lambda,A}\}.$$

It follows from Bezuidenhout and Grimmett [BG90] (see also Durrett [Dur91]) that $\overline{\mathbb{P}}_\lambda(t_\lambda^A(x) < +\infty) = 1$ as soon as $A \neq \varnothing$. The set of points infected before time t is then

$$H_t^\lambda = \{x \in \mathbb{Z}^d : \ t_\lambda^{\{0\}}(x) \leq t\}$$

and we define a fattened version of it by

$$\tilde{H}_t^\lambda = \{x + u \ : \ x \in H_t^\lambda \, , \, u \in [-1/2, 1/2]^d\}.$$

Combining the works of Durrett and Griffeath [DG82], Bezuidenhout and Grimmett [BG90] and Durrett [Dur91], it is proved that when the contact process on $\mathbb{Z}^d$ starting from the origin survives, the set of sites occupied before time t satisfies an asymptotic shape theorem, as in first passage percolation. The shape theorem can be stated as follows.

Theorem 16 (Definition of the time constant and shape theorem for the contact process)**.** *Suppose that $\lambda > \lambda_c(\mathbb{Z}^d)$, then there exists a norm μ_λ on $\mathbb{R}^d$ such that for every $x \in \mathbb{Z}^d$,*

$$\lim_{n\to\infty} \frac{t_\lambda^{\{0\}}(nx)}{n} \ = \ \mu_\lambda(x) \quad \overline{\mathbb{P}}_\lambda\text{-a.s. and in } L^1(\overline{\mathbb{P}}_\lambda) \, .$$

Moreover, for all $\varepsilon > 0$, a.s., there exists $t_0 \in \mathbb{R}^+$ such that

$$\forall t \geq t_0 \, , \quad (1-\epsilon)\mathcal{B}\mu_\lambda \subset \frac{\tilde{H}_t^\lambda}{t} \subset (1+\epsilon)\mathcal{B}\mu_\lambda$$

where $\mathcal{B}\mu_\lambda$ is the unit ball for μ_λ.

The growth of the contact process is thus asymptotically linear in time, and governed by the shape $\mathcal{B}\mu_\lambda$. More precisely, Durrett and Griffeath [DG82] proved the shape theorem for the contact process for large values of λ, using estimates that essential imply that the growth of the contact process is of linear order. Later, Bezuidenhout and Grimmett [BG90] showed that a supercritical contact process conditioned to survive growths at least linearly. In their proof, they use a rescaling argument and a stochastic comparison, at the level of the mesoscopic blocks, with a two-dimensional supercritical oriented percolation. They also indicated how this could be use to obtain a shape theorem for the entire supercritical regime by recovering the estimates proved by Durrett and Griffeath [DG82] in the case of large values of λ. This last step has been done by Durrett [Dur91]. The proof of the shape theorem for contact process is thus divided in two steps : first prove that the growth of the contact process is of linear order as soon as survival is possible as in [BG90], then deduce from it a shape theorem as in [DG82, Dur91].

Of course, the study of the contact process is linked to the study of the random distance in classical first passage percolation by the fact that these two models are random growth models, for which a shape theorem holds. For this reason, we have hope to transport properties of the random distance in first passage percolation to the contact process. However, the study of contact process is more difficult. First passage percolation can be seen as a permanent model, in the sense that if a vertex $z \in \mathbb{Z}^d$ is wet at time s, it remains wet at any time $t \geq s$. The set B_t of wet vertices is thus nondecreasing and extinction is impossible. Contact process is a non-permanent model: extinction is possible, the set of infected particules is not monotonic. Following ideas of classical subadditive theory, if we want to prove that the hitting times $t_\lambda^{\{0\}}(x)$ are such that $t_\lambda^{\{0\}}(nx)/n$ converges, Kingman's theory requires that the family $(t_\lambda^{\{0\}}(x))$ has subadditivity, stationarity and integrability properties. Since extinction is possible, the hitting times may be infinite, thus integrability does not hold. Moreover, conditioning on the survival of the process can break independence, stationarity and even subadditivity properties. Instead of subadditivity theory, Durrett and Griffeath [DG82] and Durrett [Dur91] rely on the theory of superconvolutive distributions to prove the shape theorem for the contact process.

Since then, a lot more has been proved concerning the contact process. See for instance Liggett's book [Lig99] for a review on this model. We could not present here all the known properties of contact process, thus we just focus on recent advances that will be useful in this dissertation. Garet and Marchand [GM12] extended the shape theorem to the case of contact process in a random environment, *i.e.*, when the fixed parameter e is replaced by a stationary and ergodic family $(\lambda_e)_{e\in\mathbb{E}^d}$ of random variables, λ_e giving the infection rate between the extremities of the edge e, under the assumption that $(\lambda_e)_{e\in\mathbb{E}^d}$ takes values in $[\lambda_{\min}, \lambda_{\max}]^{\mathbb{E}^d}$ where $\lambda_c(d) < \lambda_{\min} \leq \lambda_{\max} < +\infty$. The key idea of their work is the use of what they call the essential hitting time $\sigma_\lambda^{\{0\}}(x)$ for $x \in \mathbb{Z}^d$ instead of the hitting time $t_\lambda^{\{0\}}(x)$ - we present here these essential hitting times in the case of a deterministic environment λ. The time $\sigma_\lambda^{\{0\}}(x)$ can bee seen as a regeneration time, at which the site x is infected and its infection survives. Rather than giving a precise definition, we prefer to propose an informal description. Consider a point $x \in \mathbb{Z}^d$ and a contact process starting at 0. First wait for the first time t at which the vertex x is infected. If the infection of origin x at time t survives, then $\sigma_\lambda^{\{0\}}(x) = t$. If it does not, wait until this infection dies, and then wait again for the time t' at which the vertex x is infected again. If the infection of origin x at time t' survives, then $\sigma_\lambda^{\{0\}}(x) = t'$, otherwise repeat the process again until you find a time with such properties. It is the case that $\sigma_\lambda^{\{0\}}(x) < \infty$ $\overline{\mathbb{P}}_\lambda$-a.s.. The essential hitting times have good moment, stationarity and almost subadditive properties that the hitting times lack. Garet and Marchand [GM12] proved a general almost subadditive ergodic theorem that they can apply to essential hitting times, and this way they define the norm μ_λ as the limit of the rescaled essential hitting times:

$$\forall x \in \mathbb{Z}^d\,, \quad \lim_{n\to\infty} \frac{\sigma_\lambda^{\{0\}}(nx)}{n} \,=\, \mu_\lambda(x) \quad \overline{\mathbb{P}}_\lambda\text{-a.s. and in } L^1(\overline{\mathbb{P}}_\lambda)\,.$$

Then they compare the essential hitting time $\sigma_\lambda^{\{0\}}(x)$ to the hitting time $t_\lambda^{\{0\}}$ to prove the convergence of the rescaled hitting times towards the same limit. For more informations about contact process in random environment, we refer to [GM12, GM14].

1.4 Structure of the dissertation

Chapters 2, 3 and 4 are devoted to the study of maximal flows, maximal streams and minimal cutsets in first passage percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$. In Chapter 2 we present results we obtained in [RT10a, RT10b, RT13, Thé08] concerning the convergence of rescaled maximal flows through cylinders and the lower large deviations of these flows, whereas in Chapter 3 we present results we obtained in [RT13, Thé07, Thé14] concerning their upper large deviations. These two chapters are complementary and the results presented therein are intertwined. Chapter 4 deals with the asymptotic behavior of maximal flows through very general domain of $\mathbb{R}^d$, and the corresponding maximal streams and minimal cutsets. A result of convergence of these objects is stated, together with upper and lower large deviations for the maximal flows, as proved in [CT11a, CT11b, CT11c, CT14a]. This chapter relies heavily on the two previous ones. Chapter 5 is devoted to the study of the random distance defined in the classical interpretation of first passage percolation, and two related models. We present in this chapter a definition of the time constant and state a weak shape theorem for first passage percolation on the infinite cluster of a supercritical Bernoulli percolation without any moment condition, and we state the continuity of this time constant with regard to the parameter of the underlying supercritical percolation and the law of the passage times. We also present two other results of continuity that are inspired by this first one, namely the continuity of the Cheeger constant in dimension $d = 2$ with regard to the parameter of the underlying supercritical percolation, and the continuity of the time constant of the contact process in dimension $d \geq 2$. These results were proved in [CT14b, GMPT15, GMT15]. In Chapter 6 we gather some open questions we are interested in.

The articles [RT10a, RT10b, RT13] are joint works with Raphaël Rossignol. The articles [CT11a, CT11b, CT11c, CT14a, CT14b] are joint works with Raphaël Cerf. The article [GMT15] is a joint work with Olivier Garet and Régine Marchand. The article [GMPT15] is a joint work with Olivier Garet, Régine Marchand and Eviatar B. Procaccia. I am the single author of the articles [Thé08, Thé07, Thé14].

Chapter 2

Maximal flow through cylinders: convergence and lower large deviations

We gather in this chapter results we obtained concerning the convergence of rescaled maximal flows through cylinders, and lower large deviations for these maximal flows. As we will see, these two questions are closely related.

2.1 A first attempt

In this section we present the result proved in [Thé08]. Consider a dimension $d \geq 2$, the graph $(\mathbb{Z}^d, \mathbb{E}^d)$, and associate with the edges of the graph a family of i.i.d. capacities $(t(e))_{e\in\mathbb{E}^d}$ of common distribution F. Let $\mathrm{cyl}(nA, h(n))$ be a straight cylinder in $\mathbb{R}^d$, *i.e.*, the hyperrectangle A is of the form $A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{c\}$ with $k_i < l_i \in \mathbb{Z}$ and $c \in \mathbb{Z}$ while the height function $h : \mathbb{N} \to \mathbb{R}^+$ satisfies $\lim_{n\to\infty} h(n) = +\infty$. We study the maximal flow $\phi(nA, h(n))$ from the top to the bottom of $\mathrm{cyl}(nA, h(n))$. Under some additional assumptions, Kesten and Zhang proved that

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}_0) \quad \text{a.s. and in } L^1$$

(see Theorems 9 and 12). This implies that for all $\varepsilon > 0$, the probability $\mathbb{P}[\phi(nA, h(n)) \leq (\nu(\vec{v}_0) - \varepsilon)\mathcal{H}^{d-1}(nA)]$ goes to 0 when n goes to infinity, and we want to know at what speed. If for instance $F(\{0\}) > 0$, the probability that all the edges that have an endpoint at the basis of the cylinder have null capacity is of order $F(\{0\})^{\mathcal{H}^{d-1}(nA)}$, thus the probability that $\phi(nA, h(n))$ is null decays to 0 at a speed which is at most exponential in n^{d-1}. Our goal is to prove that this is exactly the right order of decay. However, the approach we follow in [Thé08] does not allow us to prove such a result. We obtain only the following partial result.

Theorem 17. *Suppose that $F(\{0\}) < 1 - p_c(d)$. Let $A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{c\}$ with $k_i < l_i \in \mathbb{Z}$ and $c \in \mathbb{Z}$. Then there exists a constant $\varepsilon(F, d, A) > 0$ and a constant $C(d) > 0$ such that for all height function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$, we have*

$$\forall \varepsilon \leq \varepsilon_0 , \quad \liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\phi(nA, h(n)) \leq \varepsilon \mathcal{H}^{d-1}(nA)\right] \geq C > 0 .$$

In fact [Thé08] deals only with the case $A = [0,1]^{d-1} \times \{0\}$, but the proof can be readily transposed to any straight cylinder.

Even if Theorem 17 is not as good as we want, it supports the conjecture that the lower large deviations of $\phi(nA, h(n))$ are of surface order, *i.e.*, that the probability that this maximal flow is abnormally small decays exponentially fast with n^{d-1}. Moreover, it states that the constant $\nu(\vec{v}_0)$, when it is known to be the limit of the rescaled flows $\phi(nA, h(n))$, is strictly positive when $F(\{0\}) < 1 - p_c(d)$. This result, obtained before Zhang's Theorem 12, was new, since Kesten proved in [Kes87] such a positivity only when $F(\{0\}) < p_0$ for some positive but not relevant p_0. Theorem 17 is in fact a generalization of a result obtained by Chayes and Chayes [CC86] when F is a Bernoulli distribution and under a stronger assumption on the height function h.

The proof of Theorem 17 is based on a coarse graining argument in the spirit of Pisztora [Pis96]. Choose a positive $\eta > 0$ small enough so that $F([0, \eta)) < 1 - p_c(d)$, thus the percolation $(\mathbb{1}_{t(e)>\eta}, e \in \mathbb{E}^d)$ is supercritical. Consider boxes inside the cylinder at a mesoscopic scale. Declare such a box to be good if the following good event occurs: there exists inside the box a cluster of edges that are open for the percolation $(\mathbb{1}_{t(e)>\eta}, e \in \mathbb{E}^d)$ such that this cluster crosses the box (*i.e.*, it intersects its $2d$ sides) and there is no other open cluster inside the box of diameter bigger than the third of the sidelength of the box. Since the percolation we consider is supercritical, it is well known (see [Cer06] for instance) that this event is typical, *i.e.*, the probability that a box is good goes to one when the dimension of the box goes to infinity. Moreover, considering adjacent boxes that overlap a little, the existence of a path of good boxes at the mesoscopic scale from the top to the bottom of the cylinder guarantees at the microscopic scale the existence of a path of open edges, *i.e.*, of edges of capacities bigger than η. The problem boils down to control the number of disjoint open path of good boxes between the top and the bottom of the box, *i.e.*, to bound the probability that the maximal flow from top to bottom of a cylinder is small when the distribution of the capacities is a Bernoulli distribution of parameter p arbitrarily close to 1.

2.2 Subadditive flows

In this section and in Section 2.3, we present the results we proved together with Rapahël Rossignol in [RT10b]. Let $\vec{v} \in \mathbb{S}^{d-1}$ be a unit vector, A be a non degenerate hyperrectangle in $\mathbb{R}^d$ normal to $\vec{v}$ and $h : \mathbb{N} \to \mathbb{R}^+$ be an height function satisfying $\lim_{n\to\infty} h(n) = +\infty$. In this section we study the maximal flows $\tau(nA, h(n))$, using their quasi-subadditive properties.

The first result we obtain in [RT10b] is the convergence of the mean of these rescaled flows.

Proposition 18. *Suppose that F admits a finite mean*

$$\int_{\mathbb{R}^+} x \, dF(x) \; < \; \infty \, .$$

For all $\vec{v} \in \mathbb{S}^{d-1}$, there exists a constant $\nu(\vec{v})$ (depending also on F and d) such that for every non degenerate hyperrectangle A in $\mathbb{R}^d$ normal to $\vec{v}$ and every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, we have

$$\lim_{n\to\infty} \frac{\mathbb{E}[\tau(nA, h(n))]}{\mathcal{H}^{d-1}(nA)} \; = \; \nu(\vec{v}) \, .$$

The proof of this proposition is not hard, and based on the classical proof of convergence of subadditive sequences. Consider two integers $n \leq N$, and pave the hypersquare NA with translates

$t_i(nA)$ of nA, for $i \in \{1, \dots, N^{d-1}/n^{d-1}\}$ roughly speaking. Remember that the maximal flow $\tau(nA, h(n))$ is equal by the max-flow min-cut theorem to the minimal capacity of a cutset whose boundary is pinned along $\partial(nA)$. Two such cutsets in adjacent cylinders can be glued together up to adding a small number of edges, thus the following almost subadditive property holds:

$$\tau(NA, h(N)) \le \sum_{i=1}^{N^{d-1}/n^{d-1}} \tau(t_i(nA), h(n)) + \text{error term}\,. \tag{2.1}$$

However, when $\vec{v}$ is not rational, we cannot hope that the translations t_i have vectors with integer coordinates, thus $\tau(t_i(nA), h(n))$ does not have the same distribution as $\tau(nA, h(n))$. We can fix this problem by moving again a little bit each cylinder $\mathrm{cyl}(t_i(nA), h(n))$ and improving a little bit the error term accordingly. From (2.1), we deduce straightforwardly Proposition 18 since the error term is easy to control in expectation. From (2.1) and Proposition 18, we also deduce immediately that

$$\limsup_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \le \nu(\vec{v})\,. \tag{2.2}$$

This follows the first steps of the classical proofs of the subadditive ergodic theorem, see for instance Durrett [Dur96]. To obtain the convergence of $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ towards $\nu(\vec{v})$, it remains to control the deviations of these rescaled flows below their expectations.

At this point, we appeal to concentration estimates to bound these deviations. We use Zhang's Theorem 11 to state that with high probability, if $F(\{0\}) < 1 - p_c(d)$ (*i.e.*, if $\nu(\vec{v}) > 0$) the number of edges in a minimal cutset for $\tau(nA, h(n))$ has cardinality of order n^{d-1}. Thanks to a concentration inequality stated by Boucheron, Lugosi and Massart [BLM03], we obtain a control on the lower large deviations of $\tau(nA, h(n))$ below its mean, thus below $\mathcal{H}^{d-1}(nA)\nu(\vec{v})$.

Theorem 19. *Suppose that $F(\{0\}) < 1 - p_c(d)$ and that F has finite mean*

$$\int_{\mathbb{R}^+} x\, dF(x) < \infty\,.$$

Then for every $\varepsilon > 0$ there exists a positive constant $C(d, F, \varepsilon)$ such that for every $\vec{v} \in \mathbb{S}^{d-1}$, every non-degenerate hyperrectangle A normal to $\vec{v}$, for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, there exists a constant $\tilde{C}(d, F, A, h, \varepsilon)$ such that

$$\mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \le \nu(\vec{v}) - \varepsilon\right] \le \tilde{C} e^{-C\mathcal{H}^{d-1}(nA)}\,.$$

Combining Proposition 18, Equation (2.2) and Theorem 19, we obtain easily the convergence of the rescaled maximal flows $\tau(nA, h(n))$ towards $\nu(\vec{v})$.

Theorem 20. *Suppose that F admits a finite mean*

$$\int_{\mathbb{R}^+} x\, dF(x) < \infty\,.$$

Then for all $\vec{v} \in \mathbb{S}^{d-1}$, for every non degenerate hyperrectangle A in $\mathbb{R}^d$ normal to $\vec{v}$ and every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, we have

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}) \quad \text{in } L^1\,. \tag{2.3}$$

Moreover, if the origin of the graph belongs to A, or if

$$\int_{\mathbb{R}^+} x^{1+1/(d-1)}\, dF(x) \,<\, \infty\,,$$

then the convergence (2.3) holds a.s..

We go further into the study of the lower large deviations of $\tau(nA, h(n))$ by proving a corresponding large deviation principle. We define the function $\mathcal{I}_{\vec{v}}$ on $\mathbb{R}^+$ as follows:

$$\mathcal{I}_{\vec{v}}(x) \,=\, \lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\tau(nA, h(n)) \leq \left(x - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right].$$

The existence of $\mathcal{I}_{\vec{v}}$ (that depends only on $\vec{v}$ but not on A nor h) follows from Inequality 2.1. The term $1/\sqrt{n}$ that appears in the definition of $\mathcal{I}_{\vec{v}}$ exists to absorb the error term in the subadditivity of τ. We modify a little bit the function $\mathcal{I}_{\vec{v}}$ to obtain the rate function $\mathcal{J}_{\vec{v}}$ mainly by defining $\mathcal{J}_{\vec{v}}(x) = +\infty$ if $x > \nu(\vec{v})$, and we prove the following large deviation principle.

Theorem 21. *Suppose that $F(\{0\}) < 1 - p_c(d)$ and*

$$\forall \lambda \in \mathbb{R}\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) \,<\, \infty\,.$$

Then for all $\vec{v} \in \mathbb{S}^{d-1}$, for every non degenerate hyperrectangle A in $\mathbb{R}^d$ normal to $\vec{v}$ and every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, the sequence

$$\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)}\right)_{n\in\mathbb{N}}$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the good rate function $\mathcal{J}_{\vec{v}}$. Moreover, we know that $\mathcal{J}_{\vec{v}}$ is convex on $\mathbb{R}^+$, infinite on $[0, \delta\|v\|_1) \cup (\nu(\vec{v}), +\infty)$ where $\delta = \inf\{r \,:\, F([0, r]) > 0\}$, equal to 0 at $\nu(\vec{v})$, and if $\delta\|v\|_1 < \nu(\vec{v})$ we also know that $\mathcal{J}_{\vec{v}}$ is finite on $(\delta\|v\|_1, \nu(\vec{v})]$, continuous and strictly decreasing on $[\delta\|v\|_1, \nu(\vec{v})]$ and strictly positive on $[\delta\|v\|_1, \nu(\vec{v}))$.

The positivity of the rate function $\mathcal{J}_{\vec{v}}$ on $[\delta\|v\|_1, \nu(\vec{v}))$ is a consequence of Theorem 19. The fact that we define $\mathcal{J}_{\vec{v}}(x) = +\infty$ if $x > \nu(\vec{v})$ is a consequence of the study of the upper large deviations of $\tau(nA, h(n))$ that will be presented in Chapter 3: the probability that $\tau(nA, h(n))$ is abnormally big decays strictly faster than $e^{-Cn^{d-1}}$ for any constant C, at least under the assumption that the capacities admit exponential moments of all order (which is true for instance if the capacities are bounded).

2.3 Non subadditive flows: some particular cases in dimension $d \geq 2$

2.3.1 Thin cylinders

We go back to the study of the maximal flows $\phi(nA, h(n))$ between the top and the bottom of the cylinder $\mathrm{cyl}(nA, h(n))$. A first case which is easy to deal with is the case of cylinders that are thin, in the sense that

$$\lim_{n\to\infty} \frac{h(n)}{n} \,=\, 0\,.$$

In any cylinder, the following inequalities are true

$$\phi(nA, h(n)) \leq \tau(nA, h(n)) \leq \phi(nA, h(n)) + T(E_n) \tag{2.4}$$

where E_n is the set of edges that are at a fixed small distance ζ of the vertical faces of $\mathrm{cyl}(nA, h(n))$ (*i.e.*, the faces of $\mathrm{cyl}(nA, h(n))$ that are not normal to $\vec{v}$). Indeed, any cutset in the cylinder $\mathrm{cyl}(nA, h(n))$ that has fixed boundary conditions pinned at $\partial(nA)$ also cuts the top from the bottom of the cylinder. Conversely, if you consider a set of edges with free boundary condition that cuts the top from the bottom of a cylinder, it is enough to add to it all the edges near the vertical faces of the cylinder to recover a cutset pinned at $\partial(nA)$. In general, Inequality (2.4) does not allow us to study the asymptotic behavior of $\phi(nA, h(n))$, but it is the case when $\lim_{n\to\infty} h(n)/n = 0$. In this case, the cardinality $|E_n|$ of E_n, which is of order $n^{d-2}h(n)$, is negligible in comparison with n^{d-1}. We deduce then from the convergence of $\mathbb{E}[\tau(nA, h(n))]/\mathcal{H}^{d-1}(nA)$, Proposition 18 above, that the same result holds for $\mathbb{E}[\phi(nA, h(n))]/\mathcal{H}^{d-1}(nA)$.

To get more precise informations on the asymptotic behavior of $\phi(nA, h(n))$ in thin cylinders, we state the same concentration estimate as for τ, using the same method.

Theorem 22. *Suppose that $F(\{0\}) < 1 - p_c(d)$ and that F has finite mean*

$$\int_{\mathbb{R}^+} x \, dF(x) < \infty .$$

Then for every $\varepsilon > 0$ there exists a positive constant $C'(d, F, \varepsilon)$ such that for every $\vec{v} \in \mathbb{S}^{d-1}$, every non-degenerate hyperrectangle A normal to $\vec{v}$, for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} h(n)/n = 0$, there exists a constant $\tilde{C}'(d, F, A, h, \varepsilon)$ such that

$$\mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v}) - \varepsilon\right] \leq \tilde{C}' e^{-C'\mathcal{H}^{d-1}(nA)} .$$

Combining the comparaison (2.4) between $\phi(nA, h(n))$ and $\tau(nA, h(n))$, the convergence of $\tau(nA, h(n))$, Theorem 20, and the lower large deviations of $\phi(nA, h(n))$ in thin cylinders, Theorem 22, we obtain the convergence of $\phi(nA, h(n))$ in thin cylinders.

Theorem 23. *Suppose that F admits a finite mean*

$$\int_{\mathbb{R}^+} x \, dF(x) < \infty .$$

Then for all $\vec{v} \in \mathbb{S}^{d-1}$, for every non degenerate hyperrectangle A in $\mathbb{R}^d$ normal to $\vec{v}$ and every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} h(n)/n = 0$, we have

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}) \quad \text{in } L^1 . \tag{2.5}$$

Moreover, if the origin of the graph belongs to A, or if

$$\int_{\mathbb{R}^+} x^{1+1/(d-1)} \, dF(x) < \infty ,$$

then the convergence (2.5) holds a.s..

When the distribution F admits an exponential moment, the comparison (2.4) implies that $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ and $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$ are exponentially equivalent with regard to n^{d-1}, thus the large deviation principle proved for $\tau(nA,h(n))/\mathcal{H}^{d-1}(nA)$ can be transposed to $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$:

Theorem 24. *Suppose that $F(\{0\})<1-p_c(d)$ and*

$$\forall \lambda \in \mathbb{R}\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) \,<\, \infty\,.$$

Then for all $\vec{v}\in\mathbb{S}^{d-1}$, for every non degenerate hyperrectangle A in $\mathbb{R}^d$ normal to $\vec{v}$ and every function $h:\mathbb{N}\to\mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n)=+\infty$ and $\lim_{n\to\infty} h(n)/n=0$, the sequence

$$\left(\frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA))}\right)_{n\in\mathbb{N}}$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the same good rate function $\mathcal{J}_{\vec{v}}$ as in Theorem 21.

2.3.2 Straight cylinders

The case of straight cylinders has already been studied by Kesten [Kes87] and Zhang [Zha07]. Our contribution is to relax the moment assumption on F required to state the convergence of $\phi(nA,h(n))/\mathcal{H}^{d-1}(nA)$ towards $\nu(\vec{v}_0)$ in straight cylinders and to study the lower large deviations of these flows. As in the two previous sections, the results we obtain are of three kinds: convergence, lower large deviations estimates and large deviation principle. We first state these three results before saying a few words about their proofs. Concerning the convergence of the rescaled maximal flows, we state the following theorem.

Theorem 25. *Suppose that F admits a finite mean*

$$\int_{\mathbb{R}^+} x\, dF(x) \,<\, \infty\,.$$

Then for every non degenerate hyperrectangle $A=\prod_{i=1}^{d-1}[k_i,l_i]\times\{c\}$ ($k_i<l_i, c\in\mathbb{Z}$) in $\mathbb{R}^d$ normal to $\vec{v}_0=(0,\dots,0,1)$ and every function $h:\mathbb{N}\to\mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n)=+\infty$ and $\lim_{n\to\infty}\log h(n)/n^{d-1}=0$, we have

$$\lim_{n\to\infty} \frac{\phi(nA,h(n))}{\mathcal{H}^{d-1}(nA)} \,=\, \nu(\vec{v}) \quad \text{in } L^1\,. \tag{2.6}$$

Moreover, if the origin of the graph belongs to A, or if

$$\int_{\mathbb{R}^+} x^{1+1/(d-1)}\, dF(x) \,<\, \infty\,,$$

then the convergence (2.5) *holds a.s..*

Concerning the lower large deviations, we obtain this result.

Theorem 26. *Suppose that $F(\{0\}) < 1 - p_c(d)$ and that F has finite mean*

$$\int_{\mathbb{R}^+} x\, dF(x) \,<\, \infty\,.$$

Then for every $\varepsilon > 0$ there exists a positive constant $C''(d, F, \varepsilon)$ such that every non-degenerate hyperrectangle $A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{c\}$ ($k_i < l_i, c \in \mathbb{Z}$) normal to $\vec{v}_0$, for every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$, there exists a constant $\tilde{C}''(d, F, A, h, \varepsilon)$ such that

$$\mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu(\vec{v}_0) - \varepsilon\right] \,\leq\, \tilde{C}'' e^{-C''\mathcal{H}^{d-1}(nA)}\,.$$

When the distribution F admits an exponential moment, we obtain the following large deviation principle.

Theorem 27. *Suppose that $F(\{0\}) < 1 - p_c(d)$ and*

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) \,<\, \infty\,.$$

Then for every non degenerate hyperrectangle $A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{c\}$ ($k_i < l_i, c \in \mathbb{Z}$) in $\mathbb{R}^d$ normal to $\vec{v}_0$ and every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$, the sequence

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)}\right)_{n\in\mathbb{N}}$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the same good rate function $\mathcal{J}_{\vec{v}_0}$ as in Theorem 21.

The proof of all these results rely on the following proposition.

Proposition 28. *Suppose that $F(\{0\}) < 1 - p_c(d)$ and*

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) \,<\, \infty\,.$$

Then for every non degenerate hyperrectangle $A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{c\}$ ($k_i < l_i, c \in \mathbb{Z}$) in $\mathbb{R}^d$ normal to $\vec{v}_0 = (0, \dots, 0, 1)$ and every function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$, for every $x \in \mathbb{R}^+$ we have

$$\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\phi(nA, h(n)) \leq \left(x - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right] \,=\, \mathcal{I}_{\vec{v}_0}(x)\,.$$

The proof of this proposition relies on the same idea as in Kesten [Kes87] and Zhang [Zha07]: subadditivity is recovered by considering in each cylinder only the minimal capacity of a cutset whose boundary is pinned along a given curve $\mathcal{C}$ on the vertical faces of the cylinder, or a curve obtained from $\mathcal{C}$ by an adequate symmetry (see Figure 1.5). This "symmetric-subadditive" argument, which was developed by Kesten [Kes87], does not work in non-straight cylinder. Indeed, it relies on the fact that the distribution of the minimal capacity of a cutset whose boundary is pinned along a given curve $\mathcal{C}$ is the same as the distribution of the minimal capacity of a cutset whose boundary is pinned along a given curve $\mathcal{C}^*$, where $\mathcal{C}^*$ is the image of $\mathcal{C}$ by one of the symmetries of hyperplane spanned by a face of the cylinder $\mathrm{cyl}(nA, h(n))$. This is true only for straight cylinders since the graph $\mathbb{Z}^d$ is invariant in this case by these symmetries.

From Proposition 28 we can deduce easily all the other results we stated concerning straight cylinders. The only additional argument is the fact that under the assumption of an exponential moment for the capacities of the edges, the probability that $\phi(nA, h(n))$ is abnormally big decays strictly faster than $e^{-Cn^{d-1}}$ for any constant C. This result will be presented in Chapter 3.

The assumption $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$ in Theorem 25 is relevant. Indeed, suppose that $F(\{0\}) > 0$, then the probability that all the vertical edges inside $\mathrm{cyl}(nA, h(n))$ that intersect a given horizontal hyperplane have null capacity is $F(\{0\})^{Cn^{d-1}}$ for some constant C, thus by independence we obtain that

$$\mathbb{P}\left[\phi(nA, h(n)) \neq 0\right] \,\leq\, \left[1 - F(\{0\})^{Cn^{d-1}}\right]^{2h(n)} . \tag{2.7}$$

If $h(n) \geq \exp(kn^{d-1})$ for k large enough, the right hand side of (2.7) is summable, thus the maximal flow $\phi(nA, h(n))$ is a.s. null for n large enough.

2.4 Non subadditive flows in dimension 2

In this section we present some results we proved together with Raphaël Rossignol in [RT10a, RT13]. We restrict ourselves to the case of dimension $d = 2$. Thanks to planar duality, we can see cutsets as paths in the dual graph, which is of great help. We prove the following result concerning the convergence of $\phi(nA, h(n))$ in dimension 2. For short, we denote by $\vec{v}_\theta$ the vector of coordinates $(\cos\theta, \sin\theta)$, and by ν_θ the value of $\nu(\vec{v}_\theta)$.

Theorem 29. *Let $d = 2$. Suppose that $F(0) < 1 - p_c(2) = 1/2$ and that F admits a moment of ordre $2 + \varepsilon$ for a positive $\varepsilon > 0$:*

$$\exists \varepsilon > 0 \quad \int_{\mathbb{R}^+} x^{2+\varepsilon}\, dF(x) < \infty\,.$$

For every segment A of length $l(A) > 0$ normal to the unit vector $\vec{v}_\theta$ of coordinates $(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi[$, for very function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \log h(n)/n = 0$, we define

$$\overline{\mathcal{D}} \,=\, \limsup_{n\to\infty} \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)}{nl(A)}\right)\right]$$

and

$$\underline{\mathcal{D}} \,=\, \liminf_{n\to\infty} \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)}{nl(A)}\right)\right].$$

Then we have

$$\limsup_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} \,=\, \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \,|\, \widetilde{\theta} \in \overline{\mathcal{D}} \right\} \qquad a.s.$$

and

$$\liminf_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} \,=\, \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \,|\, \widetilde{\theta} \in \underline{\mathcal{D}} \right\} \qquad a.s.$$

We get a necessary and sufficient condition for the convergence of $\phi(nA, h(n))/l(nA)$ to hold, and we obtain an expression of its limit $\eta_{\theta,h}$ as an infimum when it exists.

Corollary 30. *Let $d = 2$. Under the hypotheses of Theorem 29, if there exists $\alpha \in [0, \pi/2]$ such that*

$$\lim_{n\to\infty} \frac{2h(n)}{nl(A)} \,=\, \tan\alpha \,\in\, [0,+\infty]\,,$$

then

$$\lim_{n\to\infty} \frac{\phi(nA,h(n))}{nl(A)} \,=\, \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \,|\, \widetilde{\theta} \in [\theta-\alpha, \theta+\alpha] \right\} \,:=\, \eta_{\theta,h} \qquad a.s.\,.$$

We also obtain the large deviation principle that describes the lower large deviations of the rescaled maximal flow $\phi(nA, h(n))/l(nA)$ below its limit.

Theorem 31. *Let $d = 2$. Suppose that $F(0) < 1 - p_c(2) = 1/2$ and that F admits an exponential moment*

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) < \infty\,.$$

For every segment A of length $l(A) > 0$ normal to the unit vector $\vec{v}_\theta$ of coordinates $(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi[$, for very function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \log h(n)/n = 0$, and such that

$$\lim_{n\to\infty} \frac{2h(n)}{nl(A)} \,=\, \tan\alpha$$

exists in $[0, +\infty]$, the sequence

$$\left(\frac{\phi(nA,h(n))}{nl(A)} \right)_{n\in\mathbb{N}}$$

satisfies a large deviation principle of speed $nl(A)$ with a good rate function $\mathcal{K} = \mathcal{K}_{\theta,h}$. If we define

$$\delta_{\theta,h} \,=\, \inf\{\lambda \,|\, \mathbb{P}(t(e) \leq \lambda) > 0\} \times \inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} \frac{|\cos\theta| + |\sin\theta|}{\cos(\widetilde{\theta}-\theta)}\,,$$

we can list the following properties that the function $\mathcal{K}$ satisfies: it is infinite on $[0, \delta_{\theta,h}[\cup]\eta_{\theta,h}, +\infty[$, finite on $]\delta_{\theta,h}, \eta_{\theta,h}]$, strictly positive on $[\delta_{\theta,h}, \eta_{\theta,h}[$ if $\delta_{\theta,h} < \eta_{\theta,h}$, nul on $\eta_{\theta,h}$ and strictly decreasing where it is finite, i.e., *if $\mathcal{K}(\lambda) < \infty$, then for every $\varepsilon > 0$ we have $\mathcal{K}(\lambda - \varepsilon) > \mathcal{K}(\lambda)$.*

The rate function $\mathcal{K}$, that depends on θ and h, is defined through an optimization involving the rate functions $\mathcal{I}_{\vec{v}'}$ for different values of $\vec{v}'$. The function $\mathcal{K}$ is less explicit than $\mathcal{I}_{\vec{v}}$ and thus less understood, in particular we do not know if it is convex or continuous.

In the case of straight cylinders, "symmetric-subadditivity" was recovered for the maximal flows $\phi(nA, h(n))$ by looking at minimal cutsets with fixed boundary conditions. In dimension 2, the boundary of a minimal cutset, *i.e.*, of a geodesic path in the dual graph, is just made of two points (x, y), one on each vertical side of $\mathrm{cyl}(nA, h(n))$. Thus a boundary condition $\kappa = (x, y)$ can also be given by one point x (on the left side of the cylinder) and a direction $\widetilde{\theta}$ (such that $\widetilde{\theta} - \pi/2$ is the direction in which the second point y is located, seen from x), see Figure 2.1. The more the direction $\widetilde{\theta}$ differs from the orientation θ of the cylinder, the longest any cutset of boundary condition κ in the cylinder has to be: a cutset of boundary condition $\kappa = (x, \widetilde{\theta})$ inside $\mathrm{cyl}(nA, h(n))$ with A normal to $\vec{v}_\theta$ must contain at least $nl(A)/\cos(\widetilde{\theta} - \theta)$ edges. The typical minimal cutset for $\phi(nA, h(n))$ will thus be oriented in a direction $\widetilde{\theta}$ that minimizes the quotient of the asymptotic unit maximal flow in direction $\widetilde{\theta}$, namely $\nu_{\widetilde{\theta}}$, by $\cos(\widetilde{\theta} - \theta)$. Of course, this optimization can hold only among the possible boundary conditions $\kappa = (x, \widetilde{\theta})$, that are perfectly described by the rate $h(n)/n$.

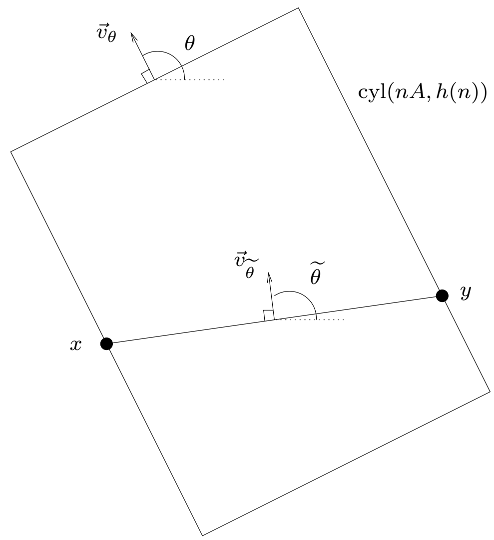

Figure 2.1: The boundary condition $\kappa = (x, \widetilde{\theta})$ inside the cylinder cyl$(nA, h(n))$.

If $\lim_{n\to\infty} h(n)/n = 0$, the only direction of boundary conditions that is asymptotically possible is $\widetilde{\theta} = \theta$, thus we get the convergence of $\phi(nA, h(n))/l(nA)$ towards ν_θ. If $\lim_{n\to\infty} h(n)/n = +\infty$, any direction $\widetilde{\theta}$ is taken into account in the optimization. Between these two limit cases, any intermediate regime can be observed. If $h(n)/n$ does not converge, the maximal flows $\phi(nA, h(n))/l(nA)$ may not converge neither.

This gives an intuitive idea of why the limit $\eta_{\theta,h}$ has such an expression. We give now a few more details about the proofs. We use a concentration inequality as presented in the previous section to prove that $\phi(nA, h(n))$ is asymptotically close to its expectation. The proof of Theorem 29 boils down to the proof of the convergence of $\mathbb{E}[\phi(nA, h(n))]/l(nA)$. We then compare $\mathbb{E}[\phi(nA, h(n))]$ to $\mathbb{E}[\phi^\kappa(nA, h(n))]$, where $\phi^\kappa(nA, h(n))$ is the minimal capacity of a cutset of boundary condition $\kappa = (x, \widetilde{\theta})$ inside cyl$(nA, h(n))$. This variable $\phi^\kappa(nA, h(n))$ is compared itself to the maximal flow $\tau(nA', h(n))$ inside an adequate cylinder cyl$(nA', h(n))$ oriented towards the direction $\widetilde{\theta}$, *i.e.*, A' is normal to $\vec{v}_{\widetilde{\theta}}$. These comparisons also allow us to deduce the large deviation principle, Theorem 31, from the corresponding large deviation principle Theorem 21 for $\tau(nA, h(n))/l(nA)$. The comparison between $\phi^\kappa(nA, h(n))$ and a maximal flow $\tau(nA', h(n))$ in a cylinder oriented towards the direction $\widetilde{\theta}$ is made thanks to a "translation-subadditivity" argument. The boundary condition $\kappa = (x, \widetilde{\theta})$ draws a line inside cyl$(NA, h(N))$ (for a large N) that can be paved with translates $t_i(nA')$ of nA' (see Figure 2.2), and the subadditivity of the maximal flows τ can be used to obtain an inequality of the type $\phi^\kappa(NA, h(N)) \leq \sum_{i=1}^{N/n} \tau(t_i(nA'), h(n))$, up to an error term. Conversely, one cannot argue that maximal flows ϕ^κ are subadditive in the classical way. Indeed, if the boundary conditions κ correspond to two points x_1, x_2 in a cylinder, and x_3, x_4 in an adjacent cylinder, it is generically the case that $x_2 \neq x_3$ thus cutsets with those boundary conditions cannot be glued together. However, a simple translation of the second cylinder in the direction $\vec{v}_\theta$ can put the boundary conditions inside the second cylinders at positions x'_3, x'_4 such that $x'_3 = x_2$, and then a kind of subadditivity is recovered. By considering such translates $t'_i(nA)$ of nA (see Figure 2.3), we can recover an inequality of the type $\tau(NA', h(N)) \leq \sum_{i=1}^{N/n} \phi^\kappa(t'_i(nA), h(n))$, up to an error term.

We finish this section with a few words about the expression of the limit $\eta_{\theta,h}$ appearing in

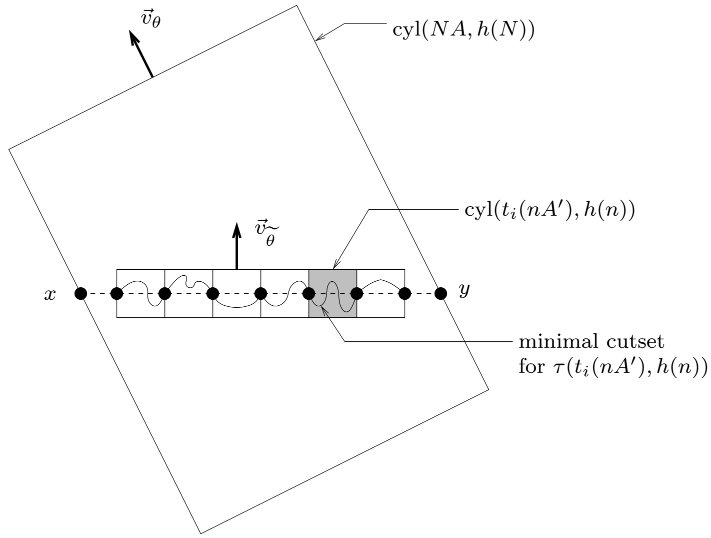

Figure 2.2: The comparison between $\phi^\kappa(NA, h(N))$ and $\sum_{i=1}^{N/n} \tau(t_i(nA'), h(n))$.

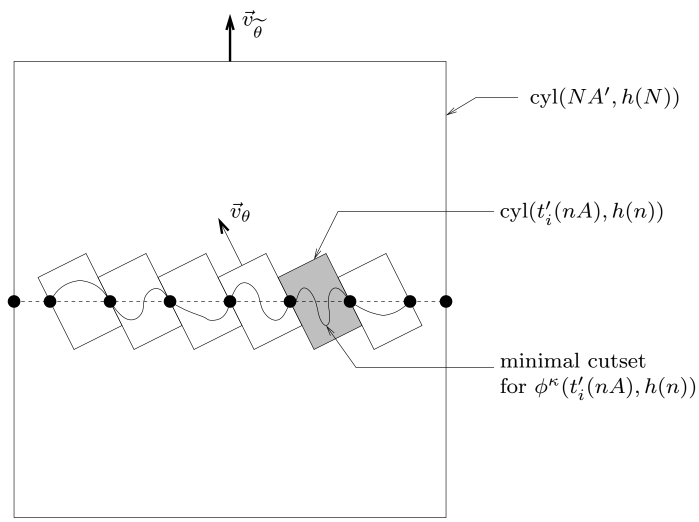

Figure 2.3: The comparison between $\tau(NA', h(N))$ and $\sum_{i=1}^{N/n} \phi^\kappa(t'_i(nA), h(n))$.

Corollary 30. This limit is expressed as an infimum, as it was the case implicitly for the limit $\mathcal{I}(A)$ that appeared in Garet's work [Gar06]. The use of a variational problem will be crucial in the study of flows through more general domains, as we will see in Chapter 4.

Chapter 3

Maximal flow through cylinders: upper large deviations

3.1 Order of the large deviations

We present in this section results we obtained in [Thé07, Thé14]. We are interested in the upper large deviations of the maximal flows through cylinders in any dimension $d \geq 2$. We obtain the following result concerning the maximal flows $\tau(nA, h(n))$.

Theorem 32. *Let A be a non degenerate hyperrectangle, and $\vec{v}$ one of the two unit vectors normal to A. Let $h : \mathbb{N} \to \mathbb{R}^+$ be a height function satisfying $\lim_{n\to\infty} h(n) = +\infty$. The upper large deviations of $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ depend on the tail of the distribution F of the capacities. Indeed, we obtain that:*
i) if F has a bounded support, then for every $x > \nu(\vec{v})$ we have

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)\min(h(n), n)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq x\right] > 0\,; \tag{3.1}$$

the upper large deviations are then of volume order for height functions h such that $h(n)/n$ is bounded, and of order n^d if $\lim_{n\to\infty} h(n)/n = +\infty$.
ii) if F is the exponential law of parameter 1, then there exists $n_0(d, A, h)$, and for every $x > \nu(\vec{v})$ there exists a positive constant D depending only on d and x such that for all $n \geq n_0$ we have

$$\frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq x\right] \leq D\,. \tag{3.2}$$

iii) if F admits exponential moments of all orders:

$$\forall \lambda > 0\,, \qquad \int_{[0,+\infty[} e^{\lambda x} dF(x) < \infty\,,$$

then for all $x > \nu(\vec{v})$ we have

$$\lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq x\right] = +\infty\,. \tag{3.3}$$

We also prove the following partial result concerning the variables $\phi(nA, h(n))$.

Theorem 33. *Let A be a non degenerate hyperrectangle in $\mathbb{R}^d$, of normal unit vector $\vec{v}$, and $h : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\lim_{n\to\infty} h(n) = +\infty$. We suppose that F admits an exponential moment:*

$$\exists \lambda > 0\,, \quad \int_{[0,+\infty[} e^{\lambda x}\, dF(x) \,<\, \infty\,.$$

Then for every $x > \nu(\vec{v})$, we have

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)h(n)} \log \mathbb{P}[\phi(nA, h(n)) \geq x\mathcal{H}^{d-1}(nA)] \,>\, 0\,.$$

The second result is partial, in the sense that $\nu(\vec{v})$ is not the a.s. limit of $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ for general hyperrectangles A and height functions h. However, it is known to be the case if the cylinders are straight or thin, thus in these two cases we prove that the upper large deviations of $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ are of volume order, as soon as the distribution of the capacities admits an exponential moment. Indeed, if all the edges inside $\mathrm{cyl}(nA, h(n))$ have an abnormally big capacity, then $\phi(nA, h(n))$ would be abnormally big. The number of edges inside $\mathrm{cyl}(nA, h(n))$ is of order $n^{d-1}h(n)$, thus the speed of the exponential decay of $\mathbb{P}[\phi(nA, h(n)) \geq (\nu(\vec{v})+\varepsilon)\mathcal{H}^{d-1}(nA)]$ is exactly $n^{d-1}h(n)$.

The speed of the upper large deviations of the variable $\tau(nA, h(n))$ is more sensitive to the tail distribution of the capacities. Indeed, $\tau(nA, h(n))$ is equal to the minimal capacity of a custset with boundary pinned along $\partial(nA)$. Because of this boundary condition, such a cutset must contain some edges near $\partial(nA)$, thus if these edges have huge capacities, then the maximal flow $\tau(nA, h(n))$ can explode. In other terms, a small amount of edges (negligible compared to the volume of the cylinder) may improve abnormally the maximal flow $\tau(nA, h(n))$ if the tail of F is not good enough. When F has bounded support, this cannot happen, thus we recover upper large deviations of volume order. In fact, the exact speed of the exponential decay of $\mathbb{P}[\tau(nA, h(n)) \geq (\nu(\vec{v}) + \varepsilon)\mathcal{H}^{d-1}(nA)]$ is not proportional to $n^{d-1}h(n)$, as the volume of the cylinder $\mathrm{cyl}(nA, h(n))$, but to $n^{d-1}\min(n, h(n))$. Intuitively, it is linked with the fact that the minimal cutset corresponding to $\tau(nA, h(n))$ cannot explore the whole cylinder $\mathrm{cyl}(nA, h(n))$ since it is pinned along $\partial(nA)$, it should be located in a sub-cylinder of $\mathrm{cyl}(nA, h(n))$ of height of order at most n. We would like to transform this intuitive idea into some rigorous result about the location of a minimal cutset, but this question is a lot more subtle. For a partial answer, we refer to the study of minimal cutsets in Chapter 4.

The idea of the proof of Theorem 33 is the following. For a large N and a smaller n, we divide $\mathrm{cyl}(NA, h(N))$ in its height into slabs $(S_i)_{i\in\{1,\dots,h(N)/h(n)\}}$ (see Figure 3.1), and we denote by $\phi(S_i)$ the maximal flow from the top to the bottom of S_i in the direction of $\vec{v}$. We fill each slab S_i with translates $\mathrm{cyl}(t_{i,j}(nA), h(n))$ of $\mathrm{cyl}(nA, h(n))$, for $j \in \{1, \dots, (N/n)^{d-1}\}$. Thanks to the subadditivity of $\tau(nA, h(n))$ we have

$$\phi(NA, h(N)) \,\leq \min_{i\in\{1,\dots,h(N)/h(n)\}} \phi(S_i) \,\leq \min_{i\in\{1,\dots,h(N)/h(n)\}} \sum_{j=1}^{(N/n)^{d-1}} \tau(t_{i,j}(nA), h(n))\,.$$

We can use classical Cramér Theorem, together with Proposition 18, to conclude. Concerning the variable $\tau(NA, h(N))$, the idea is globally the same, but we need to add edges along the vertical sides of the cylinder $\mathrm{cyl}(NA, h(N))$ to recover a cutset for $\tau(NA, h(N))$ from a cutset inside the slab S_i. If the slab S_i is at distance k from NA, the number of edges we have to add is of order

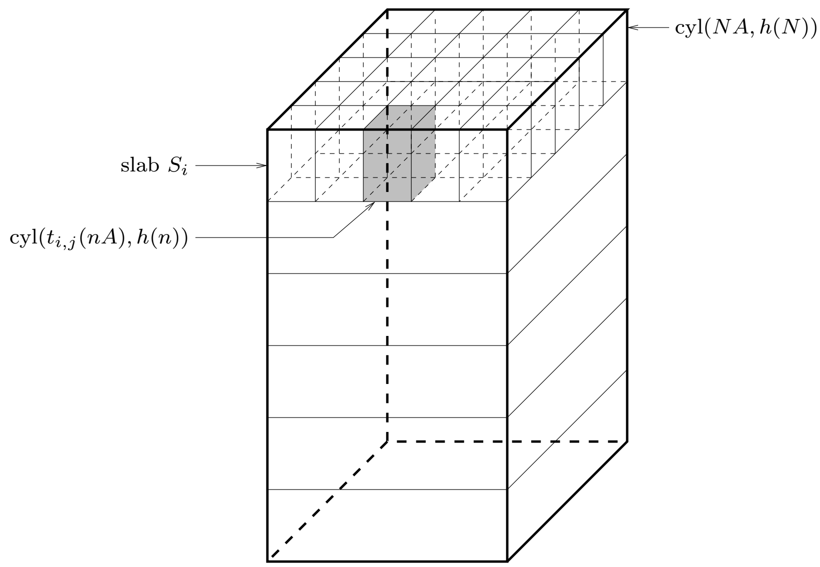

Figure 3.1: The cylinder cyl$(NA, h(N))$, the slabs S_i and the cylinders cyl$(t_{i,j}(nA), h(n))$ *(the construction can be done in a tilted cylinder).*

$N^{d-2}k$. To control this term, we cannot allow k to be of order bigger than N, thus instead of filling the entire cylinder cyl$(NA, h(N))$ with slabs of height $h(n)$, we only fill the smaller cylinder cyl$(NA, \min(N, h(N))$. This gives birth to the speed of decay $\mathcal{H}^{d-1}(nA)\min(h(n), n)$ that appears in Theorem 32.

3.2 Refinement in dimension 2

In this section we present a result we obtained in [RT13] in collaboration with Raphaël Rossignol. In dimension 2, the limit of $\phi(nA, h(n))/l(nA)$ is known even when cyl$(nA, h(n))$ is not straight or thin, at least under some assumptions on F and h. Let $\vec{v}_\theta$ be a unit vector normal to the segment A. When $\lim_{n\to\infty} 2h(n)/l(nA) = \tan\alpha \in [-\infty, +\infty]$, the limit of $\phi(nA, h(n))/l(nA)$ is equal to

$$\eta_{\theta,h} \;=\; \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \,|\, \widetilde{\theta} \in [\theta-\alpha, \theta+\alpha] \right\} \,,$$

as stated in Theorem 29 and Corollary 30. We define

$$\widetilde{\theta}^* \;=\; \operatorname{argmin} \left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \,|\, \widetilde{\theta} \in [\theta-\alpha, \theta+\alpha] \right\} \,.$$

In the sketch of the proof we gave in the previous section, we divided cyl$(NA, h(N))$ into slabs that where normal to the direction $\vec{v}_\theta$, just like A is. We can choose instead to divide cyl$(NA, h(N))$ into slabs that are normal to the direction $\vec{v}_{\widetilde{\theta}^*}$. An easy adaptation of the rest of the proof of Theorem 33 leads to the following result.

Theorem 34. *Let $d = 2$. Suppose that $F(0) < 1 - p_c(2) = 1/2$ and that F admits an exponential moment:*

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) < \infty\,.$$

For every segment A of length $l(A) > 0$ normal to the unit vector $\vec{v}_\theta = (\cos\theta, \sin\theta)$ for $\theta \in [0, \pi[$, for very height function $h : \mathbb{N} \to \mathbb{R}^+$ satisfyling $\lim_{n\to\infty} h(n) = +\infty$ and $\lim_{n\to\infty} \log h(n)/n = 0$ such that

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{l(nA)} = \eta_{\theta,h}$$

exists a.s., for all $x > \eta_{\theta,h}$ we have

$$\liminf_{n\to\infty} \frac{-1}{l(nA)h(n)} \log \mathbb{P}\left[\phi(nA, h(n)) \geq x l(nA)\right] > 0\,.$$

This proves in dimension $d = 2$ that the upper large deviations of $\phi(nA, h(n))$ are indeed of volume order.

3.3 Large deviation principle

We go back to the case of a general dimension $d \geq 2$. We present here a result proved in [Thé07]. In the case of straight cylinders, Theorem 33 states that $\mathbb{P}[\phi(nA, h(n)) \geq (\nu(\vec{v}) + \varepsilon)\mathcal{H}^{d-1}(nA)]$ decays exponentially fast with $n^{d-1}h(n)$, under some moment assumption. We go one step further and prove the corresponding large deviation principle. It has been proved in [Thé07] only for straight cylinders of the form $[0,n]^{d-1} \times [0, h(n)]$ but the proof can be extended without difficulties to any straight cylinder.

Theorem 35. *Let $A = \prod_{i=1}^{d-1}[k_i, l_i] \times \{c\}$ ($k_i < l_i, c \in \mathbb{Z}$) be a non degenerate hyperrectangle in $\mathbb{R}^d$ normal to $\vec{v}_0 = (0, \dots, 0, 1)$ and let $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n)/\log n = +\infty$. Then for every $x \in \mathbb{R}^+$, the limit*

$$\psi(x) = \lim_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)h(n)} \log \mathbb{P}\left[\phi(nA, h(n)) \geq x\mathcal{H}^{d-1}(nA)\right]$$

exists in $[0, +\infty]$ (it may be infinite) and is independent of A and h. Moreover, ψ is convexe on $\mathbb{R}^+$, finite and continuous on the set $\{x \in \mathbb{R}^+ : F([x, +\infty[) > 0\}$. If F admits a finite mean

$$\int_{\mathbb{R}^+} x\, dF(x) < \infty\,,$$

then ψ is null on $[0, \nu(\vec{v}_0)]$. If F admits an exponential moment

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) < \infty\,,$$

then ψ is strictly positive on $]\nu(\vec{v}_0), +\infty[$ and the sequence

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)}\right)_{n\in\mathbb{N}}$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)h(n)$ with good rate function ψ.

The hard part of the proof of Theorem 35 is the proof of the existence of ψ. The strict positivity of ψ is a consequence of Theorem 33 (in fact, Theorem 33 has been proved first in [Thé07] in the case of straight cylinders), whereas the proof of the large deviation principle given the existence of ψ

uses classical ideas in large deviations theory. The proof of the existence of ψ requires to study the maximal streams inside cylinders, contrary to all the results that have been presented previously that relied on the study of the minimal cutsets. Consider a large N and a smaller n. We divide the cylinder $\mathrm{cyl}(NA, h(N))$ into columns C_i for $i \in \{1, \dots, N^{d-1}/n^{d-1}\}$ whose basis are translates of nA and whose heights are equal to $2h(N)$ (see Figure 3.2), and we denote by $\phi(C_i)$ the maximal flow from the top to the bottom of C_i in the direction of $\vec{v}_0$. Then the maximal flow through

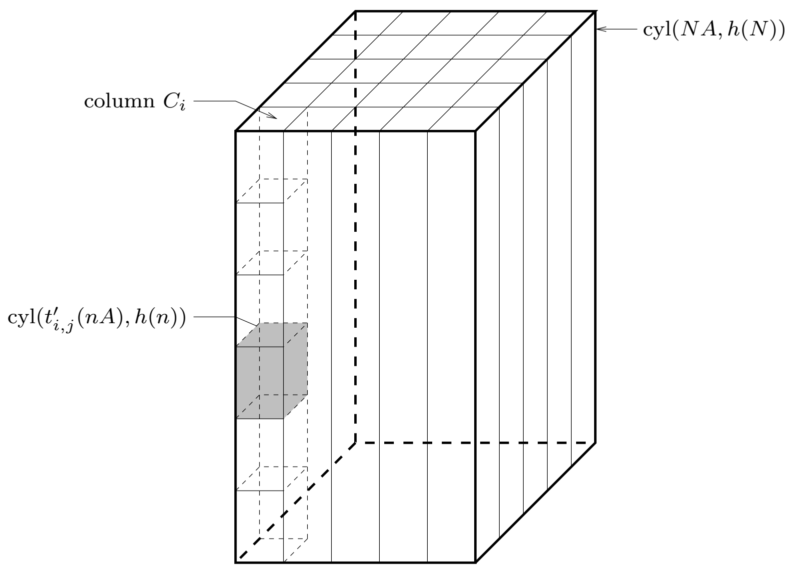

Figure 3.2: The straight cylinder $\mathrm{cyl}(NA, h(N))$, the columns C_i and the cylinders $\mathrm{cyl}(t'_{i,j}(nA), h(n))$.

$\mathrm{cyl}(NA, h(N))$ from its top to its bottom is bigger than the sum of the maximal flows through each column:

$$\phi(NA, h(N)) \geq \sum_{i=1}^{N^{d-1}/n^{d-1}} \phi(C_i)\,.$$

We divide now each column C_i into $h(N)/h(n)$ smaller cylinders of height $2h(n)$, each of them being a translate $\mathrm{cyl}(t'_{i,j}(nA), h(n))$ of $\mathrm{cyl}(nA, h(n))$. Imagine that the water that crosses the cylinder of height $2h(n)$ at the top of the column C_i can enter into the second cylinder of the pile, and so on, so that

$$\phi(C_i) \geq \min_{j \in \{1, \dots, h(N)/h(n)\}} \phi(t'_{i,j}(nA), h(n))\,. \tag{3.4}$$

Then it would be the case that for all $x \in \mathbb{R}^+$,

$$\begin{aligned}
\mathbb{P}\left[\phi(NA, h(N)) \geq x\mathcal{H}^{d-1}(NA)\right] \;&\geq\; \mathbb{P}\left[\forall i \in \{1, \dots, N^{d-1}/n^{d-1}\}\,,\, \phi(C_i) \geq x\mathcal{H}^{d-1}(nA)\right] \\
&\geq \prod_{i=1}^{N^{d-1}/n^{d-1}} \mathbb{P}\left[\phi(C_i) \geq x\mathcal{H}^{d-1}(nA)\right] \\
&\geq \prod_{i=1}^{N^{d-1}/n^{d-1}} \mathbb{P}\left[\forall j \in \{1, \dots, h(N)/h(n)\,,\, \phi(t'_{i,j}(nA), h(n)) \geq x\mathcal{H}^{d-1}(nA)\right] \\
&\geq \prod_{i=1}^{N^{d-1}/n^{d-1}} \prod_{j=1}^{h(N)/h(n)} \mathbb{P}\left[\phi(t'_{i,j}(nA), h(n)) \geq x\mathcal{H}^{d-1}(nA)\right] \\
&\geq \mathbb{P}\left[\phi(nA, h(n)) \geq x\mathcal{H}^{d-1}(nA)\right]^{\frac{N^{d-1}h(N)}{n^{d-1}h(n)}}\,. \qquad (3.5)
\end{aligned}$$

This would be enough to conclude that $\psi(x)$ exists by letting N and then n go to infinity. However, we have no hope to prove an inequality as strong as (3.4). Inspired by the symmetry argument already used in the study of minimal cutsets in straight boxes, we restrict ourselves to study maximal flows through the small cylinders $\mathrm{cyl}(t'_{i,j}(nA), h(n))$ that are achieved by streams of specific boundary conditions $\mathcal{D}$. A boundary condition for a stream is a constraint that fixes the amount of water that crosses each edge at the top and at the bottom of the cylinder. It is possible to glue together streams through cylinders that are piled in a column if you impose to these streams boundary conditions $\mathcal{D}$ and $\mathcal{D}^*$ that are symmetric, so that the water that escapes from a cylinder through an edge can directly enter in the next cylinder of the pile. If we denote by $\phi^{\mathcal{D}}(nA, h(n))$ the maximal flow through $\mathrm{cyl}(nA, h(n))$ from its top to its bottom for a stream with prescribed boundary conditions $\mathcal{D}$, we use the invariance of the graph by the symmetry with regard to one of the hyperplanes of the axis to state that $\phi^{\mathcal{D}}(nA, h(n))$ and $\phi^{\mathcal{D}^*}(nA, h(n))$ have the same law. Inequality (3.5) is thus replaced by

$$\mathbb{P}\left[\phi(NA, h(N)) \geq x\mathcal{H}^{d-1}(NA)\right] \;\geq\; \mathbb{P}\left[\phi^{\mathcal{D}}((nA), h(n)) \geq x\mathcal{H}^{d-1}(nA)\right]^{(N^{d-1}h(N))/(n^{d-1}h(n))}\,.$$

It remains to control the number of possible boundary conditions for streams to conclude the proof.

The study of the maximal flow $\phi(nA, h(n))$ through straight cylinders (convergence, lower and upper large deviations) relies heavily on symmetry properties that are not true if one consider a tilted cylinder $\mathrm{cyl}(nA, h(n))$. The study of maximal flows $\phi(nA, h(n))$ through tilted cylinders in dimension $d \geq 3$ is in fact not really easier than the study of maximal flows through general domains of $\mathbb{R}^d$. This is the object of the next Chapter.

Chapter 4

Maximal flow through a domain of $\mathbb{R}^d$

4.1 Some notations

In this chapter, we study maximal flows through general domains Ω of $\mathbb{R}^d$. Since no direction is particularly pertinent when regarding Ω, we want that the dimensions of Ω go to infinity isotropically with regard to the underlying graph. For simplicity of notation, we choose in this chapter to fix the domain Ω and to consider the rescaled graph $(\mathbb{Z}^d/n, \mathbb{E}^d/n)$. We consider an open bounded connected subset Ω of $\mathbb{R}^d$. Let Γ^1, Γ^2 be two disjoint subsets of $\Gamma = \partial\Omega$ that are open in Γ. We want to study the maximal flow from Γ^1 to Γ^2 through Ω for the capacities $(t(e), e \in \mathbb{E}^d/n)$. We consider a discrete version $(\Omega_n, \Gamma_n, \Gamma_n^1, \Gamma_n^2)$ of $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ defined by:

$$\begin{cases} \Omega_n = \{x \in \mathbb{Z}_n^d \,|\, d_\infty(x, \Omega) < 1/n\}\,, \\ \Gamma_n = \{x \in \Omega_n \,|\, \exists y \notin \Omega_n\,,\ (x,y) \in \mathbb{E}_n^d\}\,, \\ \Gamma_n^i = \{x \in \Gamma_n \,|\, d_\infty(x, \Gamma^i) < 1/n\,,\ d_\infty(x, \Gamma^{3-i}) \geq 1/n\} \text{ for } i = 1, 2\,, \end{cases}$$

where d_∞ is the L^∞-distance and (x, y) is the edge of endpoints x and y (see Figure 4.1). For short

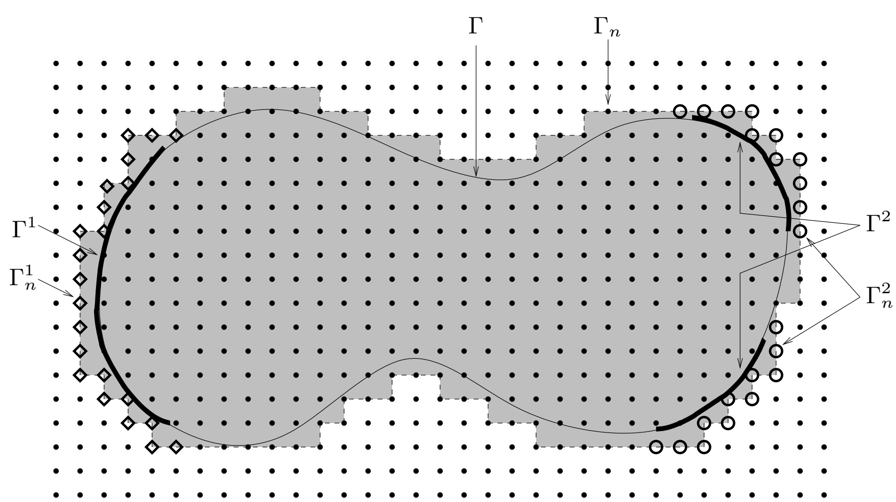

Figure 4.1: The domain $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ and its discrete version $(\Omega_n, \Gamma_n, \Gamma_n^1, \Gamma_n^2)$.

we denote by ϕ_n the maximal flow $\phi(\Gamma_n^1 \to \Gamma_n^2 \text{ in } \Omega_n)$ in the graph $(\mathbb{Z}^d/n, \mathbb{E}^d/n)$. To study the maximal flow ϕ_n, we need to impose some regularity to the domain $(\Omega, \Gamma^1, \Gamma^2)$.

Definition. *We say that a domain* $(\Omega, \Gamma^1, \Gamma^2)$ *is* nice *if it satisfies the following conditions:* Ω *is a bounded open connected subset of* $\mathbb{R}^d$*, that is a Lipschitz domain; its boundary* Γ *is included in the union of a finite number of oriented hypersurfaces of class* $\mathcal{C}^1$ *that intersect each other transversally; the sets* Γ^1 *and* Γ^2 *are open subsets of* Γ *satisfying* $\inf\{\|x-y\|_2\,,\, x\in\Gamma^1\,,\, y\in\Gamma^2)>0$*, and their relative boundaries* $\partial_\Gamma\Gamma^1$ *and* $\partial_\Gamma\Gamma^2$ *have null* $\mathcal{H}^{d-1}$ *measure.*

We have two examples in mind: Ω can be a very regular domain whose boundary Γ is a hypersurface of class $\mathcal{C}^1$, or Ω can be a tilted cylinder of top Γ^1 and bottom Γ^2.

4.2 Large deviation estimates

This section gathers the results we proved with Raphaël Cerf in the companion papers [CT11a, CT11b, CT11c]. We state in these articles that, under some hypotheses on $(\Omega, \Gamma^1, \Gamma^2)$ and F, the rescaled maximal flow ϕ_n/n^{d-1} converges a.s. towards a constant ϕ_Ω (that depends on the dimension $d\geq 2$, on the domain $(\Omega, \Gamma^1, \Gamma^2)$ and on the distribution F of the capacities), that the lower large deviations of ϕ_n/n^{d-1} are of surface order, and that its upper large deviations are of volume order. More precisely, we obtain the following theorem.

Theorem 36. *Let* $(\Omega, \Gamma^1, \Gamma^2)$ *be a nice domain. If* F *admits an exponential moment*

$$\exists \lambda > 0\,, \quad \int_{\mathbb{R}^+} e^{\lambda x}\, dF(x) < \infty\,,$$

then there exists a constant ϕ_Ω *(that depends on* $d, (\Omega, \Gamma^1, \Gamma^2)$ *and* F*) such that*

$$\lim_{n\to\infty} \frac{\phi_n}{n^{d-1}} \,=\, \phi_\Omega \quad a.s.\,.$$

Moreover, we know that

$$\phi_\Omega > 0 \iff F(\{0\}) < 1 - p_c(d)$$

and we have the following control on the large deviations of ϕ_n*:*

$$\forall x \in [0, \phi_\Omega)\,, \quad \limsup_{n\to\infty} \frac{1}{n^{d-1}} \log \mathbb{P}\left[\phi_n \leq x n^{d-1}\right] \,<\, 0\,, \tag{4.1}$$

$$\forall x \in (\phi_\Omega, +\infty)\,, \quad \limsup_{n\to\infty} \frac{1}{n^{d}} \log \mathbb{P}\left[\phi_n \geq x n^{d-1}\right] \,<\, 0\,. \tag{4.2}$$

The convergence of ϕ_n/n^{d-1} is a consequence of the large deviation estimates (4.1) and (4.2). In fact, the strategy of the proof is the following.

- We prove in [CT11b] that there exists a constant ϕ_Ω such that for every $x<\phi_\Omega$ we have

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}} \log \mathbb{P}\left[\phi_n \leq x n^{d-1}\right] \,<\, 0\,. \tag{4.3}$$

- We prove in [CT11c] that there exists a constant $\widetilde{\phi}_\Omega$ such that for every $x>\widetilde{\phi}_\Omega$ we have

$$\limsup_{n\to\infty} \frac{1}{n^{d}} \log \mathbb{P}\left[\phi_n \geq x n^{d-1}\right] \,<\, 0\,. \tag{4.4}$$

- We finally prove in [CT11a] that $\widetilde{\phi}_\Omega = \phi_\Omega$.

We believe that the hypothesis of an exponential moment is not needed for the a.s. convergence of ϕ_n/n^{d-1} to hold, but it is needed to get the upper large deviation estimate (4.2), thus we cannot hope to weaken this hypothesis with this strategy of proof.

The constants ϕ_Ω and $\widetilde{\phi}_\Omega$ can be expressed as infimums over capacities of continuous cutsets, similarly to the limit $\eta_{\theta,h}$ that appeared in dimension $d=2$ (see Theorem 29 and Corollary 30). If A is a subset of Ω, let ∂A be its boundary and $\partial^* A$ its reduced boundary - see [Cer06] for a definition of this term, what we need here is the fact that A admits an exterior normal unit vector $\vec{v}_A(x)$ at each point $x \in \partial^* A$. We can complete the boundary of any subset A of Ω to obtain a continuous cutset $\mathcal{S}(A)$, *i.e.*, a hypersurface that intersects any path from Γ^1 to Γ^2 in $\overline{\Omega}$, in this way (see Figure 4.2):

$$\mathcal{S}(A) \,=\, (\partial A \cap \Omega) \cup \big(\Gamma^2 \cap \partial (A\cap \Omega)\big) \cup \big(\Gamma^1 \cap \partial(\Omega \smallsetminus A)\big)\,.$$

We denote by $\mathcal{S}^*(A)$ the reduced set A built from the reduced boundaries of A and Ω, and we

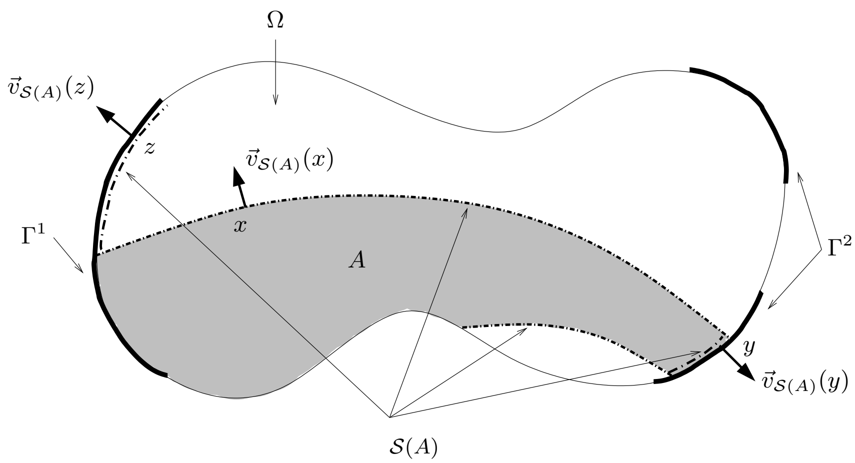

Figure 4.2: The continuous cutset $\mathcal{S}(A)$ associated with a subset A of Ω.

denote by $\vec{v}_{\mathcal{S}(A)}(x)$ the unit vector normal to $\mathcal{S}(A)$ at $x\in \mathcal{S}^*(A)$ (more precisely, $\vec{v}_{\mathcal{S}(A)}(x) = \vec{v}_A(x)$ if $x \in (\partial^* A\cap \Omega)\cup (\Gamma^2 \cap \partial^*(A\cap\Omega))$, and $\vec{v}_{\mathcal{S}(A)}(x) = \vec{v}_\Omega(x)$ if $x\in \Gamma^1 \cap \partial^*(\Omega\smallsetminus A))$. Since the constant $\nu(\vec{v})$ can be seen as the asymptotic rescaled minimal capacity of a unit continuous surface globally normal to $\vec{v}$, it is natural to associate with a continuous cutset $\mathcal{S}(A)$ the following capacity:

$$\mathrm{capa}^{\mathrm{cont}}(\mathcal{S}(A)) \,=\, \int_{\mathcal{S}^*(A)} \nu(\vec{v}_{\mathcal{S}(A)}(x))\, d\mathcal{H}^{d-1}(x)\,.$$

The constant ϕ_Ω is defined as the infimum of the continuous capacity of continuous cutsets that have enough regularity:

$$\phi_\Omega \,=\, \inf\{\mathrm{capa}^{\mathrm{cont}}(\mathcal{S}(A)) \,:\, A\subset \Omega\,,\, A \text{ has finite perimeter}\}\,.$$

The property that A has finite perimeter is equivalent to the property that $\mathbb{1}_A$ has bounded variation. This property guarantees that $\vec{v}_A(x)$ is well defined $\mathcal{H}^{d-1}$-almost everywhere on ∂A, *i.e.*, that $\mathcal{H}^{d-1}(\partial A \smallsetminus \partial^* A) = 0$. The constant $\widetilde{\phi}_\Omega$ has the same form as ϕ_Ω but the infimum is taken over polyhedral sets A. The proof of the equality $\widetilde{\phi}_\Omega = \phi_\Omega$ is thus entirely geometric, and we choose not to talk about it. We just notice that Garet [Gar06] already used a polyhedral approximation, but it was easier to prove since he worked in dimension $d = 2$.

Let us say a few words about the proof of inequality (4.3). We are studying the lower large deviations of ϕ_n/n^{d-1}: they are controlled by what happens around a minimal cutset. First, we use the estimate of the number of edges in a minimal cutset obtained by Zhang [Zha07] (see Theorem 11 in this dissertation) to restrict the problem to cutsets having a number of edges at most equal to cn^{d-1} for a constant c; we can then conclude that the set of plaquettes corresponding to the minimal cutset is "near" the boundary of a subset A of Ω of perimeter smaller than c. The subsets of Ω of perimeter smaller than a constant c is compact. By making an adequate covering of this space, we need only to deal with a finite number of sets A and their neighborhoods. We then cover the boundary of such a set A by balls of very small radius, such that ∂A is "almost flat" in each ball; we also show that if ϕ_n is smaller than $\phi_\Omega(1-\varepsilon)n^{d-1}$ for some positive ε, then some local event happens in each ball of the covering of ∂A. After that, we construct a link between this local event in a ball and the fact that the maximal flow through a cylinder (included in the ball) is abnormally small. Since the lower large deviations for the maximal flow through a cylinder are already known to be of surface order (see Theorem 19), we can conclude. This proof is largely inspired by the methods used to study the Wulff crystal in Ising model in dimension $d \geq 3$ (see [Cer06] for instance).

The proof of the upper large deviation estimate (4.4) is complicated by geometric issues, but the idea is roughly speaking very similar. Consider a polyhedral set P such that $\mathrm{capa}^{\mathrm{cont}}(P)$ is very close to $\widetilde{\phi}_\Omega$, so that if $\phi_n \geq (\widetilde{\phi}_\Omega + 2\varepsilon)n^{d-1}$, then $\phi_n \geq (\mathrm{capa}^{\mathrm{cont}}(P)+\varepsilon)n^{d-1}$. We almost cover the boundary of P by small cylinders, up to a part of of small $\mathcal{H}^{d-1}$-measure. If $\phi_n \geq (\mathrm{capa}^{\mathrm{cont}}(P) + \varepsilon)n^{d-1}$, then either the maximal flow through one of these cylinders must be abnormally big, and we know by Theorem 33 that the probability that this happens decays exponentially fast with n^d, or the maximal flow that crosses the part of ∂P forgotten during the covering is abnormally big. We control the probability that this second event happens thanks to an optimization among different choices of cutsets in this part of the domain.

We do not give more details about the proof of the upper large deviations, since this proof is less relevant than the proof of the lower large deviations. Indeed, the lower large deviations of the maximal flow ϕ_n are linked with what happens around a cutset, and this is exactly what the proof of Inequality (4.3) is looking at. Conversely, the proof of the upper large deviation principle for maximal flows through straight cylinders, Theorem 35, tells us that the study of the upper large deviations of maximal flows is more linked with the understanding of the maximal streams than with the understanding of the minimal cutsets. The proof of inequality (4.4) is thus efficient, but we have no hope to prove this way for instance that $\lim_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}\left[\phi_n \geq xn^{d-1}\right]$ exists, and thus to obtain a corresponding large deviation principle. In a sense, the proof of Theorem 36 gives to us many informations about the behavior of minimal cutsets, but no informations about the behavior of maximal streams.

4.3 Maximal stream and minimal cutset

This section is devoted to the presentation of the results we obtained with Raphaël Cerf in [CT14a]. Our goal is now to understand the asymptotic behavior of maximal streams and minimal cutsets inside Ω. A stream and a cutset inside Ω_n on $(\mathbb{Z}^d/n, \mathbb{E}^d/n)$ are discrete objects. We associate with them some objects that can be considered at the same time in a discrete and in a continuous setting.

Concerning the cutsets, if E_n is a cutset inside Ω_n between Γ_n^1 and Γ_n^2, we see its dual set of plaquettes E_n^* as the boundary inside Ω of a subset $\mathcal{E}_n$ defined as

$$\mathcal{E}_n \,=\, \left\{ x+u \,\middle|\, \begin{array}{c} x \in \mathbb{Z}_n^d \cap \Omega \,, \text{ there exists a path from } \Gamma_n^1 \text{ to } x \text{ in } (\mathbb{Z}^d/n, (\mathbb{E}^d/n \cap \Omega) \smallsetminus E_n) \\ u \in \left[\frac{-1}{2n}, \frac{1}{2n}\right]^d \end{array} \right\}.$$

We have already defined in the previous section the corresponding continuous objects: with a subset A of Ω we associate the continuous cutset $\mathcal{S}(A)$ (see Figure 4.2) and the continuous capacity $\mathrm{capa}^{\mathrm{cont}}(\mathcal{S}(A))$. The first variational problem we define is thus as previously

$$\phi_\Omega \,=\, \inf\{\mathrm{capa}^{\mathrm{cont}}(\mathcal{S}(A)) \,:\, A \subset \Omega \,,\, A \text{ has finite perimeter}\}\,,$$

and we define the corresponding set of minimizers

$$\Sigma^{\mathrm{cutset}} \,=\, \{A \subset \Omega \,:\, A \text{ has finite perimeter and } \mathrm{capa}^{\mathrm{cont}}(\mathcal{S}(A)) = \phi_\Omega\}\,.$$

Concerning the streams, if $\vec{f}_n$ is an admissible stream inside Ω_n between Γ_n^1 and Γ_n^2, we define the corresponding vector measure $\vec{\mu}_n$ by

$$\vec{\mu}_n \,=\, \frac{1}{n^d} \sum_{e \in \mathbb{E}^d/n \cap \Omega} \vec{f}_n(e) \delta_{c(e)}$$

where $c(e)$ is the center of e. The vector $\vec{\mu}_n$, that we call a stream by extension, is a rescaled measure version of the stream function $\vec{f}_n$. It is defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ where $\mathcal{B}(\mathbb{R}^d)$ is the collection of the Borel sets of $\mathbb{R}^d$, and it takes values in $\mathbb{R}^d$. Since $\vec{\mu}_n$ is rescaled, we associate with it a flow which is equal to the rescaled flow $\mathrm{flow}(\vec{f}_n)/n^{d-1}$:

$$\mathrm{flow}^{\mathrm{disc}}(\vec{\mu}_n) \,=\, \frac{\mathrm{flow}(\vec{f}_n)}{n^{d-1}}\,.$$

We say that a (vector) stream $\vec{\mu}_n$ is maximal if $\mathrm{flow}^{\mathrm{disc}}(\vec{\mu}_n) = \phi_n/n^{d-1}$, and if according to this stream no water escapes from Ω_n through Γ_n^1 nor enters in Ω_n through Γ_n^2. We now turn to the definition of a continuous stream. A continuous stream is for us a measure $\vec{\sigma}\mathcal{L}^d$ which is absolutely continuous with regard to the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$ and whose density $\vec{\sigma} \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d, \mathcal{L}^d)$ is null outside Ω, *i.e.*, $\vec{\sigma} = 0$ $\mathcal{L}^d$-a.e. on Ω^c, and has the following properties:

- boundary conditions: $\vec{\sigma} \cdot \vec{v}_\Omega = 0$ $\mathcal{H}^{d-1}$-a.e. on $\Gamma \smallsetminus (\Gamma^1 \cup \Gamma^2)$ and $\vec{\sigma} \cdot \vec{v}_\Omega \leq 0$ $\mathcal{H}^{d-1}$-a.e. on Γ^1;
- conservation law: $\mathrm{div}\vec{\sigma} = 0$ $\mathcal{L}^d$-a.e. on Ω;
- capacity constraint: $\vec{\sigma} \cdot \vec{v} \leq \nu(\vec{v})$ for all $\vec{v} \in \mathbb{S}^{d-1}$, $\mathcal{L}^d$-a.e. on Ω .

Here we interpret $\nu(\vec{v})$ as the asymptotic rescaled maximal flow that can flow through the media in direction $\vec{v}$. Since we suppose only that $\vec{\sigma} \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d, \mathcal{L}^d)$, the meaning of those properties is not obvious. The derivation must be understood in terms of distributions. We refer to Nozawa [Noz90] for precise definitions giving mathematical sense to all these properties. With any continuous stream $\vec{\sigma}\mathcal{L}^d$, we associate its continuous flow defined as

$$\mathrm{flow}^{\mathrm{cont}}(\vec{\sigma}\mathcal{L}^d) \,=\, \int_{\Gamma^1} -\vec{\sigma} \cdot \vec{v}_\Omega \, d\mathcal{H}^{d-1}\,.$$

This is the amount of water that enters into Ω along Γ^1 according to the stream $\vec{\sigma}\mathcal{L}^d$. The second variational problem we define is thus

$$\hat{\phi}_\Omega \,=\, \sup\left\{ \mathrm{flow}^{\mathrm{cont}}(\vec{\sigma}\mathcal{L}^d) \,\middle|\, \begin{array}{c} \vec{\sigma} \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d, \mathcal{L}^d)\,,\ \vec{\sigma} = 0\ \mathcal{L}^d\text{-a.e. on } \Omega^c\,, \\ \mathrm{div}\vec{\sigma} = 0\ \mathcal{L}^d\text{-a.e. on } \Omega\,, \\ \vec{\sigma}\cdot\vec{v} \le \nu(\vec{v}) \text{ for all } \vec{v} \in \mathbb{S}^{d-1}\,,\ \mathcal{L}^d\text{-a.e. on } \Omega\,, \\ \vec{\sigma}\cdot\vec{v}_\Omega \le 0\ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma^1\,, \\ \vec{\sigma}\cdot\vec{v}_\Omega = 0\ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma \smallsetminus (\Gamma^1 \cup \Gamma^2) \end{array} \right\}. \tag{4.5}$$

We also define the corresponding set of minimizers

$$\Sigma^{\mathrm{stream}} \,=\, \left\{ \vec{\sigma}\mathcal{L}^d \,\middle|\, \begin{array}{c} \vec{\sigma} \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d, \mathcal{L}^d)\,,\ \vec{\sigma} = 0\ \mathcal{L}^d\text{-a.e. on } \Omega^c\,, \\ \mathrm{div}\vec{\sigma} = 0\ \mathcal{L}^d\text{-a.e. on } \Omega\,, \\ \vec{\sigma}\cdot\vec{v} \le \nu(\vec{v}) \text{ for all } \vec{v} \in \mathbb{S}^{d-1}\,,\ \mathcal{L}^d\text{-a.e. on } \Omega\,, \\ \vec{\sigma}\cdot\vec{v}_\Omega \le 0\ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma^1\,, \\ \vec{\sigma}\cdot\vec{v}_\Omega = 0\ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma \smallsetminus (\Gamma^1 \cup \Gamma^2)\,, \\ \mathrm{flow}^{\mathrm{cont}}(\vec{\sigma}\mathcal{L}^d) = \hat{\phi}_\Omega \end{array} \right\}.$$

A continuous max-flow min-cut has been proved by Nozawa [Noz90] in this setting

Theorem 37 (Continuous max-flow min-cut theorem)**.** *Suppose that Ω is a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary Γ, and that Γ^1 and Γ^2 are two disjoint Borel subsets of Γ. The following equality holds:*

$$\phi_\Omega \,=\, \hat{\phi}_\Omega \,<\, \infty\,.$$

Moreover, there exists a maximal continuous stream, i.e., *there exists a vector field $\vec{\sigma}$ as required in (4.5) such that* $\mathrm{flow}^{cont}(\vec{\sigma}\mathcal{L}^d) = \hat{\phi}_\Omega$.

We define the distance $\mathfrak{d}$ between subsets of $\mathbb{R}^d$ by

$$\forall A, B \subset \mathbb{R}^d\,, \quad \mathfrak{d}(A, B) \,=\, \mathcal{L}^d(A \triangle B)$$

where $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$ is the symmetric difference of A and B. We prove the convergence of a sequence of discrete minimal custsets (resp. maximal streams) towards the set of continuous minimal cutsets Σ^{cutset} (resp. of continuous maximal streams Σ^{stream}).

Theorem 38. *We suppose that the domain $(\Omega, \Gamma^1, \Gamma^2)$ is nice and that the law F of the capacities has a bounded support.*
(i) For all $n \ge 1$, let $\vec{\mu}_n^{\max}$ be a random maximal discrete stream (seen as a vector measure) from Γ_n^1 to Γ_n^2 in Ω_n. Then $(\vec{\mu}_n^{\max})_{n\ge1}$ converges weakly a.s. towards the set Σ^{stream}, i.e.,

$$a.s.\,,\ \forall f \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})\,,\ \lim_{n\to\infty} \inf_{\vec{\sigma}\mathcal{L}^d \in \Sigma^{stream}} \left\| \int_{\mathbb{R}^d} f\, d\vec{\mu}_n^{\max} - \int_{\mathbb{R}^d} f \vec{\sigma}\, d\mathcal{L}^d \right\| \,=\, 0\,.$$

(ii) We suppose also that $F(\{0\}) < 1 - p_c(d)$. For all $n \ge 1$, let $\mathcal{E}_n^{\min}$ be a minimal cutset (seen as a subset of $\mathbb{R}^d$) in Ω_n from Γ_n^1 to Γ_n^2 with minimal cardinality (i.e., with minimal perimeter). Then the sequence $(\mathcal{E}_n^{\min})_{n\ge1}$ converges a.s. for the distance $\mathfrak{d}$ towards the set Σ^{cutset}, i.e.,

$$a.s.\,,\ \lim_{n\to\infty} \inf_{A \in \Sigma^{cutset}} \mathfrak{d}(\mathcal{E}_n^{\min}, A) \,=\, 0\,.$$

As a by-product of the proof of Theorem 38, we recover two known results:

- The continuous max-flow min-cut theorem proved by Nozawa [Noz90]

$$\phi_\Omega = \hat{\phi}_\Omega .$$

- The convergence of the rescaled maximal flows we presented in Section 4.2, restricted to the case of bounded capacities

$$\lim_{n\to\infty} \frac{\phi_n}{n^{d-1}} = \phi_\Omega \quad \text{a.s.}.$$

If the set of minimizers Σ^{stream} is reduced to a single continuous maximal stream, then Theorem 38 states that the sequence of discrete maximal streams converges weakly towards this maximizer. The same holds for the minimal cutsets. However, the question of the uniqueness of a continuous maximal stream, or of a continuous minimal cutset, is not trivial and we think that the answer should depend on the domain $(\Omega, \Gamma^1, \Gamma^2)$.

The strategy of the proof is the following. We first study a sequence of discrete maximal streams $(\vec{\mu}_n^{\max})_{n\geq 1}$. By compactness, we prove that from each subsequence of $(\vec{\mu}_n^{\max})_{n\geq 1}$ we can extract a sub-subsequence which is weakly convergent. If we denote by $\vec{\mu}$ its limit, we prove that a.s. $\vec{\mu} = \vec{\sigma}\,\mathcal{L}^d$ and that $\vec{\sigma}\,\mathcal{L}^d$ is a continuous stream which is admissible for the max-flow problem $\hat{\phi}_\Omega$: the boundary conditions and the conservation law satisfied by $\vec{\sigma}$ are derived directly from the corresponding properties satisfied by $\vec{\mu}_n^{\max}$, the delicate part of the proof is to state that $\vec{\sigma}$ satisfies the capacity constraint

$$\vec{\sigma}\cdot\vec{v} \leq \nu(\vec{v}) \quad \text{for all } \vec{v}\in\mathbb{S}^{d-1},\ \mathcal{L}^d\text{-a.e. on } \Omega . \tag{4.6}$$

Moreover, we prove that along the converging subsequence,

$$\lim_{n\to\infty} \text{flow}^{\text{disc}}(\vec{\mu}_n^{\max}) = \text{flow}^{\text{cont}}(\vec{\sigma}\mathcal{L}^d) \quad \text{a.s.} \tag{4.7}$$

Independently, we study a sequence of minimal cutsets $(\mathcal{E}_n^{\min})_{n\geq 1}$. By compactness and using Zhang's Theorem 11, we prove that from each subsequence of $(\mathcal{E}_n^{\min})_{n\geq 1}$ we can extract a sub-subsequence which is convergent for the distance $\mathfrak{d}$. If we denote by A its limit, we prove that $A \subset \Omega$ and has finite perimeter, *i.e.*, A is admissible for the min-cut problem ϕ_Ω. Moreover, we prove that along the converging subsequence,

$$\liminf_{n\to\infty} \frac{T(\mathcal{E}_n^{\min})}{n^{d-1}} \geq \text{capa}^{\text{cont}}(A) \quad \text{a.s.} \tag{4.8}$$

where $T(\mathcal{E}_n^{\min})$ is the capacity of the minimal cutset $E_n^{\min}$ corresponding to $\mathcal{E}_n^{\min}$. Finally we establish that for any subset A of Ω with finite perimeter and any admissible stream $\vec{\sigma}\,\mathcal{L}^d$, we have

$$\text{capa}^{\text{cont}}(F) \geq \text{flow}^{\text{cont}}(\vec{\sigma}\mathcal{L}^d) . \tag{4.9}$$

Then combining Equations (4.7), (4.8) and (4.9) we derive the results presented in Theorem 38.

The most original part of this work is the study of maximal streams. The study of minimal cutsets relies largely on the techniques used in [CT11b] to prove that the lower large deviations of ϕ_n are of surface order. To complete the proofs we also use the result of polyhedral approximation proved in [CT11a]. In the proof of the law of large numbers for ϕ_n we present here, we have replaced the study of the upper large deviations of ϕ_n performed in [CT11c] by the study of the maximal streams, which is more natural, and we have adapted the arguments given in the study of the lower large deviations of ϕ_n in [CT11b] to obtain informations on the behavior of minimal cutsets.

Let us say a few word about the proof of (4.6), which relies on the comparison between the integral of a stream inside a cylinder and the flow that crosses this cylinder according to that stream. This comparison is a key argument in our study, and is used again in the proof of (4.7) and (4.9). If $(\vec{\mu}_n)_{n\in\mathbb{N}}$ is a sequence of discrete admissible streams that converges towards a limit $\vec{\sigma}\mathcal{L}^d$, we have

$$\int_D \vec{\sigma}\cdot\vec{v}\,d\mathcal{L}^d \,=\, \lim_{n\to\infty}\int_D d\vec{\mu}_n\cdot\vec{v}\,,$$

for every Borel set D such that $\mathcal{L}^d(\partial D)=0$. On one hand, using Lebesgue differentiation theorem, we know that for $\mathcal{L}^d$-a.e. x,

$$\frac{1}{\mathcal{L}^d(D(x,\varepsilon))}\int_{D(x,\varepsilon)}\vec{\sigma}\cdot\vec{v}\,d\mathcal{L}^d$$

converges towards $\vec{\sigma}(x)\cdot\vec{v}$ when ε goes to zero, where $D(x,\varepsilon)$ is a good sequence of neighborhoods of x of diameter ε. To conclude that $\vec{\sigma}\cdot\vec{v}$ is bounded by $\nu(\vec{v})$, it remains to compare $\int_D d\vec{\mu}_n\cdot\vec{v}$ with $\nu(\vec{v})$. Let us admit for a moment that if D is a cylinder of height h in the direction $\vec{v}$, $\int_D d\vec{\mu}_n\cdot\vec{v}$ is close to $h\Psi(\vec{\mu}_n,D,\vec{v})/n^{d-1}$, where $\Psi(\vec{\mu}_n,D,\vec{v})$ is the amount of fluid that crosses D from the lower half part to the upper half part of its boundary in the direction $\vec{v}$ according to the stream $\vec{\mu}_n$. Since $\Psi(\vec{\mu}_n,D,\vec{v})\leq\tau_n(D,\vec{v})$, where $\tau_n(D,\vec{v})$ is the maximal value of such a flow, we can conclude the proof by using the convergence of the rescaled flow $\tau_n(D,\vec{v})/n^{d-1}$ towards $\nu(\vec{v})$. The delicate step is the proof of the property we admitted. In fact, if $\vec{l}$ is a $\mathcal{C}^1$ vector field on D with null divergence and such that $\vec{l}\cdot\vec{v}_D=0$ $\mathcal{H}^{d-1}$-a.e. on the vertical faces of D (the ones that are not normal to $\vec{v}$), if we denote by B the basis of D, then by Fubini theorem we have

$$\int_D \vec{l}\cdot\vec{v}\,d\mathcal{L}^d=\int_0^h\left(\int_{B+u\vec{v}}\vec{l}\cdot\vec{v}\,d\mathcal{H}^{d-1}\right)du$$

and we have for all u

$$\int_{B+u\vec{v}}\vec{l}\cdot\vec{v}\,d\mathcal{H}^{d-1}=\int_B\vec{l}\cdot\vec{v}\,d\mathcal{H}^{d-1}$$

by the Gauss-Green Theorem since $\operatorname{div}\vec{l}=0$ $\mathcal{L}^d$-a.e. on Ω. We obtain that

$$\int_D\vec{l}\cdot\vec{v}\,d\mathcal{L}^d=h\int_B\vec{l}\cdot\vec{v}\,d\mathcal{H}^{d-1}$$

and $\int_B\vec{l}\cdot\vec{v}\,d\mathcal{H}^{d-1}$ is indeed the flow that goes from the bottom to the top of D according to $\vec{l}$. In [CT14a] we adapt this argument to a discrete stream $\vec{\mu}_n$, and we consider a cylinder thin enough (*i.e.*, h small enough) to control the amount of fluid that enters in D or escapes from D through its vertical faces.

The assumption that the capacities are bounded in Theorem 38 is not relevant in all likelihood. However, this technical assumption gives us the compactness of the family of the discrete maximal streams, without which we do not know how to prove the convergence of these streams.

Chapter 5

Time constant

5.1 Shape theorem in a supercritical Bernoulli percolation cluster

In this section we present the results we obtained with Raphaël Cerf in [CT14b]. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$, equipped with a family of i.i.d. non-negative random variables $(t(e))_{e\in\mathbb{E}^d}$ with common distribution F. We change our point of view and interpret now $t(e)$ as the time needed to cross the edge e. The time $T(x,y)$ needed to go from a vertex x to a vertex $y \in \mathbb{Z}^d$ is defined by

$$T(x,y) \,=\, \inf\{T(\gamma) \,:\, \gamma \text{ is a path from } x \text{ to } y\}\,.$$

Suppose that the passage time $t(e)$ of an edge e can be infinite, *i.e.*, that the support of F is $[0,+\infty]$. Equivalently, perform an i.i.d. Bernoulli bound percolation of parameter $p_\infty = F([0,+\infty))$, associate an infinite passage time each closed edge, and then independently associate with each open edge a finite passage time (whose distribution is F conditioned to be finite). We want to study the behavior of $T(0,x)$ when $\|x\|_1$ is large. If 0 and x are not connected by a path of edges with finite passage time, then $T(0,x) = +\infty$. To have hope to deal with finite passage times $T(0,x)$ for arbitrarily large $\|x\|_1$, one should at least suppose that the percolation $(\mathbb{1}_{t(e)<\infty}, e \in \mathbb{E}^d)$ of edges with finite passage time percolates. In fact, we make the assumption that this percolation is supercritical, *i.e.*, we suppose that $p_\infty = F([0,+\infty)) > p_c(d)$.

We extend to this setting the definition of the time constant and the shape theorem without any moment assumption, at the price of weakening the convergence we prove. In fact, we follow Cox, Durrett and Kesten [CD81, Kes86] and define regularized passage times that converge in a stronger sense towards the time constant and the asymptotic shape. Since the only assumption we make on $F(\{+\infty\})$ is that $F(\{+\infty\}) < 1 - p_c(d)$, it is not the case that any point x is surrounded by a shape of edges with finite passage time that disconnect x from infinity. Thus we need to propose a definition of regularized passage times that is different of the one proposed by Cox, Durrett and Kesten [CD81, Kes86]. Let $M > 0$ be such that $F([0,M]) > p_c(d)$. We denote by $\mathcal{C}_\infty$ (resp. $\mathcal{C}_M$) the a.s. unique infinite cluster of the percolation $(\mathbb{1}_{\{t(e)<\infty\}}, e \in \mathbb{E}^d)$ (resp. $(\mathbb{1}_{\{t(e)\leq M\}}, e \in \mathbb{E}^d)$), *i.e.*, the percolation obtained by keeping only edges with finite passage time (resp. with passage time less than M). For any $x \in \mathbb{Z}^d$, we define $\widetilde{x}^{\mathcal{C}_\infty}$ (resp. $\widetilde{x}^{\mathcal{C}_M}$) as the point y of $\mathcal{C}_\infty$ (resp. $\mathcal{C}_M$) which minimizes $\|x-y\|_1$ (with a deterministic rule to break ties). For any $x, y \in \mathbb{Z}^d$, we define two regularized passage times, namely

$$\widetilde{T}^{\mathcal{C}_\infty}(x,y) \,=\, T(\widetilde{x}^{\mathcal{C}_\infty}, \widetilde{y}^{\mathcal{C}_\infty})$$

and

$$\widetilde{T}^{\mathcal{C}_M}(x,y) \,=\, T(\widetilde{x}^{\mathcal{C}_M}, \widetilde{y}^{\mathcal{C}_M})\,.$$

For this second time, the parameter M only plays a role in the choice of $\widetilde{x}^{\mathcal{C}_M}$ and $\widetilde{y}^{\mathcal{C}_M}$. Once these points are chosen, the optimization in $\widetilde{T}_G^{\mathcal{C}_M}(x,y)$ is on all paths between $\widetilde{x}^{\mathcal{C}_M}$ and $\widetilde{y}^{\mathcal{C}_M}$, paths using edges with passage time larger than M included. But as $\widetilde{x}^{\mathcal{C}_M} \in \mathcal{C}_M$ and $\widetilde{y}^{\mathcal{C}_M} \in \mathcal{C}_M$, we know that there exists a path using only edges with passage time less than M between these two points. To be more precise, we denote by $D^{\mathcal{C}}(x,y)$ the chemical distance (or graph distance) between two vertices x and y on a cluster $\mathcal{C}$:

$$\forall x,y \in \mathbb{Z}^d\,, \quad D^{\mathcal{C}}(x,y) \,=\, \inf\{|r| \,:\, r \text{ is a path from } x \text{ to } y\,,\, r \subset \mathcal{C}\}\,,$$

where $\inf \varnothing = +\infty$. Then, for any $x,y \in \mathbb{Z}^d$,

$$\widetilde{T}^{\mathcal{C}_M}(x,y) \,\leq\, M D^{\mathcal{C}_M}(\widetilde{x}^{\mathcal{C}_M}, \widetilde{y}^{\mathcal{C}_M})\,.$$

Antal and Pisztora [AP96] proved a control on the tail distribution of the chemical distance in a supercritical Bernoulli percolation cluster. Together with a control on the size of the holes of the infinite cluster of a supercritical percolation, this gives us good integrability properties for the regularized passage time $\widetilde{T}^{\mathcal{C}_M}$.

Proposition 39. *Let F be a probability measure on $[0,+\infty]$ such that $F([0,+\infty)) > p_c(d)$. For every $M \in \mathbb{R}^+$ such that $F([0,M]) > p_c(d)$, there exist positive constants C_1, C_2 and C_3 such that*

$$\forall x \in \mathbb{Z}^d\,,\ \forall l \geq C_3\|x\|_1\,, \quad \mathbb{P}\left[\widetilde{T}^{\mathcal{C}_M}(0,x) > l\right] \,\leq\, C_1 e^{-C_2 l}\,.$$

Proposition 39 implies in particular that the times $\widetilde{T}^{\mathcal{C}_M}(0,x)$ are integrable. A classical application of a subadditive ergodic theorem gives the existence of a time constant defined as the a.s. limit of the rescaled passage times regularized at level M. By a comparison between $\widetilde{T}^{\mathcal{C}_M}(0,x)$, $\widetilde{T}^{\mathcal{C}_\infty}(0,x)$ and $T(0,x)$, we prove that the regularized times $\widetilde{T}^{\mathcal{C}_\infty}(0,x)$ and the non-regularized times $T(0,x)$ also converge to this time constant, but in a weaker sense.

Theorem 40. *Let F be a probability measure on $[0,+\infty]$ such that $F([0,+\infty)) > p_c(d)$. There exists a deterministic function $\mu : \mathbb{R}^d \to [0,+\infty)$ such that for every $M \in \mathbb{R}^+$ satisfying $F([0,M]) > p_c(d)$, we have the following convergences:*

$$\forall x \in \mathbb{Z}^d\,, \quad \mu(x) \,=\, \inf_{n\in\mathbb{N}^*} \frac{\mathbb{E}\left[\widetilde{T}^{\mathcal{C}_M}(0,nx)\right]}{n} \,=\, \lim_{n\to+\infty} \frac{\mathbb{E}\left[\widetilde{T}^{\mathcal{C}_M}(0,nx)\right]}{n}\,, \tag{5.1}$$

$$\forall x \in \mathbb{Z}^d\,, \quad \lim_{n\to\infty} \frac{\widetilde{T}^{\mathcal{C}_M}(0,nx)}{n} \,=\, \mu(x) \quad \textit{a.s. and in } L^1, \tag{5.2}$$

$$\forall x \in \mathbb{Z}^d\,, \quad \lim_{n\to\infty} \frac{\widetilde{T}^{\mathcal{C}_\infty}(0,nx)}{n} \,=\, \mu(x) \quad \textit{in probability}\,, \tag{5.3}$$

$$\forall x \in \mathbb{Z}^d\,, \quad \lim_{n\to\infty} \frac{T(0,nx)}{n} \,=\, \theta^2\delta_{\mu(x)} + (1-\theta^2)\delta_{+\infty} \quad \textit{in distribution}, \tag{5.4}$$

where $\theta = \mathbb{P}[0 \in \mathcal{C}_\infty]$.

Note that even if the definition (5.1) of $\mu(x)$ requires to introduce a parameter M, the constant $\mu(x)$ does not depend on M. Note also that if instead of taking the point $\widetilde{x}^{\mathcal{C}_M}$ in the infinite cluster $\mathcal{C}_M$ of edges with passage time less than M, we take the point $\widetilde{x}^{\mathcal{C}_\infty}$ in the infinite cluster $\mathcal{C}_\infty$ of edges with finite passage time, the almost sure convergence is weakened into the convergence in probability (5.3). Without any regularization, the convergence in (5.4) is only in law. As in the classical first passage percolation model, the function μ can be extended, by homogeneity, into a pseudo-norm on $\mathbb{R}^d$. We also know that either $\mu = 0$, or μ is indeed a norm. Proposition 39 gives strong enough integrability properties for $\widetilde{T}^{\mathcal{C}_M}(0,x)$ to ensure that the convergence (5.2) is uniform in all directions.

Theorem 41. *Let F be a probability measure on $[0,+\infty]$ such that $F([0,+\infty)) > p_c(d)$. Then for every $M \in \mathbb{R}^+$ such that $F([0,M]) > p_c(d)$, we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^d\,,\,\|x\|_1\geq n} \left| \frac{\widetilde{T}^{\mathcal{C}_M}(0,x) - \mu(x)}{\|x\|_1} \right| = 0 \quad a.s.$$

In the case where μ is a norm, Theorem 41 is equivalent to the classical a.s. shape theorem for the regularized times $\widetilde{T}^{\mathcal{C}_M}$ (see Theorem 43 below for a precise statement). Under the only assumption that μ is a norm, this shape theorem is a tool to study the lower large deviations of the point-to-line passage times, *i.e.*, the infimum of the passage time of a path from the point 0 to the hyperplane $\{x = (x_1,\dots,x_d) \,:\, x_1 = n\}$ for large n, at least for particular probability measures F. This allow us to extend Kesten's study on the positivity of the time constant.

Theorem 42. *Let F be a probability measure on $[0,+\infty]$ such that $F([0,+\infty)) > p_c(d)$. Then either μ is identically equal to 0 or $\mu(x) > 0$ for all $x \neq 0$, and we know that*

$$\mu = 0 \iff F(\{0\}) \geq p_c(d)\,.$$

Suppose now that $F(\{0\}) < p_c(d)$, then μ is a norm and we denote by $\mathcal{B}_\mu$ the unit ball for this norm:

$$\mathcal{B}_\mu = \{x\in\mathbb{R}^d \,:\, \mu(x)\leq 1\}\,.$$

We define B_t (resp. $\widetilde{B}_t^{\mathcal{C}_M}, \widetilde{B}_t^{\mathcal{C}_\infty}$) as the enlarged set of all points reached from the origine within a time t :

$$B_t = \{z+u \,:\, z\in\mathbb{Z}^d\,,\, T(0,z)\leq t\,,\, u\in[-1/2,1/2]^d\}\,.$$

(resp. $\widetilde{T}^{\mathcal{C}_M}, \widetilde{T}^{\mathcal{C}_\infty}$). Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$, and $A\triangle B$ be the symmetric difference between two sets A and B. Roughly speaking, $\mathcal{B}_\mu$ is the limit of the sets B_t, $\widetilde{B}_t^{\mathcal{C}_M}$ and $\widetilde{B}_t^{\mathcal{C}_\infty}$.

Theorem 43. *Let F be a probability measure on $[0,+\infty]$ such that $F([0,+\infty)) > p_c(d)$ and $F(\{0\}) < p_c(d)$. Then*

$$\forall\varepsilon>0\,,\ a.s.,\ \exists t_0\in\mathbb{R}^+\,,\ \forall t\geq t_0\,, (1-\varepsilon)\mathcal{B}_\mu \subset \frac{\widetilde{B}_t^{\mathcal{C}_M}}{t} \subset (1+\varepsilon)\mathcal{B}_\mu\,, \tag{5.5}$$

$$a.s., \quad \lim_{t\to\infty} \mathcal{L}^d\left(\frac{\widetilde{B}_t^{\mathcal{C}_\infty}}{t}\triangle\mathcal{B}_\mu\right) = 0 \tag{5.6}$$

and on the event $\{0\in\mathcal{C}_\infty\}$,

$$a.s., \quad \frac{1}{t^d}\sum_{x\in B_t\cap\mathbb{Z}^d} \delta_{x/t} \ \text{ converges weakly towards } \ \theta\mathbb{1}_{\mathcal{B}_\mu}\mathcal{L}^d \ \text{ when } t\to\infty\,. \tag{5.7}$$

As we said previously, the convergence (5.5) is in fact equivalent to the uniform convergence, Theorem 41, once the positivity of μ is known from Theorem 42. The convergence (5.6) can be deduced from (5.5) by a comparison between $\widetilde{T}^{\mathcal{C}_M}(0,x)$ and $\widetilde{T}^{\mathcal{C}_\infty}(0,x)$ for any x in a compact set and a wise use of an ergodic theorem. The convergence (5.7) can be deduced from (5.6) by using the fact that, at a mesoscopic level, the density of points inside $\mathcal{C}_\infty$ is asymptotically θ.

5.2 Continuity of the time constant in a supercritical Bernoulli percolation cluster

In this section and in the next one, we present the results we obtained with Olivier Garet, Régine Marchand and Eviatar B. Procaccia in [GMPT15]. The time constant is now defined for general i.i.d. first passage percolation with possibly infinite passage time, and we want to extend the property of the continuity of the time constant with respect to the distribution of the passage times in this setting. Since we now have to deal with different probability measures, we emphasize the dependence on the distribution F we consider by adding a subscript on all the quantities we manipulate: $t_F(e), T_F, \mathcal{C}_{F,M}, \mathcal{C}_{F,\infty}, \mu_F$. We obtain the following result.

Theorem 44. *Let $F,(F_n)_{n\in\mathbb{N}}$ be probability measures on $[0,+\infty]$ such that for every $n\in\mathbb{N}$, $F_n([0,+\infty))>p_c(d)$ and $F([0,+\infty))>p_c(d)$. If F_n converges weakly towards F, then*

$$\lim_{n\to+\infty}\sup_{x\in\mathbb{S}^{d-1}}|\mu_{F_n}(x)-\mu_F(x)| \,=\, 0\,.$$

This implies immediately the convergence of the corresponding asymptotic shapes, *i.e.*, the convergence of the unit balls $\mathcal{B}_{\mu_{F_n}}$ towards $\mathcal{B}_{\mu_F}$, when they are defined.

Corollary 45. *Let $F,(F_n)_{n\in\mathbb{N}}$ be probability measures on $[0,+\infty]$ such that for every $n\in\mathbb{N}$, $F_n([0,+\infty))>p_c(d)$, $F([0,+\infty))>p_c(d)$ and $F(\{0\})<p_c(d)$. If F_n converges weakly towards F, then*

$$\lim_{n\to+\infty} d_H(\mathcal{B}_{\mu_{F_n}},\mathcal{B}_{\mu_F}) \,=\, 0\,,$$

where d_H is the Hausdorff distance between non-empty compact sets of $\mathbb{R}^d$.

Particularly, when $F_p=p\delta_1+(1-p)\delta_{+\infty}$, the norm μ_{F_p} governs the asymptotic distance in the infinite cluster of a supercritical Bernoulli percolation (see [GM04, GM07, GM10]). We get the following corollary.

Corollary 46. *For $p>p_c(d)$, let us denote by $\mathcal{B}_p$ the unit ball for the norm that is associated to the chemical distance in supercritical bond percolation with parameter p. Then,*

$$p\in(p_c(d),1]\mapsto\mathcal{B}_p$$

is continuous for the Hausdorff distance between non-empty compact sets of $\mathbb{R}^d$.

The structure of the proof follows the approach initiated by Cox and Kesten [CK81, Kes86]. It is divided into three steps:

Step 1 - Prove that if F_n, F are probability measures on $[0,+\infty]$ such that $F_n([0,+\infty)) > p_c(d)$ and $F([0,+\infty)) > p_c(d)$, if F_n converges weakly towards F and if F_n dominates stochastically F for all $n \in \mathbb{N}$, then for every $x \in \mathbb{Z}^d$,

$$\limsup_{n\to\infty} \mu_{F_n}(x) \;\leq\; \mu_F(x)\,.$$

Step 2 - Prove the continuity of the time constant, Theorem 44, for probability measures F_n, F with compact support on $\mathbb{R}^+$.

Step 3 - Prove that if F is a probability measure on $[0,+\infty]$ such that $F([0,+\infty)) > p_c(d)$, and if $F^K = \mathbb{1}_{[0,K)}F + F([K,+\infty])\delta_K$ is the distribution of the passage times truncated a level K, then for every $x \in \mathbb{Z}^d$,

$$\lim_{K\to\infty} \mu_{F^K}(x) \;=\; \mu_F(x)\,.$$

By a coupling between probability measures, it is easy to see in the classical finite first passage percolation that when a probability measure G stochastically dominates a probability measure F, then $\mu_G(x) \geq \mu_F(x)$ for every $x \in \mathbb{Z}^d$. The same result is not obvious in the case of possibly infinite passage times because of the use of the regularized passage time $\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,x) = T_G(\widetilde{0}^{\mathcal{C}_{G,M}}, \widetilde{x}^{\mathcal{C}_{G,M}})$ and $\widetilde{T}_F^{\mathcal{C}_{F,M}}(0,x) = T_F(\widetilde{0}^{\mathcal{C}_{F,M}}, \widetilde{x}^{\mathcal{C}_{F,M}})$. Indeed, the starting and ending point of the geodesics considered here depend on the distributions G and F. A preliminary step is thus to get rid of this dependance, by proving that

$$\forall x \in \mathbb{Z}^d\,, \quad \mu_F(x) \;=\; \inf_{n\to\infty} \frac{\mathbb{E}\left[T_F(\widetilde{0}^{\mathcal{C}_{H,M}}, \widetilde{nx}^{\mathcal{C}_{H,M}})\right]}{n} \;=\; \lim_{n\to\infty} \frac{T_F(\widetilde{0}^{\mathcal{C}_{H,M}}, \widetilde{nx}^{\mathcal{C}_{H,M}})}{n} \quad \text{a.s.}\,, \tag{5.8}$$

for an alternative probability measure H and a sufficiently large constant M. Using the alternative definition (5.8) of the norm μ_F, it is easy to recover the monotonicity of $\mu_F(x)$ with respect to F.

The proof of Step 1 is a straightforward adaptation of Cox and Kesten's arguments in [Cox80, CK81, Kes86]. It relies on the coupling between laws, on the definition of $\mu_F(x)$ as an infimum and on the integrability of the regularized passage times without moment assumption on the passage times of the edges. We obtained all the needed ingredients in [CT14b] (see the previous section). Step 2 remains the same as in Kesten [Kes86], since it deals only with passage times with compact support on $\mathbb{R}^+$. The delicate part of the proof is Step 3. Following the strategy of Cox and Kesten [CK81, Kes86], we consider a fixed x in $\mathbb{Z}^d$ and a geodesic γ for the time $T_{F^K}(0,x)$ (we omit here intentionally the question of the regularization of the times). This geodesic contains bad edges e such that $T_F(e) > K$. To control the difference $T_F(0,x) - T_{F^K}(0,x)$, we transform the path γ into a path γ' by removing these bad edges and bypass each one of them, so that $T_F(\gamma') - T_{F^K}(\gamma)$ is small. Since the only assumption we make on $F(\{+\infty\})$ is that $F(\{+\infty\}) < 1 - p_c(d)$, it is not the case that any edge e is surrounded by a shape of edges with bounded passage time that disconnect e from infinity, thus we have no hope to construct the bypass around a bad edge e of γ inside such a shape, as Cox and Kesten did. This is the reason why we cannot perform our construction of γ' at the microscopic scale. We use instead a coarse graining argument, in the spirit of the work of Antal and Pisztora [AP96], to construct the bypasses at the scale of good blocks.

5.3 Continuity in related models

5.3.1 Cheeger constant

In this section we restrict ourselves to the dimension $d = 2$. As presented in Section 1.3.1, the study of the Cheeger constant is closely linked with the study of first passage percolation by the mere definition of the underlying norms in these two models: the norm β_p in the isoperimetric problem and the norm μ_F in first passage percolation. We recall the definition on β_p for a fixed parameter $p > p_c(2) = 1/2$. We denote by $\mathcal{C}_p$ is the a.s. unique infinite cluster of the supercritical Bernoulli percolation on $\mathbb{Z}^2$ of parameter p. With any path γ we associate

$$\mathbf{b}_p(\gamma) \,=\, |\{e \in \partial^+\gamma \,:\, e \text{ is } p\text{-open}\}| \,,$$

where $\partial^+\gamma$ is the right-boundary of γ. For all $x, y \in \mathcal{C}_p$ we define

$$b_p(x,y) \,=\, \inf\{\mathbf{b}_p(\gamma) \,:\, \gamma \text{ is a right-most path from } x \text{ to } y\} \,.$$

The norm β_p is finally defined via the ergodic subadditive theorem as the following limit:

$$\forall x \in \mathbb{R}^2 \,, \quad \beta_p(x) \,=\, \lim_{n\to\infty} \frac{b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p})}{n} \quad \text{a.s. and in } L^1 \,.$$

The Cheeger constant $\lim_{n\to\infty} n\varphi_n(p)$ is defined as the solution of a continuous isoperimetric problem associated with the norm β_p, and the shape $\widehat{W}_p$ is defined as the Wulff crystal associated with β_p (for more definitions, we refer to Section 1.3.1). Together with Olivier Garet, Régine Marchand and Eviatar B. Procaccia, we prove in [GMPT15] the following result.

Theorem 47. *Let $d = 2$. The applications*

$$p \in (p_c(2), 1] \mapsto \lim_{n\to\infty} n\varphi_n(p) \quad \textit{and} \quad p \in (p_c(2), 1] \mapsto \widehat{W}_p$$

are continuous, the last one for the Hausdorff distance between non-empty compact sets of $\mathbb{R}^2$.

Theorem 47 is a straightforward consequence of the continuity of the norm β_p.

Proposition 48. *Let $d = 2$. For every $p \in (p_c(2), 1]$,*

$$\lim_{q\to p} \sup_{x\in\mathbb{S}^1} |\beta_q(x) - \beta_p(x)| \,=\, 0 \,.$$

As in the previous section, we consider a coupling of the percolations with different parameters. If $p_c(2) < p < q$, there is no trivial comparison between $\beta_p(x)$ and $\beta_q(x)$:

- if γ is a q-open path from 0 to x such that $b_q(0,x) = \mathbf{b}_q(\gamma)$, then γ may not be p-open;
- if γ is a p-open path from 0 to x such that $b_p(0,x) = \mathbf{b}_p(\gamma)$, then γ is q-open by coupling but $\mathbf{b}_q(\gamma)$ can be bigger than $\mathbf{b}_p(\gamma)$ since some right-most edges of γ can be p-closed but q-open.

However, the proof of Proposition 48 is very similar to the step 3 of the proof of the continuity of the time constant in first passage percolation, Theorem 44. Let $p_c(2) < p < q$, and consider a q-open path γ from 0 to x such that $b_q(0,x) = \mathbf{b}_q(\gamma)$. Some edges of γ are bad, in the sense that they are q-open but p-closed. We construct a modification γ' of γ which is p-open by removing these bad edges, and bypassing them at the scale of good mesoscopic blocks, with the same construction as in the proof of Theorem 44. We control $b_p(0,x) - b_q(0,x)$ with $\mathbf{b}_p(\gamma') - \mathbf{b}_q(\gamma)$, and this allows us to conclude the proof of Proposition 48.

5.3.2 Contact process

We consider again the case of any dimension $d \geq 2$. This section is devoted to the result we obtain with Olivier Garet and Régine Marchand in [GMT15] about the contact process. Let $\lambda > \lambda_c$ be a supercritical rate of infection, the rate of recovering being 1. We recall that the time constant μ_λ associated with a supercritical contact process $(\xi_t^{\lambda,\{0\}})_{t\geq 0}$ is defined by Garet and Marchand in [GM12, GM14] via an almost subadditive ergodic theorem as

$$\forall x \in \mathbb{Z}^d, \quad \mu_\lambda(x) \,=\, \lim_{n\to\infty} \frac{\sigma_\lambda^{\{0\}}(nx)}{n} \quad \overline{\mathbb{P}}_\lambda\text{-a.s. and in } L^1(\overline{\mathbb{P}}_\lambda)\,,$$

where $\sigma_\lambda^{\{0\}}(x)$ is the essential hitting time of $x \in \mathbb{Z}^d$, that can be seen as a renewal point at which x is infected and this infection survives. We state the following result.

Theorem 49. *For every $\lambda > \lambda_c(\mathbb{Z}^d)$, we have*

$$\lim_{\lambda'\to\lambda} \sup_{x\in\mathbb{S}^{d-1}} |\mu_{\lambda'}(x) - \mu_\lambda(x)| \,=\, 0\,.$$

Let $\mathcal{B}_{\mu_\lambda}$ (resp. $\mathcal{B}_{\mu_{\lambda'}}$) be the unit ball associated with the norm μ_λ (resp. $\mu_{\lambda'}$). As a straightforward consequence of Theorem 49, we obtain the convergence of the corresponding asymptotic shapes.

Corollary 50. *For every $\lambda > \lambda_c(\mathbb{Z}^d)$, we have*

$$\lim_{\lambda'\to\lambda} d_H(\mathcal{B}_{\mu_{\lambda'}}, \mathcal{B}_{\mu_\lambda}) \,=\, 0\,,$$

where d_H denotes the Hausdorff distance between non-empty compact sets of $\mathbb{R}^d$.

In the context of the contact process we recover by coupling the monotonicity of the norm μ_λ with respect to the parameter λ: $\lambda \mapsto \mu_\lambda$ is non increasing on $(\lambda_c(\mathbb{Z}^d), +\infty)$.

The first part of the proof is, exactly as in the proof of Theorem 44, to establish the left-continuity $\lambda \mapsto \mu_\lambda(x)$ for any $x \in \mathbb{Z}^d$. Similarly to the proof of Theorem 44, this left-continuity is a consequence of a coupling of contact processes with different parameters, of the expression of $\sigma_\lambda^{\{0\}}$ as an infimum through an almost subadditive ergodic theorem and of the integrability of $\sigma_\lambda^{\{0\}}$. The proof is a little bit complicated by the necessity to condition on the survival of the processes, but remains quite easy.

The study of the right-continuity is more delicate, and is rather different from the corresponding part of the proof of Theorem 44. It relies on ideas used by Garet and Marchand in [GM14] to study the lower large deviations of the contact process in random environment. For a fixed parameter $\lambda > \lambda_c(\mathbb{Z}^d)$, their argument can be roughly summarize as follows. Suppose that the infected region is abnormally big at a time t, *i.e.*, some essential hitting times $\sigma_\lambda^{\{0\}}(x)$ are abnormally small. They consider boxes at a mesoscopic scale, and say that a box is good if the infection propagates at a typical speed in this box. The fact that $\sigma_\lambda^{\{0\}}(x)$ is abnormally small for some x implies that there exists a too fast path of infection from 0 to x, along which a positive proportion of boxes cannot be good. Since the probability that a box is good can be chosen as close to 1 as we need (by considering boxes large enough), the probability that there exists x such that $\sigma_\lambda^{\{0\}}(x)$ is abnormally small decays exponentially fast with $\|x\|_1$.

Consider now $\lambda_c < \lambda \leq \lambda'$, thus $\mu_{\lambda'} \leq \mu_\lambda$. If $\mu_{\lambda'} \leq \mu_\lambda - \varepsilon$ for a positive ε, then $\mathcal{B}_{\mu_\lambda}$ is strictly included in $\mathcal{B}_{\mu_{\lambda'}}$. We consider the rescaling by blocks introduced by Garet and Marchand in the study of the large deviations for the contact process of parameter λ. Since the boxes have finite size, we can suppose that with high probability the infection events are exactly the same for the contact processes of parameters λ and λ' in such boxes, simply by choosing λ' close enough to λ. In this case, the contact process $\xi^{\lambda',\{0\}}$ is very similar to $\xi^{\lambda,\{0\}}$ in all such good boxes, and thus its asymptotic shape should be given by $\mathcal{B}_{\mu_\lambda}$. The fact that the infected region looks like $\mathcal{B}_{\mu_{\lambda'}}$, if $\mathcal{B}_{\mu_{\lambda'}}$ is perceptibly bigger than $\mathcal{B}_{\mu_\lambda}$, can be seen as a large deviation event for the process $\xi^{\lambda',\{0\}}$, and the probability that it happens can be controlled in the same way as the lower large deviations for the contact process $\xi^{\lambda,\{0\}}$.

Chapter 6

Open problems

We try to summarize in this section some open questions related to our works, and in which we are interested. Some of them are on-going works, alone or in collaboration with colleagues.

6.1 Maximal flow in first passage percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$

Some questions are directly linked with the large deviations of maximal flows on $(\mathbb{Z}^d, \mathbb{E}^d)$.

Question 1. *Prove a large deviation principle of speed $\mathcal{H}^{d-1}(nA)h(n)$ corresponding to the upper large deviations of the maximal flows $\phi(nA, h(n))$ through tilted cylinders.*

Such an upper large deviation principle is probably the missing piece we need to tackle the problem of proving a more general large deviation principle on the environment for the streams in a domain.

Question 2. *Prove a large deviation principle for the environment in a general domain $(\Omega, \Gamma^1, \Gamma^2)$, from which can be derived large deviation principles for the maximal flows ϕ_n from Γ_n^1 to Γ_n^2 in Ω_n, for the corresponding maximal streams and the corresponding minimal cutsets.*

This could give us a general picture of all the large deviations for flows through domains.

Another direction we are interested in is the study of the behavior of maximal flows for capacities with heavy tail distribution. A first question is the definition of the asymptotic rescaled maximal flow ν without moment assumption.

Question 3. *Prove the existence of an asymptotic rescaled maximal flow ν without moment assumption. Give some necessary and sufficient conditions for the a.s. convergence of the rescaled flows $\tau(nA, h(n))$ and $\phi(nA, h(n))$ towards this constant.*

As we have seen in Chapter 5, answering this question could give us tools to study the continuity of ν.

Question 4. *Prove that ν is continuous with respect to the distribution of the capacities of the edges.*

Another approach to this question is to understand the link between the tail distribution of F and that of $\phi(A, h)$, in the spirit of the work of Ahlberg [Ahl15] concerning the distance in classical first passage percolation.

Question 5. *Understand the link between* $\mathbb{P}[t(e) \geq x]$ *and* $\mathbb{P}[\phi(A,h) \geq x]$.

A third direction we are interested in is the study of maximal flows through a non-homogeneous domain $(\Omega, \Gamma^1, \Gamma^2)$. Let F_x be the distribution the passage times of the edges near $x \in \Omega$. If $x \mapsto F_x$ varies smoothly enough, thanks to the (likely) continuity of $F \mapsto \nu_F$ and the approach we developed to study the maximal flows through $(\Omega, \Gamma^1, \Gamma^2)$ in the homogeneous case using boxes of mesoscopic size (see Chapter 4), we have hope to be able to study maximal flows in a non-homogeneous setting.

Question 6. *Prove the a.s. convergence of maximal flows, maximal streams and minimal cutsets through a domain* $(\Omega, \Gamma^1, \Gamma^2)$ *with independent capacities whose distribution varies smoothly inside the domain.*

6.2 Random distance in first passage percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$

Large deviations for the first passage times $T(0,x)$ and related estimates for the set $B(t)$ of points reached within time t have already been studied, see Kesten's paper [Kes93] for instance. However, concerning the set $B(t)$, these large deviations are somehow frustrating, since they only control the probability that one point which is abnormally far away belongs to $B(t)$, or that a point which is abnormally close to the origin does not belong to $B(t)$. We would like to understand how goes to zero the probability that $B(t)/t$ diverges from its asymptotic shape $\mathcal{B}_\mu$ in volume.

Question 7. *Study* $\mathbb{P}[\mathcal{L}^d(B(t), tA) \leq \varepsilon t]$ *for* $A \neq \mathcal{B}_\mu$.

The next step would be to prove a corresponding large deviation principle in volume.

Question 8. *Prove a large deviation principle for* $B(t)/t$ *in volume.*

Another direction that interests us is the study of first passage percolation in the non i.i.d. but ergodic and stationary case. Garet and Marchand [GM04] generalized results of Boivin [Boi90] in this setting. However, Garet and Marchand study the case of a stationary and ergodic probability measure for the passage times on the infinite cluster of an i.i.d. supercritical Bernoulli percolation. What if the underlying percolation itself is not i.i.d. ? Drewitz, Ráth and Sapozhnikhov [DRS14] made recently great advances in this direction by proving the existence of a time constant and a shape theorem for the chemical distance in percolation models with long-range correlations. However, many questions remain open, like the continuity of the time constant with respect to the parameter of the percolation in this setting.

Question 9. *Prove the continuity of the time constant with respect to the parameter of the percolation in percolation models with long-range correlations.*

6.3 Isotropic models of first passage percolation

When we study the model of first passage percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$, the time constant $\mu(\vec{v})$ (resp. the asymptotic rescaled maximal flow $\nu(\vec{v})$) we obtain depends on the direction $\vec{v}$ we consider. This can be an issue for technical reasons - the properties of the shape $\mathcal{B}\mu$, for instance the question of its strict convexity in any given direction, are not well understood - and when we have in mind some applications - when first passage percolation is seen as a model for communication networks, it may not be relevant to consider a non-isotropic setting.

It is possible to define an isotropic model of first passage percolation by considering i.i.d. passage times associated with the edges of an underlying random graph such as the Delaunay graph. This graph is constructed as follows. Let $Q \subset \mathbb{R}^d$ be a homogeneous and stationary Poisson point process on $\mathbb{R}^d$. The cell associated with a point $x \in Q$ is defined as the set of all points in $\mathbb{R}^d$ that are closest to x than to any other point of Q. The Delaunay graph has for vertices the points of Q, and two such vertices are linked by an edge if and only if the boundaries of their cells has a common face of dimension $d-1$. Given such a graph, one can associated independently with the edges a family of i.i.d. passage times. First passage percolation on the Delaunay graph has already been studied, see the review of Howard [How04] and the works of Wahidi-Asl and Wierman [VAW90, VAW92]. However some particular properties of this model are still not yet fully understood, and we are particularly interested in the large deviations aspect.

Question 10. *Complete the study of the large deviations of the distance in first passage percolation on the Delaunay graph.*

Moreover, maximal flows in first passage percolation on the Delaunay graph have never been studied.

Question 11. *Study the maximal flows in first passage percolation on the Delaunay graph (convergence, large deviations...).*

Another approach is to consider the Boolean model. Consider a homogeneous and stationary Poisson point process Q on $\mathbb{R}^d$, with intensity λ with respect to the Lebesgue measure. We center at each point $x \in Q$ a ball of random radius r_x, such that the radii are i.i.d. with common distribution the probability measure G on $\mathbb{R}^+$. The union of all these balls is denoted by Σ, which is thus a random subset of $\mathbb{R}^d$. This defines the Poisson Boolean model, see for instance the book of Meester and Roy [MR96] on the subject. It is well known that given the dimension d and the law G of the radii, a phase transition occurs: there exists a parameter $\lambda_c \in \mathbb{R}^+$ such that if $\lambda > \lambda_c$ then a.s. the set Σ is not bounded - we say that Σ percolates -, and if $\lambda < \lambda_c$ then a.s. the set Σ is bounded. Moreover Gouéré [Gou08] proved that $\lambda_c > 0$ if and only if G admits a moment of order d.

On this Boolean model, Gouéré and Marchand [GM08] defined a first passage percolation by allowing a particle to travel at speed 1 outside Σ and at infinite speed inside Σ. This leads to the definition of a random pseudo-metric T_λ, and by an application of a classical ergodic subadditive theorem they define a time constant μ_λ as the a.s. and L^1 limit of the rescaled times $T(0,x)/\|x\|_2$. Since the model is isotropic, this time constant does not depend on x. The constant μ_λ^{-1} is the asymptotic speed at which a particle can travel in this model. If $\lambda > \lambda_c$, since Σ percolates, it is easy to see that $\mu_\lambda = 0$. We define $\lambda_\mu = \inf\{\lambda : \mu_\lambda = 0\}$. Gouéré and Marchand gave in [GM08] some necessary and some sufficient conditions on G to have $\lambda_\mu > 0$. We want to prove that this critical parameter is the same as λ_c.

Question 12. *Prove that $\lambda_\mu = \lambda_c$. Prove that $\mu_{\lambda_c} = 0$.*

The case of unbounded radii is of particular interest since we have to deal with long distance correlations.

We would also like to define and study a maximal flow through the vacant set (*i.e.*, Σ^c) of a Boolean model, since it could be an interesting continuous generalization of maximal flows on $(\mathbb{Z}^d, \mathbb{E}^d)$.

Question 13. *Define a maximal flow through the vacant set of a Boolean model and study its asymptotic behavior.*

6.4 Related models

As we have seen in Section 5.3, methods developed to study first passage percolation can be useful to study other growth models. We are interested in generalizing the continuity results we obtained in first passage percolation and for the contact process.

Question 14. *Give a general set of hypotheses that assure that the time constant associated with a growth process is continuous with respect to the law of the process.*

The study of the Cheeger constant in Section 1.3.1 is restricted to the case of dimension 2. However, Gold [Gol16] extended recently the works of Biskup, Louidor, Procaccia and Rosenthal [BLPR12] to any dimension d, proving the existence of the Cheeger constant via a generalized definition of the norm β_p. We are interested in extended the proof of the continuity of the Cheeger constant in higher dimension.

Question 15. *Prove the continuity of the Cheeger constant with regard to the parameter of the underlying percolation in any dimension $d \geq 3$.*

This question is closely linked with the study of the continuity of the rescaled maximal flow ν in first passage percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$ since the generalized norm β_p is defined as an infimum of a certain quantity over cutsets.

Consider now the random walk on $\mathbb{Z}^d$ evolving in a i.i.d. random potential $(V(x))_{x\in\mathbb{Z}^d}$. An object of interest in this setting is the Lyapounov exponent. The model of (vertex) first passage percolation on $\mathbb{Z}^d$ can be seen as a zero-temperature limit of this model, the Lyapounov exponent being equal in this case to the time constant. See for instance Mourrat [Mou12] for a shape theorem in this setting, and Le [Le13] for a proof of the continuity of the Lyapounov exponent with respect to the distribution of the potentials. We would like to study the corresponding positive-temperature generalization of minimal cutsets in first passage percolation.

Question 16. *Study a positive-temperature generalization of minimal cutsets in first passage percolation.*

This problem is difficult to tackle. The study of random surfaces is in general very hard, and mathematicians have mainly studied so far more tractable models of random surfaces, namely the effective interface models (see for instance Giacomin [Gia01] and Velenik [Vel06]) where the random surface can be seen as the graph of a random function from $\mathbb{Z}^d$ to $\mathbb{R}$.

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